Galilei invariant nonlinear Schrödinger type equations and their exact solutions

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In this paper we describe wide classes of nonlinear Schrödinger-type PDEs which are invariant under the Galilei group and its generalizations. We construct sets of ansatzes for Galilei invariant equations, and exact classes solutions are found for some nonlinear Schrödinger equations.

1. Introduction

Let us consider the following nonlinear equations

$$L_1(\psi,\psi^*) \equiv S\Psi - F_1(x,\psi,\psi^*),$$
$$S = p_0 - \frac{p_a^2}{2m}, \quad p_0 = i\frac{\partial}{\partial x_0}, \quad p_a = -i\frac{\partial}{\partial x_a}, \quad a = 1,2,3, \tag{1}$$

$$L_2 \equiv p_0\psi + g_{ab}(x_0,\vec{x},\psi,\psi^*)\frac{\partial^2\psi}{\partial x_a\partial x_b} + F_2(x_0,\vec{x},\psi,\psi^*,\underset{1}{\psi},\underset{1}{\psi}^*) = 0,$$
$$\psi \equiv \psi(x_0,x_1,x_2,x_3), \quad x_0 \equiv t, \quad \underset{1}{\psi}(x) = \left\{\frac{\partial\psi}{\partial x_a}\right\}, \quad \underset{1}{\psi}^* = \left\{\frac{\partial\psi^*}{\partial x_a}\right\}, \tag{2}$$

where F_1, F_2, g_{ab} are some smooth functions,

$$L_3\psi \equiv S\psi - F_3(\psi,\psi^*,\underset{1}{\psi},\underset{1}{\psi}^*,\underset{2}{\psi},\underset{2}{\psi}^*),$$
$$\underset{2}{\psi}(x) \equiv \left\{\frac{\partial^2\psi}{\partial x_a\partial x_b}\right\}, \quad \underset{2}{\psi}^*(x) \equiv \left\{\frac{\partial^2\psi^*}{\partial x_a\partial x_b}\right\}. \tag{1$'$}$$

In the present paper we consider the following problems.

Problem 1. Describe all nonlinear equations (1), (2) which are invariant with respect to the Galilei group and its various generalizations.

Problem 2. Study the conditional symmetry of equation (l).

Problem 3. Construct classes of exact solutions for Galilei invariant equations.

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2. Galilei invariant quazilinear equations

Theorem 1 [1]. *There are only three types of equations of the form (1)*

$$S\psi = \lambda F(|\psi|)\psi, \tag{3}$$

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$$S\psi = \lambda|\psi|^k\psi, \quad k \in \mathbb{R}, \tag{4}$$

$$S\psi = \lambda|\psi|^{4/n}\psi, \quad n = 1, 2, 3\ldots, \tag{5}$$

which are invariant, correspondingly, with respect to the following algebras:

$$\begin{gathered} AG(1,n) = \langle P_0, P_a, J_{ab}, G_a, Q\rangle, \quad a = 1, 2, \ldots, n, \\ P_0 = i\frac{\partial}{\partial x_0} \equiv p_0, \quad P_a = -i\frac{\partial}{\partial x_a} \equiv p_a, \\ J_{ab} = x_a p_b - x_b p_a, \quad G_a = x_0 p_a - m x_a Q, \quad Q = i\left(\psi\frac{\partial}{\partial\psi} - \psi^*\frac{\partial}{\partial\psi^*}\right); \end{gathered} \tag{6}$$

$$\begin{gathered} AG(1,n) = \langle AG(1,n), D\rangle, \\ D = 2x_0 p_0 - x_a p_a - kI, \quad k \in \mathbb{R}, \quad I = \psi\frac{\partial}{\partial\psi} + \psi^*\frac{\partial}{\partial\psi^*}; \end{gathered} \tag{7}$$

$$\begin{gathered} AG(1,n) = \langle AG_1(1,n), \Pi\rangle, \\ \Pi = x_0^2 p_0 + x_0 x_a p_a + \frac{m}{2}x^2 Q + \frac{n}{2}x_0 I, \end{gathered} \tag{8}$$

λ is arbitrary parameter, n is the number of space variables.

Note 1. If we put $F = 0$ in (1) we obtain the standard linear Schrödinger equation and its maximal invariance algebra is $AG_2(1,n)$.

Corollary 1. *There is only one nonlinear equation in the class of Schrödinger equations (1)*

$$\left(p_0 - \frac{p_a^2}{2m}\right)\psi = \lambda|\psi|^{4/n}\psi \tag{9}$$

which has the same symmetry as the linear Schrödinger equation.

Let us answer the following question: whether there exist other equations in the class (1) invariant under the Galilei algebra $AG(1,n)$ but not invariant under operators D and Π (7), (8).

The following theorem answers this question.

Theorem 2 [2]. *There is only one equation of the form (1)*

$$\left(p_0 - \frac{p_a^2}{2m}\right)\psi = \lambda\ln(\psi\psi^*)\psi, \quad \lambda = \lambda_1 + i\lambda_2 \tag{10}$$

which is invariant with respect to the following algebras:

$$AG_3(1,n) \equiv \langle AG(1,n), B_1\rangle, \quad \lambda_1 \neq 0, \quad \lambda_2 = 0, \quad B_1 = I - 2\lambda_1 x_0 Q;$$

$$AG_4(2,n) = \langle AG(1,n), B_2\rangle, \quad \lambda_3 \neq 0, \quad B_2 = \exp(2\lambda_2 x_0)(I + i\lambda_1\lambda_2^{-1}Q).$$

Note 2. The maximal invariance algebra of equation (10) with logarithmic nonlinearity contains operators not admitted by the linear equation (1).

Corollary 2. *Operators B_1, B_2 generate the following transformations for ψ:*

$$\begin{gathered} \psi \to \psi' = \exp\{(1 - 2i\lambda_1 x_0)\theta_1\}\psi \quad \textit{for } \lambda_2 = 0, \\ \psi \to \psi' = \exp\{\theta_2[2x_0\lambda_2(1 - i\lambda_1\lambda_2^{-1})\}\psi, \quad \lambda_2 \neq 0, \\ \psi \to \psi' = \exp\{\theta_3\exp(2\lambda_1 x_0)\}\psi, \quad \lambda_1 \neq 0, \quad \lambda_2 \neq 0, \end{gathered}$$

where θ_1, θ_2, θ_3 are group parameters.

The following equation is widely used for description of dissipative systems

$$i\frac{\partial\psi}{\partial t}=\frac{1}{2m}\Delta\psi-i\beta\ln(\psi(\psi^*)^{-1})\psi+F_2(\psi\psi^*)\psi. \tag{11}$$

Equation (11) is usually called the phase Schrödinger equation or the Schrödinger–Langevin equation [4].

The main difference of equation (11) from equations (3), (4), (5), (10), (11) is that it is not invariant with respect to the Galilei transformations. This equation does not the standard Galilei relativity principle. However equation (11) has interesting symmetry properties.

Theorem 3 [2]. *The maximal invariance algebra of equation (11) is a 11-dimensional Lie algebra*

$$A=\langle P_0,P_a,J_{ab},G_a^{(1)},Q\rangle,$$
$$G_a^{(1)}=\exp\{2\beta x_0\}\left(p_a+\frac{\beta m}{2}x_a\right)Q,\quad Q_1=\exp\{2\beta x_0\}Q.$$

Corollary 3. *Operators $G_a^{(1)}$ generate the transformations*

$$x_0\to x_0'=x_0,\quad x_a\to x_a'=x_a+\exp\{2\beta x_0\}\theta_a, \tag{12}$$

$$\psi\to\psi'=\psi\exp\{i[\beta m\exp(4\beta x_0)\theta^2+\exp(2\beta x_0)x_a\theta_a]\}, \tag{13}$$

where $\theta^2=\theta_a\theta_a$, θ_a are group parameters.

So operators $G_a^{(1)}$ as distinguished from the Galilei operators, generate nonlinear transformations (12). In the first approximation by β (12) coincides with the Galilei transformations. It is known that the Galilei transformations are of the form

$$\begin{gathered}x_0\to x_0'=x_0,\quad x_a\to x_a'=x_a+x_0\theta_a,\\ \psi\to\psi'=\exp\left\{im\left(\vec\theta\vec x+\frac{1}{2}(\vec\theta)^2x_0\right)\right\}\psi(x').\end{gathered} \tag{14}$$

3. Galilei invariant nonlinear equations with first order derivatives

Let us consider equations

$$S\psi=F(x,\psi,\psi^*,\underset{1}{\psi},\underset{1}{\psi}^*). \tag{15}$$

Theorem 4 [5]. *There exist four classes of equations of the form (15) which are invariant with respect to Galilei algebras:*

$$AG_1(1,n):\quad S\psi=F_1(|\psi|,(\vec\nabla|\psi|)^2)\psi; \tag{16}$$

$$AG_1(1,n):\quad S\psi=|\psi|^{-2/k}F_2(|\psi|^{-2+2/k}(\vec\nabla|\psi|)^2)\psi, \tag{17}$$

$$S\psi=(\vec\nabla|\psi|)^2F_3(|\psi|)\psi; \tag{18}$$

$$AG_2(1,n):\quad S\psi=|\psi|^{4/n}F_4(|\psi|^{-2-4/n}(\vec\nabla|\psi|)^2). \tag{19}$$

Let us adduce some simplest $G_2(1,3)$ invariant equations:

$$S\psi = \lambda|\psi|^{4/3}\psi, \tag{20}$$

$$S\psi = \lambda|\psi|^2 \frac{\partial|\psi|\partial|\psi|}{\partial x_a \partial x_a}\psi. \tag{21}$$

4. Conditional symmetry of the nonlinear Schrödinger equation

Let us consider some nonlinear differential equation of s-th order:

$$L(x, \psi, \underset{1}{\psi}, \underset{2}{\psi}, \ldots, \underset{s}{\psi}) = 0, \tag{22}$$

$\underset{s}{\psi}$ designates the set of all s-th order derivatives.

Let us assume that equation (22) is invariant with respect to a certain Lie algebra $A = \langle X_1, X_2, \ldots, X_n\rangle$, where X_k are basis vectors of the algebra A.

This means that the following conditions must be satisfied:

$$\underset{s}{X_k} L = RL, \tag{23}$$

where $\underset{s}{X_k}$ is the s-th prolongation of the operator $X_k \in A$, $R = R(x, \psi, \underset{1}{\psi}, \ldots)$ is some differential expression.

Let us consider a set of operators which do not belong to the invariance algebra of equation (22)

$$Y = \langle Y_1, Y_2, \ldots, Y_r\rangle, \quad Y_k \in A.$$

Definition 1 [6, 7]. We shall say that equation (22) is conditionally invariant with respect to the operators Y if there exists an additional condition

$$\tilde{L}_1(x, \psi, \underset{1}{\psi}, \ldots,) = 0 \tag{24}$$

on solutions of equation (22), such that equation (22) together with (24) is invariant with respect to the set of operators Y. This means that the following conditions are satisfied:

$$\underset{s}{Y_k} L = R_0 L + R_1 \tilde{L}_1, \quad \underset{s}{Y_k} \tilde{L} = R_2 L + R_3 \tilde{L}_1,$$

where R_0, R_1, R_2, R_3 are some smooth functions, $\underset{s}{Y_k}$ is the s-th Lie prolongation of the operator $Y_k \in Y$.

It is evident that Definition 1 makes sense only if system (23), (24) is compatible.

The notion of conditional symmetry has turned out extremely efficient, and during recent years it was established that d'Alembert, Schrödinger, Maxwell, heat, Boussinesq equations possess nontrivial conditional symmetry. The problem of detailed description of conditional symmetry for principal equations of mathematical physics remains open [6, 7].

Theorem 5 [2, 8]. *Equation*

$$\left(p_0 - \frac{p_a^2}{2m}\right)\psi = F(|\psi|)\psi \tag{25}$$

is conditionally invariant with respect to the operator

$$Y = x_a p_a + r\left(\psi\frac{\partial}{\partial\psi} + \psi^*\frac{\partial}{\partial\psi^*}\right) - i\ln(\psi(\psi^*)^{-1})Q, \tag{26}$$

if

$$F = a_1|\psi|^{2r^{-1}} + a_2|\psi|^{-2r^{-1}}, \quad r \neq 0 \tag{27}$$

$$\tilde{L}_1(u) = \Delta|\psi| - a_3|\psi|^{(r-2)/r} = 0, \quad a_3 = \frac{1}{2}a_2 m, \quad r, a_1, a_2 \in \mathbb{R}. \tag{28}$$

Corollary 4. *Operator (26) generates the following finite transformations*

$$x_0 \to x_0' = x_0, \quad x_a \to x_a' = \exp\theta \cdot x_a, \tag{29}$$

$$\psi \to \psi' = \exp(r\theta)\exp\{\exp(2\theta)\}(\psi(\psi^*)^{-1})^{1/2}|\psi|, \tag{30}$$

θ is the group parameter.

Formula (30) gives nonlinear transformations for the function ψ.

So equation (25), (27) together with (28) admits an additional operator Y (26). Equation (25) with the nonlinearity (27) without the additional condition (28) is not invariant with respect to the operator (26).

Having the additional symmetry operator (26) we can construct new ansatzes.

5. Reduction and exact solutions of nonlinear equations

Let us consider the simplest equations (1), (2) which are invariant with respect to algebra $AG_2(1,3)$:

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + \lambda|\psi|^{4/3}\psi, \tag{31}$$

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + \lambda|\psi|^{-2}\frac{\partial|\psi|}{\partial x_k}\frac{\partial|\psi|}{\partial x_k}\psi. \tag{32}$$

We shall search for the solutions in the form [7]

$$\psi = f(t,\vec{x})\varphi(w), \quad w \equiv (w_1, w_2, w_3), \quad w_k = w_k(t,\vec{x}). \tag{33}$$

Definition 2. We shall say that the formula (33) is an ansatz for equations (31), (32) if functions $f(x)$, w_1, w_2, w_3 have such structure that four-dimensional equations are reduced to three-dimensional ones for the function $\varphi(w)$. Equations obtained for $\varphi(w)$ depend only on w.

The problem of reduction in the general formulation is an extremely difficult problem and it requires explicit description of functions $f(x)$, w_1, w_2, w_3 which satisfy a nonlinear system of equations. We do not think that it is possible now to construct the general solution of these equations. But in case of an equation having rich symmetry properties the problem of reduction and description of $f(x)$ and w can be partially reduced to an algebraic problem of description of inequivalent subalgebras of this equation [7].

By means of subalgebraic structure of the algebra $AG_2(1,3)$ we have constructed quite a large list of ansatzes which reduce four-dimensional equations (31), (32) to three-dimensional ones. I adduce some of them.

Ansatzes for equations (31), (32).

$$\begin{gathered}1.\quad \psi(x)=\exp\left(i\frac{x_3^2}{4t}\right)\varphi(w),\\ w_1=t,\quad w_2=x_1^2+x_2^2,\quad w_3=x_3-t\arctan\frac{x_2}{x_1}.\end{gathered}\tag{34}$$

The reduced equation

$$i\left(\frac{\partial\varphi}{\partial w_1}+\frac{\varphi}{2w_1}+\frac{w_3}{w_1}\frac{\partial\varphi}{\partial w_3}\right)=-4w_2\frac{\partial^2\varphi}{\partial w_2^2}-\left(1+\frac{w_1^2}{w_2^2}\right)\frac{\partial^2\varphi}{\partial w_3^2}+\lambda|\varphi|^{4/3}\varphi.\tag{35}$$

$$\begin{gathered}2.\quad \psi=(t^2+1)^{-3/4}\exp\left\{\frac{i}{4}\left(\frac{|\vec{x}|^2t}{1+t^2}+2\alpha\arctan t\right)\right\}\varphi(w),\\ w_1=\frac{x_1}{\sqrt{1+t^2}},\quad w_2=\frac{x_2}{\sqrt{1+t^2}},\quad w_3=\frac{x_3}{\sqrt{1+t^2}}.\end{gathered}\tag{36}$$

The reduced equation

$$-\frac{\partial^2\varphi}{\partial w_1^2}-\frac{\partial^2\varphi}{\partial w_2^2}-\frac{\partial^2\varphi}{\partial w_3^2}-\frac{(2\alpha-\vec{w}\vec{w})}{4}\varphi+\lambda|\varphi|^{4/3}\varphi=0,\tag{37}$$

where α is an arbitrary real parameter.

$$\begin{gathered}3.\quad \psi=(t^2+1)^{-3/4}\exp\left\{\frac{i}{4}\left(\frac{|\vec{x}|^2t}{1+t^2}+2\beta\frac{tx_2-x_1}{t^2+1}\arctan t\right)\right\}\varphi(w),\\ w_1=\frac{tx_1+x_2}{t^2+1}+\beta\arctan t,\quad w_2=\frac{tx_2+x_1}{t^2+1},\quad w_3=\frac{x_3}{\sqrt{t^2+1}}.\end{gathered}\tag{38}$$

The reduced equation

$$i\left(\beta\frac{\partial\varphi}{\partial w_1}+w_1\frac{\partial\varphi}{\partial w_2}-w_2\frac{\partial\varphi}{\partial w_1}\right)=\Delta\varphi+\frac{1}{4}(2\beta w_2+\vec{w}\vec{w})\varphi+\lambda|\varphi|^{4/3}\varphi.\tag{39}$$

Having investigated symmetry of reduced equations which depend on three variables and then of ones depending on two variables we come finally to ordinary differential equations of the form

$$A(w)\frac{d^2\varphi}{dw^2}+B(w)\frac{d\varphi}{dw}+C(w)\varphi+\lambda|\varphi|^{4/3}\varphi=0,\tag{40}$$

where $A(w)$, $B(w)$, $C(w)$ are second degree polynomials.

Having solved equations (40) we construct exact solutions of the four-dimensional nonlinear Schrödinger equations (31) by means of the formulae (34), (36), (38).

Solutions of equation (32) constructed by means of ansatzes (34), (36), (38).

$$\psi(t,\vec{x})=\frac{\exp(ia_0t)}{\{\sqrt{-\gamma}\cos(\vec{a}\vec{x})\}^{3/2}},\quad \lambda>0,\quad a_0<0;$$

$$\psi(t,\vec{x})=\frac{\exp(ia_0t)}{\{\sqrt{-\gamma}\,\mathrm{sh}(\vec{a}\vec{x})\}^{3/2}},\quad \lambda>0,\quad a_0>0;$$

$$\psi(t,\vec{x})=\frac{\exp(ia_0t)}{\{\sqrt{-\gamma}\,\mathrm{ch}(\vec{a}\vec{x})\}^{3/2}},\quad \lambda<0,\quad a_0>0;$$

a_k are arbitrary real parameters and what is more $\vec{a}\vec{a} = a^2 = \frac{4}{9}|a_0|$, $\gamma = 3\lambda/5a_0$.

One can see that all obtained solutions depend non-analytically on the parameter λ (constant of interaction).

The obtained three-dimensional partial solutions can be used for construction of multi-parameter families of exact solutions. Really, as equation (31) is invariant with respect to 13-parameter group $G(1,3)$, that means the following.

If $\psi_1(t,\vec{x})$ is a solution of equation (31), then functions

$$\begin{aligned}
&\psi_2(t,\vec{x}) = \exp\left\{\frac{i}{2}\left(\vec{v}\vec{x} + \frac{\vec{v}^{\,2}t}{2}\right)\right\}\psi_1(t,\vec{x}+\vec{v}t),\\
&\psi_3(t,\vec{x}) = \exp\left\{-\frac{i}{4}\frac{\theta\vec{x}^{\,2} + 2\vec{v}\vec{x} + \vec{v}^{\,2}t}{1-\theta t}\right\}(1-\theta t)^{-3/2}\psi_1\left(\frac{t}{1-\theta t}, \frac{\vec{x}-\vec{v}t}{1-\theta t}\right)
\end{aligned} \tag{41}$$

are also solutions of the same equation. $\vec{v}$, θ are real parameters.

6. Galilei invariant nonlinear equations with second order derivatives

Now we formulate one result about the equations ($1'$) which are invariant under $AG_2(1,n)$ (for more details, see [9]).

Theorem 6 [9]. *The equations*

$$\begin{aligned}
S\psi = {} & A_0\left\{\Delta\psi - \psi^{-1}\frac{\partial\psi}{\partial x_a}\frac{\partial\psi}{\partial x_a} + (\psi^*)^{-1}\psi\left[\Delta\psi^* - (\psi^*)^{-1}\frac{\partial\psi^*}{\partial x_a}\frac{\partial\psi^*}{\partial x_a}\right]\right\} + \\
& + A_1|\psi|^{4/n}\psi + A_2|\psi|^{-\frac{2n+4}{n}}\frac{\partial|\psi|}{\partial x_a}\frac{\partial|\psi|}{\partial x_b}\times \\
& \times\left\{\frac{\partial^2\psi}{\partial x_a\partial x_b} - \psi^{-1}\frac{\partial\psi}{\partial x_a}\frac{\partial\psi}{\partial x_b} + (\psi^*)^{-1}\psi\left[\frac{\partial^2\psi^*}{\partial x_a\partial x_b} - (\psi^*)^{-1}\frac{\partial\psi^*}{\partial x_a}\frac{\partial\psi^*}{\partial x_b}\right]\right\},\\
& A_0 \equiv A_0(w), \quad A_1 \equiv A_1(w), \quad A_2 \equiv A_2(w) \quad w = \frac{\partial|\psi|}{\partial x_a}\frac{\partial|\psi|}{\partial x_a}|\psi|^{-\frac{2n+4}{n}}
\end{aligned}$$

are invariant under $AG_2(1,n)$ algebra. A_0, A_1, A_2 are arbitrary smooth functions.

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1. Fushchych W., Serov M., On some exact solutions of the three-dimensional nonlinear Schrödinger equation, *J. Phys. A*, 1987, **20**, № 16, L29–L33.
2. Fushchych W., Chopyk V., Symmetry and non-Lie reduction of the nonlinear Schrödinger equation, *Ukr. Math. J.*, 1993, **45**, № 4, 581–597.
3. Schuh D., Chung K.-M., Hartman H.N., Nonlinear Schrödinger-type field equation for the description of dissipative systems, *J. Math. Phys.*, 1983, **24**, № 6, 1652–1660.
4. Briihl L., Lange H., The Schrödinger–Langevin equation: special solutions and nonexistence of solitary waves, *J. Math. Phys.*, 1984, **25**, № 4, 786–790.
5. Fushchych W., Cherniha R., Galilei invariant nonlinear equation of Schrödinger type and their exact solutions: I, II, *Ukr. Math. J.*, 1989, **41**, № 10, 1349–1357; № 12, 1687–1694.
6. Fushchych W., Nikitin A., Symmetries of Maxwell’s equations, D. Reidel, 1987.

7. Fushchych W., Shtelen W., Serov M., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Kluwer Academic Publishers, 1993.
8. Fushchych W., Chopyk V., Conditional invariance of the nonlinear Schrödinger equation, *Proc. Ukrainian Acad. Sci.*, 1990, № 4, 30–33.
9. Fushchych W., Cherniha R., Systems of nonlinear evolution equations of the second order invariant under the Galilean algebra and its extensions, *Proc. Ukrainian Acad. Sci.*, 1993, № 8, 44–51.