Ansatz ’95

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In this talk I am going to present a brief review of some key ideas and methods which were given start and were developed in Kyiv, at the Institute of Mathematics of National Academy of Sciences of Ukraine during recent years.

Plan of the talk

The simplest classification of equations.

What is ansatz? The problem of PDE reduction without symmetry.

Conditional symmetry. How can we expand symmetry of PDE?

Conditional symmmetry of Maxwell and Schrödinger systems.

Q-conditional symmetry of the nonlinear wave equation, which is not invariant with respect to the Lorentz group.

Conditional symmetry of the Poincaré–invariant d’Alembert equation.

Conditional symmetry of the nonlinear heat equation.

Reduction and Antireduction.

Antireduction of the nonlinear acoustics equation.

Antireduction of the equation for short waves in gas dynamics.

Antireduction of nonlinear heat equation.

Nonlocal symmetry, new relativity principles.

Non–Lie symmetry of the Schrödinger equation.

Time is absolute in relativistic physics.

New equations of motions.

High–order parabolic equation in Quantum Mechanics.

Nonlinear generalization of the Maxwell equations.

Equations for fields with the spin 1/2.

How to extend symmetry of on equation with arbitrary coefficients?

1 Classification of equations

Every field of science must begin from some classification. We have today a lot of classifications of differential equations: parabolic, hyperbolic, elliptic, ultrahyperbolic

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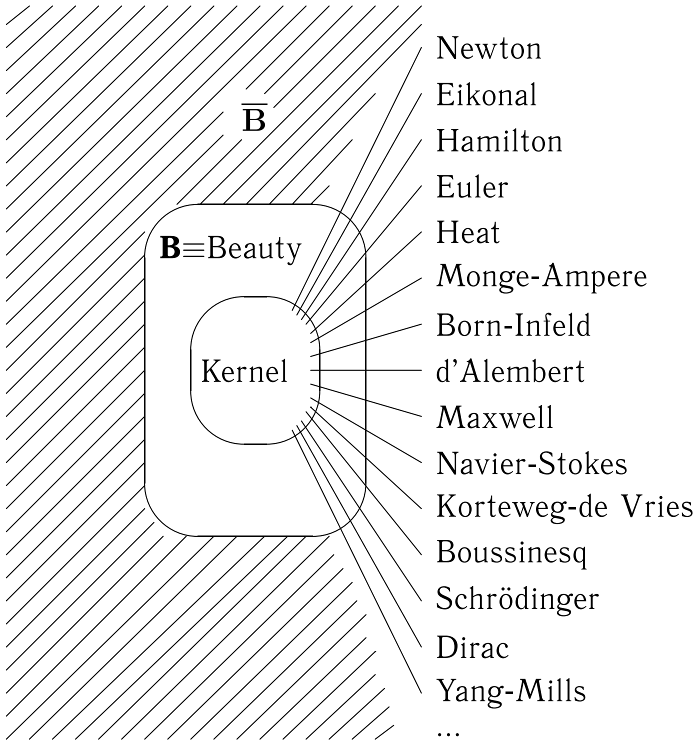
etc. I believe that it is most appropriate for our Conference to divide all equations of mathematics into two classes: B and $\bar{B}$



It is seen from the adduced picture that all fundamental equations of mathematical physics are united into one class B. From the point of view of existing now classifications they belong to essentially different classes. Equations from the class B have wide symmetry, and by this feature they are substantially different from other equations of mathematics.

It is important to point out that there are close relations among these different equations, which have not been investigated yet till now. For example, if we know solutions of the heat equation, we can construct solutions for the wave (d'Alembert) equation. By means of solutions of the Dirac equation, solutions of the Maxwell, heat, Yang–Mills, and other equations [18] can be obtained.

2 Ansatz reduction of PDE without using symmetry

Let us consider a PDE

$$\begin{aligned}
&L(x, u, u_{(1)}, u_{(2)}, \ldots u_{(n)}) = 0,\\
&u = u(x), \quad x = (x_0, x_1, \ldots, x_n), \quad u_{(1)} = (u_0, u_1, u_2, \ldots, u_n), \ u_\mu = \frac{\partial u}{\partial x_\mu},\\
&u_{(2)} = (u_{00}, u_{01}, \ldots, u_{nn}), \ u_{\mu\nu} = \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}.
\end{aligned} \tag{2.1}$$

Depending on the explicit form of L, equation (2.1) can belong to B or $\bar{B}$. In mathematical physics we often come across equations of the following type:

$$Lu \equiv \Box u - F(x, u, u_{(1)}) = 0. \tag{2.2}$$

What can we say today about solutions of equations (2.1), (2.2)? The answer is trivial: Nothing.

If equation (2.2) belongs to the class B and is invariant with respect to the Poincaré group $P(1, n)$, that is, a nonlinear function $F(x, u, u_{(1)})$ has the special form

$$F(x, u, u_{(1)}) = F\left(u, \frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu}\right) \tag{2.3}$$

then for equation (2.2) we can construct some classes of exact solutions, study Painlevé properties, construct approximate solutions, study asymptotic properties, etc.

Definition 1. (W. Fushchych, 1981, 1983 [1, 2, 3]) We shall call a formula

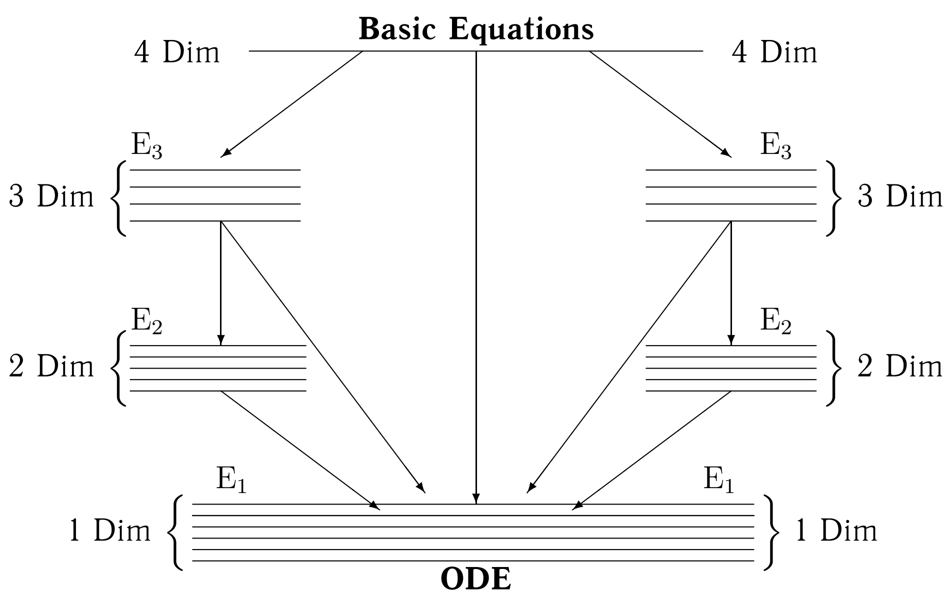
$$u = f(x)\varphi(\omega) + g(x), \tag{2.4}$$

an ansatz for equation (2.2) if after substitution of (2.4) we get an equation for the function $\varphi(\omega)$ which depends only on new variables $\omega = (\omega_1, \omega_2, \ldots, \omega_{n-1})$, where $f(x)$, $g(x)$ are given functions.

If (2.4) is an ansatz for (2.2), then the latter is reduced (the number of independent variables decreases by one) to an equation for the function $\varphi(\omega)$.

Thus the problem of reduction of an equation reduces to description of three functions $\langle f(x), g(x), \omega\rangle$ which leads to an equation for $\varphi(\omega)$ with less number of variables.

We can display schematically the process of reduction for an 4–dimensional equation in the following way:



E_3 is a set of three-dimensional equations, E_2 is a set of two-dimensional equations, E_1 is a set of one-dimensional equations with the following inclusion $E_3 \subset E_2 \subset E_1$.

That is, from one principal equation we obtain the whole set of ODE. Having solved the ODE, we find exact solutions of a multidimensional equation.

Description of ansatzes of the form (2.4) for the nonlinear wave equation is an extremely difficult nonlinear problem. In the simplest case, when we put $f(x) = 1$, $g(x) = 0$ for the nonlinear Poincaré–invariant d'Alembert equation

$$\Box u = F(u), \tag{2.5}$$

the problem of reduction of (2.5) to ODE reduces to construction of solutions for the following overdetermined system for ω (Fushchych W., Serov M. 1983 [3])

$$\begin{aligned} &\Box \omega = F_2(\omega), \\ &\frac{\partial \omega}{\partial x_\mu}\frac{\partial \omega}{\partial x^\mu} = \left(\frac{\partial \omega}{\partial x_0}\right)^2 - \left(\frac{\partial \omega}{\partial x_1}\right)^2 - \left(\frac{\partial \omega}{\partial x_2}\right)^2 - \cdots - \left(\frac{\partial \omega}{\partial x_n}\right)^2 = F_2(\omega). \end{aligned} \tag{2.6}$$

If ω is a solution of the system (2.6), then the multidimensional equation (2.5) reduces to ODE with variable coefficients

$$a_2(\omega)\ddot{\varphi}(\omega) + a_1(\omega)\dot{\varphi}(\omega) + a_0(\omega)\varphi(\omega)\ F(\varphi) = 0 \tag{2.7}$$

A solution of equation (2.5) has the form

$$u(x_0, \ldots x_n) = \varphi(\omega), \quad \omega = \omega(x_0, x_1, \ldots, x_n), \tag{2.8}$$

φ is a solution of equation (2.7).

Compatibility and general solutions of system (2.6) are described in detail in papers of Zhdanov, Revenko, Yehorchenko, Fushchych (1987–1993, [4–6]). As we see, without using explicitly the symmetry of equation (2.5), we can reduce a multidimensional wave equation to ODE. It is obvious that all ansatzes and solutions, which are constructed on the basis of the classical method by Sophus Lie, can be obtained within the framework of our approach. The subgroup analysis of the Poincaré group $P(1,n)$ (Patera J., Winternitz P., Zassenhaus H., 1975–1983, [7, 8] Fedorchuk V., Barannyk A., Barannyk L., Fushchych W., 1985–1991 [9–11]) gives only a part of possible ansatzes.

Note 1. P. Clarkson and M. Kruskal (1989 [12]) implemented the approach suggested by us in 1981–1983 [1, 2, 3] for the one-dimensional Boussinesq equation and constructed in explicit form ansatzes and solutions which cannot be obtained within the framework of the classical S. Lie method. In the literature, this approach is often called the "direct method of reduction". I believe that it would be more consistent and correct to call this method of construction of PDE solutions a method of ansatzes.

3 Conditional symmetry

The Lie symmetry, as known, is a local symmetry of the whole set of solutions. The Lie algorithm enables us to define the invariance algebra for an arbitrary given equation and to construct ansatzes.

The term and the concept “conditional symmetry” was introduced and developed in our papers (1983–1993, [2, 3, 13–18]). This extremely simple concept has appeared to be efficient and enabled us to discover a nature of many ansatzes which could not be obtained within the framework of the Lie method.

Conditional symmetry is the symmetry of subsets of equation’s solutions. Knowing conditional symmetry of an equation, we can construct non–Lie ansatzes and solutions. It is more difficult to study conditional symmetry of a given equation than to study its classical Lie symmetry. The difficulty is related to the fact that to find conditional symmetry of an equation, it is necessary to solve nonlinear determining equations.

During recent years, there are intensive studies in this promising direction, and today we can make following general conclusion:

Corollary 1. *Principal nonlinear equations of mathematical physics have conditional symmetry.*

Let us denote by the symbol

$$Q = \langle Q_1, Q_2, \ldots, Q_r \rangle \tag{3.1}$$

some set of operators which does not belong to the invariance algebra (IA) of equation (2.1).

Definition 2. (Fushchych W., Nikitin A., Shtelen W. and Serov M., 1987 [13, 14, 18], Fushchych W. and Tsyfra I. (1987 [15])). Equation (2.1) is said to be conditionally invariant under the operators Q from (3.1), if there exists a supplementary condition on the solutions of (3.1) of the form

$$L_1(x, u, u_{(1)}, \ldots, u_{(n)}) = 0 \tag{3.2}$$

such that (3.1) together with (3.2) is invariant under Q.

Thus, one has the following criterion of conditional invariance [13, 15, 18]

$$Q_s L = \lambda_0 L + \lambda_1 L_1, \tag{3.3}$$

$$Q_s L_1 = \lambda_2 L + \lambda_3 L_1, \tag{3.4}$$

where λ_0, λ_1, λ_2, λ_3 are some differential expressions, Q_s is the s-th prolongation by Lie.

Definition 3. We shall say that an equation is Q-conditionally invariant if the additional equation $L_1 = 0$ is a quasilinear equation of the first order

$$L_1(x, u, u_{(1)}) \equiv Qu = 0, \tag{3.5}$$

$$Q = \xi_\mu(x, u)\frac{\partial}{\partial x^\mu} + \eta(x, u)\frac{\partial}{\partial u}, \tag{3.6}$$

with η being a smooth function.

Thus, the problem of finding the conditional symmetry of a equation reduces to the solution of equations (3.3), (3.4). As a rule, the determining equations for calculating ξ_μ and η are nonlinear equations.

As is known, in the classical approach ξ_μ, η satisfy a linear system of differential equations which, because of being overdetermined, can be solved.

3.1 Conditional symmetry of the Maxwell equations

The first equation where we had noticed conditional symmetry was the Maxwell subsystem [13]

$$\frac{\partial \vec{E}}{\partial t} = \mathrm{rot}\, \vec{H}, \quad \frac{\partial \vec{H}}{\partial t} = -\mathrm{rot}\, \vec{E}. \tag{3.7}$$

It is possible to prove by means the standard Lie method that the maximal invariance algebra of system (3.7) is an 8-dimensional extended Euclid algebra $AE_1(4)$ with basis elements:

$$P_\mu = i\frac{\partial}{\partial x_\mu}, \quad J_{ab} = x_a p_b - x_b p_a + S_{ab}, \quad D = x_\mu P^\mu, \tag{3.8}$$

where S_{ab} are 6×6 matrices, which realize a reduced representation of the Lie algebra of the group $SU(2)$.

Thus, system (3.7) is not invariant with respect to the Lorentz transformations, which are generated by operators

$$J_{oa} = x_o P_a - x_a P_0 + S_{0a}, \tag{3.9}$$

$\langle S_{ab}, S_{0a}\rangle$ are matrices which realize a finite-dimensional representation of the Lie algebra of the Lorentz group $S(1,3)$.

Theorem 1. (Fushchych W. and Nikitin A. 1983 [13]). *System (3.7) is conditionally invariant under the Lorentz boosts (3.9) if and only if the solutions of (3.7) satisfy the conditions*

$$\mathrm{div}\, \vec{E} = 0, \quad \mathrm{div}\, \vec{H} = 0. \tag{3.10}$$

Thus, system (3.7) only together with equations (3.10) is invariant under the Lorentz group.

Note 2. 90 years ago H. Lorentz (1904, April 23), H. Poincaré (1905, June 5, July 23), A. Einstein (1905, June 30) discovered the theorem about invariance of the full Maxwell system (3.7), (3.10) with respect to rotations in the four-dimensional pseudo-Euclidean space-time. This theorem is a mathematical formulation of the fundamental Lorentz–Poincaré–Einstein principle of relativity.

3.2 Conditional symmetry of linear Schrödinger systems

Let us consider the multicomponent system of disconnected Schrödinger equations:

$$\begin{aligned}
&S\Psi = \left(p_0 - \frac{p_a^2}{2m}\right)\Psi_r = 0, \quad r = 1, 2, \ldots, n,\\
&p_0 = i\frac{\partial}{\partial x_0}, \quad p_a = -i\frac{\partial}{\partial x_a}, \quad a = 1, 2, 3,\\
&\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_n), \quad \Psi = \Psi(x_0 = t, x_1, x_2, x_3).
\end{aligned} \tag{3.11}$$

It is evident that every separate Schrödinger equation (3.11) is invariant with respect to a scalar representation of the group $G_2(1,3)$, a full Galilei group.

Let us consider a problem of existence of nontrivial vector, spinor, tensor representations of the full Galilei group, which are realized on the set of solutions of system (3.11).

We demand system (3.11) be invariant with respect to the following linear representations of the algebra $AG_2(1,3)$

$$\begin{aligned}
&P_0 = i\frac{\partial}{\partial x_0}, \quad P_a = -i\frac{\partial}{\partial x_a}, \quad M = im,\\
&J_a = x_a p_b - x_b p_a + S_a, \quad S_a = \frac{1}{2}\varepsilon_{abc}S_{bc},\\
&G_a = x_0 p_a - x_a p_0 + \lambda_a, \quad D = 2x_0 P_0 - x_k P_k + \lambda_0,\\
&A = x_0 D - x_0^2 P_0 + \frac{1}{2}m x_k^2 - \lambda_a x_a,
\end{aligned} \tag{3.12}$$

where matrices S_a, λ_0, λ_a satisfy the commutation relations [29]

$$\begin{aligned}
&[S_a, S_b] = i\varepsilon_{abc}S_c, \quad [\lambda_a, \lambda_b] = 0, \quad [\lambda_0, S_a] = 0,\\
&[\lambda_a S_b] = i\varepsilon_{abc}S_c, \quad [\lambda_0, \lambda_a] = i\lambda_a.
\end{aligned} \tag{3.13}$$

Theorem 2 (Fushchych and Shtelen, 1983, [19]). *System of equations (3.11) is conditional invariant under representation $AG_2(1,3)$ (3.12) if*

$$\left(\lambda_0 - \frac{3}{2}i - \frac{1}{m}\lambda_k P_k\right)\Psi = 0, \tag{3.14}$$

$$(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\Psi = 0. \tag{3.15}$$

3.3 Q-conditional symmetry of Lorentz noninvariant nonlinear wave equation

Let us consider the following wave equation (Fushchych and Tsyfra 1987, [15])

$$Lu \equiv \Box u + F(x, \underset{1}{u}) = 0 \tag{3.16}$$

$$\begin{aligned}
F(x, \underset{1}{u}) = &-\left(\frac{\lambda_0}{x_0}\right)^2\left(\frac{\partial u}{\partial x_0}\right)^2 + \left(\frac{\lambda_1}{x_1}\right)^2\left(\frac{\partial u}{\partial x_1}\right)^2 +\\
&+\left(\frac{\lambda_2}{x_2}\right)^2\left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\lambda_3}{x_3}\right)^2\left(\frac{\partial u}{\partial x_3}\right)^2, \quad x_\mu \neq 0,
\end{aligned} \tag{3.17}$$

λ_μ are arbitrary parameters.

Equation (3.16) is invariant only with respect to scale transformations and translations:

$$x_\mu \to x'_\mu = e^b x_\mu, \quad u \to u' = e^{2b}u, \quad u \to u' = u + c,$$

b is a real parameter.

Let us consider a Lorentz-invariant ansatz

$$u = \varphi(\omega), \quad \omega = x_\mu x^\mu = x_0^2 - x_1^2 - x_2^2 - x_3^2. \tag{3.18}$$

This ansatz, despite the fact that (3.16) is not invariant with respect to the Lorentz group, reduces equation (3.16) to ODE

$$\omega\frac{d^2\varphi}{d\omega^2}+2\frac{d\varphi}{d\omega}+\lambda^2\left(\frac{d\varphi}{d\omega}\right)^2=0 \tag{3.19}$$

whose solutions are given by the functions

$$\lambda=\lambda_0^2-\lambda_1^2-\lambda_2^2-\lambda_3^2,$$
$$\varphi(\omega)=2(-\lambda^2)^{1/2}\tan^{-1}\omega(-\lambda^2)^{-1/2} \quad \text{for } -\lambda^2>0,$$
$$\varphi(\omega)=-(\lambda^2)^{-1/2}ln\left\{\frac{(\lambda^2)^{1/2}+\omega}{(\lambda^2)^{1/2}-\omega}\right\} \quad \text{for } -\lambda^2<0.$$

What is the reason of such reduction? From the classical point of view, ansatz (3.18) must not reduce the Lorentz non–invariant equation (3.16) to ODE.

The reason of all this is the fact that equation (3.16) is conditionally invariant with respect to the Lorentz group.

Theorem 3 (Fushchych and Tsyfra, 1987 [15]). *Equation (3.16), (3.17) is conditionally invariant with respect to the Lorentz group if the following six conditions are added:*

$$J_{\mu\nu}u=0,\quad J_{\mu\nu}=x_\mu\frac{\partial}{\partial x_\nu}-x_\nu\frac{\partial}{\partial x_\mu},\quad \mu,\nu=0,1,2,3. \tag{3.20}$$

Thus, equation (3.16) together with the additional condition (3.20) is invariant with respect to the Lorentz group. The condition (3.20) picks out the subset from the whole set of solutions which is invariant with respect to the Lorentz group.

3.4 Conditionally conformal symmetry of the Poincaré-invariant d'Alembert equation

Let us consider the nonlinear d'Alembert equation with an additional condition

$$\Box u+F(u)=0, \tag{3.21}$$

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu}=F_1(u). \tag{3.22}$$

Theorem 4 (Fushchych, Zhdanov, Serov 1989 [18]). *Equation (3.21) is conditionally invariant under the conformal group if*

$$F=3\lambda(u+c)^{-1}, \tag{3.23}$$

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu}=\lambda, \tag{3.24}$$

where λ, c are arbitrary constants. The operators of conformal symmetry are

$$K_\mu=2x_\mu D-(x_\alpha x^\alpha-u^2)\frac{\partial}{\partial x^\mu},\quad \mu=0,1,2,3 \tag{3.25}$$

$$D = x^\mu \frac{\partial}{\partial x^\mu} + u\frac{\partial}{\partial u}. \tag{3.26}$$

Remark 3. Formulae (3.25), (3.26) give a nonlinear representation for the conformal algebra $AC(1,3)$.

An ansatz for the system

$$\Box u = u^{-1}, \quad \partial_\mu u \partial^\mu u = 1 \tag{3.27}$$

has the form (Fushchych and Zhdanov, 1989 [4])

$$u^2 = (a_\mu x^\mu + g_1)^2 - (b_\mu x^\mu + g_2)^2, \tag{3.28}$$

where $g_1 = g_1(\theta_\mu x^\mu)$, $g_2 = g_2(\theta_\mu x^\mu)$ are arbitrary smooth functions, a_μ, b_μ, θ_μ are arbitrary complex parameters satisfying the condition

$$a_\mu a^\mu = -b_\mu b^\mu = 1, \quad a_\mu b^\mu = a_\mu \theta^\mu = b_\mu \theta^\mu = \theta_\mu \theta^\mu = 0.$$

Remark 5. The problem of compatibility and construction of solutions of the d'Alembert–Hamilton system are considered in detail in [5, 6].

3.5 Conditional symmetry of the nonlinear heat equation

Let us consider the equation

$$u_0 + \vec{\nabla}[f(u)\vec{\nabla}u] = 0, \quad f(u) \neq \text{const}. \tag{3.29}$$

Ovsyannikov L. (1962, [20]) carried out the complete classification of the one-dimensional equation (3.29). Dorodnitsyn A., Knyaseva Z., Svirshchevskii S. (1983, [21]) carried out group classification of the three-dimensional equation (3.29) From the analysis of these results it follows.

Conclusion 1. (Fushchych 1983 [2]). *Among equations of the class (3.29), there are no nonlinear equations invariant with respect to Galilei transformations which are generated by the operators*

$$G_a = x_0 \partial_a + M(u) x_a \frac{\partial}{\partial u}, \tag{3.30}$$

$M(u)$ is constant.

Theorem 5 (Fushchych, Serov, Chopyk 1988 [16]). *The equation (3.29) is conditional invariant under the Galilean operators (3.30) if*

$$u_0 + \frac{(\vec{\nabla}u)^2}{2M(u)} = 0, \tag{3.31}$$

$$M(u) = \frac{u}{2f(u)}. \tag{3.32}$$

Conclusion 2. *The nonlinear equation (3.29) with the additional condition (3.31) is compatible with the Galilei relativity principle.*

Conclusion 3. *If*

$$f(u) = \frac{1}{2m}u^k, \qquad M(u) = \frac{2m}{kn+2}u^{1-k}, \tag{3.33}$$

$$f(u) = e^u, \qquad M(u) = 1, \tag{3.34}$$

where m, k are arbitrary constants, $kn+2 \neq 0$, then equation (3.29) is conditionally invariant with respect to Galilei transformations.

Q-conditional symmetry of the one-dimensional equation

$$u_0 - u_{11} = F(u)$$

was studied in detail (Fushchych and Serov, 1990, [22, 23]). Recently these results were obtained by Clarkson P. and Mansfield E. (1994, [24]).

4 Reduction and antireduction

Under the term "reduction–antireduction", we understand a decreasing of dimension of an equation with respect to independent variables and increasing (antireduction) by the number of dependent variables. That is we have simultaneously the process of reduction (by the number of independent variables) and antireduction (increasing the number of reduced systems with respect to the original equation) [25].

In the classical Lie approach as a rule the number of components of dependent variables for reduced systems does not increase.

Example 1. Let us consider the nonlinear acoustics equation (Khokhlov–Zabolotskaja equation)

$$u_{01} - (u_1 u)_1 - u_{22} - u_{33} = 0, \tag{4.1}$$

$$u = u(x_1, x_2, x_3).$$

The ansatz (Fushchych and Myronyuk, 1991 [26])

$$u = \frac{1}{3}x_1\varphi^{(1)}(\omega_0,\omega_2,\omega_3) + \frac{1}{6}x_1^2\varphi^{(2)}(\omega_0,\omega_2,\omega_3) + \varphi^{(3)}(\omega_0,\omega_2,\omega_3), \tag{4.2}$$

$$\omega_0 = x_0, \quad \omega_2 = x_2, \quad \omega = x_3$$

antireduces four-dimensional equation (4.1) to the system of coupled three-dimensional equations for functions $\varphi^{(1)}$, $\varphi^{(2)}$, $\varphi^{(3)}$

$$\begin{gathered}
\frac{\partial^2\varphi^{(1)}}{\partial\omega_2^2} + \frac{\partial^2\varphi^{(2)}}{\partial\omega_3^2} = (\varphi^{(2)})^2, \\
\frac{\partial^2\varphi^{(1)}}{\partial\omega_2^2} + \frac{\partial^2\varphi^{(1)}}{\partial\omega_3^2} + \frac{\partial\varphi^{(1)}}{\partial\omega_0} - \varphi^{(1)}\varphi^{(2)} = 0, \\
\frac{\partial^2\varphi^{(3)}}{\partial\omega_2^2} + \frac{\partial^2\varphi^{(3)}}{\partial\omega_3^2} - \frac{1}{3}\varphi^{(2)}\varphi^{(3)} - \frac{1}{3}\frac{\partial\varphi^{(1)}}{\partial\omega_0} + \frac{1}{9}(\varphi^{(1)})^2 = 0.
\end{gathered} \tag{4.3}$$

The formula (4.2) gives a non-Lie ansatz for equation (4.1).

Example 2. Let us consider the equation for short waves in gas dynamics

$$\begin{aligned}&2u_{01}-2(2x_1+u_1)u_{11}+u_{22}+2\lambda u_1=0,\\&u=u(x_0=t,x_1,x_2).\end{aligned}\tag{4.4}$$

The ansatz (Fushchych and Repeta 1991, [27])

$$\begin{aligned}&u=x_1\varphi^{(1)}(\omega_0,\omega_2)+x_1^2\varphi^{(2)}(\omega_0,\omega_2)+x_1^{3/2}\varphi^{(3)}+\varphi^{(4)},\\&\omega_0=x_0,\qquad \omega_2=x_2\end{aligned}\tag{4.5}$$

antireduces one three-dimensional scalar equation (4.4) to a system of two-dimensional equations for four functions

$$\begin{aligned}&\varphi^{(3)}=0,\quad \frac{\partial^2\varphi^{(1)}}{\partial\omega_2^2}=0,\quad \frac{\partial^2\varphi^{(2)}}{\partial\omega_2^2}=0,\\&\frac{\partial^2\varphi^{(4)}}{\partial\omega_2^2}=\frac{9}{4}\big(\varphi^{(1)}\big)^2,\quad \frac{\partial\varphi^{(1)}}{\partial\omega_0}=\varphi^{(1)}\left(3\varphi^{(2)}+\frac{1}{2}-\lambda\right).\end{aligned}\tag{4.6}$$

4.1 Antireduction and ansatzes for the nonlinear heat equation

Let us consider the nonlinear one-dimensional heat equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left\{a(u)\frac{\partial u}{\partial x}\right\}+F(u),\tag{4.7}$$

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+F(u).\tag{4.8}$$

We consider an implicit ansatz

$$h\big(t,x,u,\varphi^{(1)}(\omega),\varphi^{(2)}(\omega)\ldots,\varphi^{(N)}(\omega)\big)=0,\tag{4.9}$$

which reduces the two–dimensional equation (4.7) to the system of ODE for functions $\varphi^{(1)},\ldots,\varphi^{(N)}$. We have constructed a quite long list of ansatzes which reduce equation (4.7) to the system of ODE (Zhdanov R. and Fushchych W. 1994, [33]).

Example 3. If in (4.7)

$$a(u)=\lambda u^{-3/2},\quad F(u)=\lambda_1 u+\lambda_2 u^{5/2},\tag{4.10}$$

then the ansatz [33] is as follows

$$u^{-3/2}=\varphi^{(1)}(t)+\varphi^{(2)}(t)x+\varphi^{(3)}(t)x^2+\varphi^{(4)}(t)x^3,\tag{4.11}$$

$$\begin{aligned}&\dot\varphi^{(1)}=2\lambda\varphi^{(1)}\varphi^{(3)}-\frac{2}{3}\lambda(\varphi^{(2)})^2-\frac{3}{2}\lambda_1\varphi^{(1)}-\frac{3}{2}\lambda_2,\\&\dot\varphi^{(2)}=-\frac{2}{3}\lambda\varphi^{(2)}\varphi^{(3)}+6\lambda\varphi^{(1)}\varphi^{(4)}-\frac{3}{2}\lambda_1\varphi^{(2)},\\&\dot\varphi^{(3)}=-\frac{2}{3}\lambda(\varphi^{(3)})^2+2\lambda\varphi^{(2)}\dot\varphi^{(4)}-\frac{3}{2}\lambda_1\varphi^{(3)},\\&\dot\varphi^{(4)}=-\frac{3}{2}\lambda_1\varphi^{(4)}.\end{aligned}\tag{4.12}$$

Having solved the system of ODE (4.12), by formula (4.11) we construct exact solutions of the equation (4.7).

Example 4. If in (4.8)

$$F(u) = \left\{\alpha + \beta \ln u - \gamma^2 (\ln u)^2\right\} u, \tag{4.13}$$

then the ansatz

$$\ln u = \varphi^{(1)}(t) + e^{\gamma x}\varphi^{(2)}(t) \tag{4.14}$$

reduces (4.8) to the system of ODE

$$\begin{aligned} &\dot\varphi^{(1)} = 2 + \beta\varphi^{(1)} - \gamma^2(\varphi^{(1)})^2,\\ &\dot\varphi^{(2)} = \left\{\beta + \gamma^2 - 2\gamma^2\varphi^{(1)}\right\}\varphi^{(2)}. \end{aligned} \tag{4.15}$$

It is possible to construct solutions of system (4.15) in the explicit form. Depending on the sign of the quantity $d = \beta^2 + 4\alpha\gamma^2$ we get the following solutions of the nonlinear equation (4.8), (4.13).

Case 4.1 $d > 0$

$$u = c\left(\cos\frac{d^{1/2}t}{2}\right)^{-2}\exp\left(\gamma x + \gamma^2 t\right) + \frac{1}{2\gamma^2}\left(\beta - d^{1/2}\mathrm{tg}\,\frac{d^{1/2}t}{2}\right). \tag{4.16}$$

Case 4.2 $d < 0$

$$u = c\left(\mathrm{ch}\,\frac{|d|^{1/2}t}{2}\right)^{-2}\exp\left(\gamma x + \gamma^2 t\right) + \frac{1}{2\gamma^2}\left(\beta + |d|^{1/2}\mathrm{th}\,\frac{|d|^{1/2}t}{2}\right). \tag{4.17}$$

Case 4.3 $d = 0$

$$u = ct^{-2}\exp\left(\gamma x + \gamma^2 t\right) + \frac{1}{2\gamma^2 t}(\beta t + 2). \tag{4.18}$$

Example 5. If in (4.7)

$$a(u) = \lambda u^k, \quad F(u) = \lambda_1 u + \lambda_2 u^{1-k}, \tag{4.19}$$

then the ansatz

$$u^k = \varphi^{(1)}(t) + \varphi^{(2)}(t)x + \varphi^{(3)}(t)x^2 \tag{4.20}$$

antireduces (4.7) to the system of ODE

$$\begin{aligned} &\dot\varphi^{(1)} = 2\lambda\varphi^{(1)}\varphi^{(3)} + \lambda k^{-1}(\varphi^{(2)})^2 + k\lambda_2,\\ &\dot\varphi^{(2)} = 2\lambda(1 + 2k^{-1})\varphi^{(2)}\varphi^{(3)} + k\lambda_1\varphi^{(2)},\\ &\dot\varphi^{(3)} = 2\lambda(1 + 2k^{-1})(\varphi^{(3)})^2 + k\lambda_1\varphi^{(3)}. \end{aligned} \tag{4.21}$$

5 Non-Lie symmetry, new relativity principles

5.1 Non-Lie symmetry Schrödinger equation

Let us consider the Schrödinger equation

$$\left(i\frac{\partial}{\partial x_0}-\frac{p_a^2}{2n}\right)u(x_0,\vec{x})=0. \tag{5.1}$$

It is well known that the maximal (in the Lie sense) invariance algebra (5.1) is the full Galilei algebra $AG_2(1,3)=\langle P_0,P_a,J_{ab},G_a,D,A\rangle$

$$\begin{aligned}
&P_0=i\frac{\partial}{\partial x_0},\quad P_a=-i\frac{\partial}{\partial x_0},\quad a=1,2,3,\\
&J_{ab}=x_ap_b-x_bp_a,\quad G_a=x_0p_a-mx_a,\\
&D=2x_0P_0-x_kP_k,\quad A=x_0D-x_0^2P_0+\frac{1}{2}mx_a^2.
\end{aligned} \tag{5.2}$$

Operators G_a generate the standard Galilei transformations:

$$t\to t'=\exp\{iG_av_a\}\,t\exp\{-iG_av_a\}=t, \tag{5.3}$$

$$x_a\to x_a'=\exp\{iG_bv_b\}\,x_a\exp\{-iG_cv_c\}=x_a+v_at. \tag{5.4}$$

Let us put the following question: do symmetries which are not reduced for the algebra (5.2) exhaust for equation (5.1)?

Answer: The Schrödinger equation (5.1) has additional symmetries (supersymmetries, non-Lie, nonlocal) which are not reduced to the Galilei algebra $AG_2(1,3)$ [29].

One of results in this direction is the following:

Theorem 6. (Fushchych and Seheda 1977 [28]). *The Schrödinger equation (5.1) is invariant with respect to the Lorentz algebra* $AL(1,3)$

$$J_{ab}=x_ap_b-x_bp_a, \tag{5.5}$$

$$J_{0a}=\frac{1}{2m}(pG_a+G_ap),\quad p=(p_1^2+p_2^2+p_3^2)^{1/2}=(-\Delta)^{1/2}. \tag{5.6}$$

It is not difficult to check that the operators $\langle J_{ab},J_{0c}\rangle\equiv AL(1,3)$ satisfy the commutation relations

$$[J_{ab},J_{0c}]=i(g_{ac}J_{b0}-g_{bc}J_{a0}),\quad [J_{0a},J_{0b}]=-iJ_{ab}.$$

It is important to point out that J_{0a} are integral-differential symmetry operators and generate nonlocal transformations

$$x_a\to x_a'=\exp\{iJ_{ob}V_b\}\,x_a\exp\{-iJ_{0c}V_c\}\neq \text{ Galilei transform. (5.4)}, \tag{5.7}$$

$$t\to t'=\exp\{iJ_{0a}V_a\}\,t\exp\{-iJ_{0b}V_b\}=t. \tag{5.8}$$

Hence the operators J_{0a} (5.6) generate new transformations which do not coincide with the known Galilei and Lorentz transformation. Thus we have new relativity principle. It is defined by formulae (5.7), (5.8).

5.2 Time is absolute in relativistic physics

The four-component Dirac equation lies in the foundation of the modern quantum mechanics

$$\gamma_\mu p^\mu \Psi = m\Psi(x_0, x_1, x_2, x_3). \tag{5.9}$$

Here γ_μ are 4×4 Dirac matrices.

Since the time of discovery of this equation it is known that (5.9) is invariant with respect to the Poincaré algebra $AP(1,3) = \langle P_\mu, J_{\mu\nu}\rangle$ with the basis elements

$$P_\mu = i\frac{\partial}{\partial x^\mu}, \quad J^{(1)}_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \quad S_{\mu\nu} = \frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \tag{5.10}$$

Operators $J^{(1)}_{\mu\nu}$ generate the standard Lorentz transformations

$$t \to t' = \exp\left\{iJ^{(1)}_{0a}v_a\right\} t \exp\left\{-iJ_{0b}v_b\right\}, \tag{5.11}$$

$$x_a \to x'_a = \exp\left\{iJ^{(1)}_{0b}v_b\right\} x_a \exp\left\{-iJ_{0c}v_c\right\}. \tag{5.12}$$

Hence, the fundamental statement follows that time $t \in T(1)$ and space $\vec{x} \in R(3)$ are the single pseudo-Euclidean space-time with the metric

$$s^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2. \tag{5.13}$$

Let us put another question: Do there exist symmetries in equation (5.10) which cannot be reduced to the algebra AP(1,3) (5.11)?

Answer: The Dirac equation (5.9) has a wide additional symmetry (supersymmetry, non-Lie symmetry) which cannot be reduced to the algebra $AP(1,3)$ (5.10) [13, 29].

I shall say here briefly about one of such symmetries.

Theorem 7. (Fushchych 1971, 1974 [30, 31]. *The Dirac equation (5.9) is invariant with respect to the following representation of the Poincaré algebra*

$$P^{(2)}_0 = H = \gamma_0\gamma_a p_a + \gamma_0 m, \quad P^{(2)}_a = -i\frac{\partial}{\partial x_a}, \quad a = 1,2,3, \tag{5.14}$$

$$J^{(2)}_{ab} = x_a p_b - x_b p_a + S_{ab}, \qquad S_{ab} = \frac{i}{4}(\gamma_a\gamma_b - \gamma_b\gamma_a), \tag{5.15}$$

$$J^{(2)}_{0a} = x_0 p_a - \frac{1}{2}(x_a H + H x_a). \tag{5.16}$$

Thus we have two different representations of the Poincaré algebra $AP(1,3)$ (5.10) and (5.14)–(5.16).

The representation (5.15) and (5.16) generates nonlocal transformations

$$x_a \to x'_a = \exp\{iJ^{(2)}_{ab}v_b\}x_a \exp\{iJ^{(2)}_{0c}v_c\} \neq \text{Lorentz transform}, \tag{5.17}$$

$$t \to t' = \exp\{iJ^{(2)}_{0b}v_b\}t\exp\{-iJ^{(2)}_{0c}v_c\} = t. \tag{5.18}$$

Thus, time does not change in relativistic physics. Time is absolute in relativistic physics.

There are two nonequivalent possibilities (duality) for transformations of coordinates and time: Lorentz transformation (5.11), (5.12) and non-Lorentz transformation (5.17), (5.18).

The Maxwell and Klein–Gordon–Fock equations are also invariant under nonlocal transformations (5.17), (5.18) when time does not change. However energy and momentum are transformed by the Lorentz law [31,32]. We have new relativity principle (5.17), (5.18).

What is the reason of such a paradoxical statement? The reason is that the operators $J^{(2)}_{0a}$ are non-Lie symmetry operators and the standard relation (S. Lie's theorems) between Lie groups and Lie algebras is broken.

So, physics is not equivalent to geometry and geometry is not physics. Physics is Nature. Theoretical Physics is only a Model of Nature!

6 On some new motion equations

Some new motion equations are adduced in this section. These equations are generalizations of known classical equations. Symmetry of these equations has not been investigated.

6.1 High order parabolic equation in quantum mechanics

The Schrödinger equation (5.1) is not the only equation compatible with the Galilei relativity principle. A more general equation was suggested in [1, 2]

$$\begin{gathered}(\lambda_1 S + \lambda_2 S^2 + \cdots + \lambda_n S^n)u = \lambda u,\\ S \equiv p_0 - \frac{p_a^2}{2m}, \quad S^2 = S\cdot S, \quad S^n = S^{n-1}S,\end{gathered} \tag{6.1}$$

$\lambda, \lambda_1, \lambda_2, \ldots, \lambda_n$ are arbitrary parameters. Equation (6.1) as well as the classical equation (5.1) is invariant with respect to the Galilei transformations but it is not invariant with respect to scale and projective transformations.

A new equation for two particles (waves):

$$\begin{gathered}p_0 u_1 = \frac{1}{2m_1} p_a^2 u_1 + V_1\big(t, x_1, x_2, \ldots, x_6, u_1, u_2\big),\\ p_0 u_2 = \frac{1}{2m_2} p_{a+3}^2 u_2 + V_2\big(t, x_1, x_2, \ldots, x_6, u_1, u_2\big),\\ u_1 = u_1(t, x_1, x_2, x_3), \quad u_2 = u_2(t, x_4, x_5, x_6), \quad V_1 \text{ and } V_2 \text{ are potentials.}\end{gathered}$$

6.2 Nonlinear generalization of Maxwell equations

If we assume that the light velocity is not constant [34], we can suggest some generalizations of the Maxwell equations

$$\begin{gathered}\frac{\partial \vec{E}}{\partial t} = \mathrm{rot}\,\{c(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{H}\}, \qquad \frac{\partial \vec{H}}{\partial t} = -\mathrm{rot}\,\{c(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E}\},\\ \mathrm{div}\,\{a(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E}\} = 0, \qquad \mathrm{div}\,\{b(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{H}\} = 0,\end{gathered} \tag{6.2}$$

where a, b and c are some functions of electromagnetic field;

$$\begin{aligned} &\frac{\partial \vec{E}}{\partial t} = \mathrm{rot}\,\{c(\vec{B}^2, \vec{D}^2, \vec{B}\vec{D})\vec{B}\} + \vec{j}, \\ &\frac{\partial \vec{H}}{\partial t} = -\mathrm{rot}\,\{c(\vec{B}^2, \vec{D}^2, \vec{B}\vec{D})\vec{D}\}, \end{aligned} \quad \text{or} \quad \begin{aligned} &\frac{\partial \vec{D}}{\partial t} = \mathrm{rot}\,\{c(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E}\} + \vec{j}, \\ &\frac{\partial \vec{B}}{\partial t} = -\mathrm{rot}\,\{c(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E}\}, \end{aligned} \tag{6.3}$$

$$\begin{aligned} &\lambda_1 \vec{D} + \lambda_2 \Box \vec{D} = F_1(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E} + F_2(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{H}, \\ &\lambda_3 \vec{B} + \lambda_4 \Box \vec{B} = R_1(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E} + R_2(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{H}, \end{aligned} \tag{6.4}$$

$$\mathrm{div}\,\vec{D} = \rho, \quad \mathrm{div}\,\vec{B} = 0, \tag{6.5}$$

where F_1, F_2, R_1, R_2 are functions of fields $\vec{E}$ and $\vec{H}$, c in equations (6.2), (6.3) can be a function of $(t, \vec{x})$, $c = c(t, \vec{x})$, or depend on the gravity potential V, $c = C(V)$. Nonlinear wave equations for $\vec{E}$ and $\vec{H}$ have form

$$\frac{\partial^2 \vec{E}}{\partial t^2} - c^2 \Delta \vec{E} = 0, \quad \frac{\partial^2 \vec{H}}{\partial t^2} - c^2 \Delta \vec{H} = 0, \tag{6.6}$$

or

$$\frac{\partial^2 \vec{E}}{\partial t^2} - \Delta(c^2 \vec{E}) = 0, \quad \frac{\partial^2 \vec{H}}{\partial t^2} - \Delta(c^2 \vec{H}) = 0; \tag{6.7}$$

or

$$\frac{\partial^2}{\partial t^2}\left(\frac{1}{c^2}\vec{E}\right) - \Delta \vec{E} = 0, \quad \frac{\partial^2}{\partial t^2}\left(\frac{1}{c^2}\vec{H}\right) - \Delta H = 0, \tag{6.8}$$

with one of the conditions

$$c^2 = \frac{1}{2}\frac{\left(\frac{\partial \vec{E}}{\partial t}\right) + \left(\frac{\partial \vec{H}}{\partial t}\right)}{(\mathrm{rot}\,\vec{H})^2 + (\mathrm{rot}\,\vec{E})^2} \tag{6.9}$$

or

$$\frac{\partial c^2}{\partial x^\mu}\frac{\partial c^2}{\partial x^\mu} = 0. \tag{6.10}$$

or

$$c_\mu \frac{\partial c_2}{\partial x_\mu} = \lambda(E^2 \vec{H}^2, \vec{E}\vec{H}) F_{\alpha\beta} c^\beta, \tag{6.11}$$

c_α is the four-velocity of the light (electromagnetic field), $c^2 = c_\alpha c^\alpha$.

Equations of hydrodynamical type for electromagnetic field have form

$$\begin{aligned} &\frac{\partial \vec{E}}{\partial t} = a_1 \left\{\vec{\nabla} \times (\vec{c} \times \vec{H})\right\} + a_2 \left\{\vec{\nabla} \times (\vec{c} \times \vec{E})\right\}, \\ &\frac{\partial \vec{H}}{\partial t} = b_1 \left\{\vec{\nabla} \times (\vec{c} \times \vec{E})\right\} + b_2 \left\{\vec{\nabla} \times (\vec{c} \times \vec{H})\right\}, \\ &\frac{\partial \vec{c}}{\partial t} + (\vec{c}\vec{\nabla})\vec{c} = R_1 \vec{E} + R_2 \vec{H}, \end{aligned} \tag{6.12}$$

$\vec{c}$ is the three-velocity of the light, where a_1, a_2, b_1, b_2, R_1, R_2 are functions of $\vec{E}^2$, $\vec{H}^2$, $\vec{E}\vec{H}$.

Maxwell's equations in a moving frame with the velocity can be generalized in such forms

$$\frac{\partial \vec{E}}{\partial t} + \lambda_1 v_k \frac{\partial \vec{E}}{\partial x_k} + \lambda_2 \operatorname{rot} \vec{H} = 0, \quad \frac{\partial \vec{H}}{\partial t} + \lambda_3 v_k \frac{\partial \vec{H}}{\partial x_k} + \lambda_4 \operatorname{rot} \vec{E} = 0,$$

or

$$\frac{\partial \vec{E}}{\partial t} + \lambda_1 v_k \frac{\partial \vec{H}}{\partial x_k} + \lambda_2 \operatorname{rot} \vec{H} = 0, \quad \frac{\partial \vec{H}}{\partial t} + \lambda_3 v_k \frac{\partial \vec{E}}{\partial x_k} + \lambda_4 \operatorname{rot} \vec{E} = 0,$$

with the conditions $\frac{\partial v_k}{\partial t} + v_l \frac{\partial v_k}{\partial x_l} = 0$.

6.3 Equations for fields with the spin 1/2

Fields with the spin 1/2 are described, as a rule, by first-order equations, by the Dirac equation. However, such fields can be also described by second-order equations. Some of such equations are adduced below:

$$p_\mu p^\mu \Psi = F_1(\bar{\psi}\psi)\Psi, \quad \bar{\psi}\gamma_\mu p^\mu \Psi = F_2(\bar{\psi}\Psi); \tag{6.13}$$

$$p_\mu p^\mu \Psi = R_1(\bar{\psi}\psi)\Psi, \quad (\bar{\psi}\gamma_\mu \Psi) p^\mu \Psi = F_2(\bar{\psi}\psi)\Psi; \tag{6.14}$$

$$p_\mu p^\mu \Psi = F_1(\bar{\psi}\psi)\Psi, \quad (\bar{\psi}\gamma_\mu \Psi)(\bar{\psi} p^\mu \Psi) = F_3(\bar{\psi}\psi); \tag{6.15}$$

$$p_\mu p^\mu \Psi + \lambda \gamma_\mu p^\mu \Psi = F(\bar{\psi}\psi)\Psi; \tag{6.16}$$

$$p_\mu p^\mu \Psi = F_1(\bar{\psi}\psi)\Psi, \quad p_0 \Psi = \{(\bar{\psi}\gamma_0 \Psi)(\bar{\psi}\gamma_k \psi) p_k + m \bar{\Psi}\gamma_0 \Psi\}\Psi.$$

6.4 How to extend symmetry of an equation with arbitrary coefficients?

Let us consider the a second-order equation

$$a_{\mu\nu}(x) \frac{\partial^2 u}{\partial x^\mu \partial x^\nu} + b_\mu(x) \frac{\partial u}{\partial x^\mu} + F(u) = 0. \tag{6.17}$$

Equation (6.17) with arbitrary fixed coefficients has only a trivial symmetry ($x \to x' = x$, $u \to u' = u$). However, if we do not fix coefficient functions $a_{\mu\nu}(x), b_\mu(x)$, such an equation can have wide symmetry. E.g., if $a_{\mu\nu}$, b_μ satisfy the equations

$$\Box a_{\mu\nu} = \frac{\partial u}{\partial x_\mu} \frac{\partial u}{\partial x_\nu} F_1(u) \tag{6.18}$$

or

$$\Box b_\mu = F_2(u) \frac{\partial u}{\partial x_\mu}, \quad \Box a_{\mu\nu} = \frac{\partial^2 u}{\partial x_\mu \partial x_\nu} F_3(u), \tag{6.19}$$

then the nonlinear system (6.17), (6.18), (6.19) is invariant with respect to the Poincaré group P(1,3). Let us emphasize that here even if we put $F_1 = 0$, $F_2 = 0$, equations (6.17), (6.18), (6.19) are a nonlinear system of equations. With some particular functions F_1 and F_2, it is possible to construct ansatzes which reduce system (6.17), (6.18), (6.19) to the system of ordinary differential equations.

So, considering (6.17) as a nonlinear equation with additional conditions for $a_{\mu\nu}$, b_ν, we can construct the exact solution for eqation (6.17). The adduced idea about extension of the symmetry of (6.17) can be used for construction of exact solutions for motion equations in gravity theory.

The second example of equations which have wide symmetry is

$$v_\mu v_\nu \frac{\partial^2 F_{\alpha\beta}}{\partial x^\mu \partial x^\nu} = 0, \tag{6.20}$$

$$v_\mu \frac{\partial v_\nu}{\partial x^\mu} = 0. \tag{6.21}$$

If in (6.20) v_μ are fixed functions the equation, as a rule, has trivial symmetry.

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