Amplitude-phase representation for solutions of nonlinear d'Alembert equations

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We consider the nonlinear complex d'Alembert equation $\Box\Psi = F(|\Psi|)\Psi$ with Ψ represented in terms of amplitude and phase, in $(1+n)$-dimensional Minkowski space. We exploit a compatible d'Alembert–Hamilton system to construct new types of exact solutions for some nonlinearities.

1. Introduction

Let us consider the general nonlinear complex d'Alembert equation in $(1+n)$-dimensional Minkowski space

$$\Box\Psi = F(|\Psi|)\Psi, \tag{1}$$

where F is a smooth, real function of its argument, Ψ is a complex function of $1+n$ real variables, and

$$\Box = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}.$$

Equation (1) plays a fundamental role in classical and quantum field theories, and in superfluidity and liquid crystal theory. Many exact solutions have been found using Lie symmetry methods [6, 11, 12, 13, 8, 7], as well as with conditional symmetries [7].

In this paper we use the representation $\Psi = ue^{iv}$, where u is the amplitude and v is the phase (both real functions). On substituting this in (1), we find the following system:

$$\Box u - u(v_\mu v_\mu) = uF(u), \tag{2}$$

$$u\Box v + 2u_\mu v_\mu = 0. \tag{3}$$

We use the notation

$$u_\mu v_\mu = \frac{\partial u}{\partial x_0}\frac{\partial v}{\partial x_0} - \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} - \cdots - \frac{\partial u}{\partial x_n}\frac{\partial v}{\partial x_n}.$$

The system (2), (3) is obviously equivalent to the starting equation (1). However, equations (2), (3) has the advantage that it gives us the possibility of making functional and differential connections between the amplitude and phase, which substantially simplifies the problem of integrating equation (1). Moreover, in assuming the simplest possible relations between the amplitude and phase, we are able to construct exact solutions of (2), (3), and hence of (1).

We now seek solutions of (2), (3). We consider two cases: (i) the amplitude as a function of the phase, $u = g(v)$; (ii) the phase as a function of the amplitude, $v = g(u)$. This is reminiscent of the polar description of plane curves in geometry. The

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system (2), (3) then yields a pair of equations for the phase v in the first case and for the amplitude u in the second case. There then arises the question of the compatibility of the two equations obtained, and we solve it by exploiting the compatible system

$$w = \frac{\lambda N}{w}, \quad w_\mu w_\mu = \lambda, \tag{4}$$

where $\lambda = -1, 0, 1$ and $N = 0, 1, \ldots, n$. Exact solutions for the system (4) are given in table 1 in section 2.

The system (4) is a particular case of the d'Alembert–Hamilton system

$$\Box w = F_1(w), \quad w_\mu w_\mu = F_2(w), \tag{5}$$

The system (5) was studied by Smirnov and Sobolev in 1932, with $w = w(x_0, x_1, x_2)$ and $F_1 = F_2 = 0$. Collins [2–4] studied (5) with w a function of three complex variables, and obtained compatibility conditions for the functions $F_1(w)$, $F_2(w)$. For $(1+3)$ and higher dimensional Minkowski space, (5) was studied by Fushchych and co-workers [9, 10]: they obtained compatibility conditions for $F_1(w)$, $F_2(w)$ and some exact solutions.

Here, we exploit the results of Fushchych et. al [10], applying them to the system (4). Moreover, the compatibility of (4) dictates the type of nonlinearity $F(u)$ which can appear in (1). This is the novelty of our approach to finding some exact solutions of (1).

2. Solutions

2.1. $u = g(\nu)$. We now assume that the amplitude is a function of the phase: $u = g(v)$. Inserting this assumption in (2), (3), we obtain

$$\Box v = \frac{-2g\dot{g}F(g)}{g\ddot{g} - 2\dot{g}^2 - g^2} = F_1(v), \tag{6}$$

$$v_\mu v_\mu = \frac{g^2 F(g)}{g\ddot{g} - 2\dot{g}^2 - g^2} = F_2(v) \tag{7}$$

with $\dot{g} = dg/dv$.

We now deal with (6), (7) in two ways: (i) assume forms for F_1, F_2 so as to make equations (6), (7) compatible; (ii) transform equation (4) locally so as to agree with (6), (7).

First, let us make the assumption

$$F_1(v) = \frac{\lambda N}{v}, \quad F_2(v) = \lambda$$

with $N, \lambda \neq 0$. Then equations (6), (7) become a compatible system [10], and we also find that g and F must satisfy

$$g\ddot{g} - 2\dot{g}^2 - g^2 - \frac{g^2}{\lambda}F(g) = 0, \quad \frac{-2\dot{g}}{g} = N. \tag{8}$$

From (8) it now follows that

$$g(v) = \sigma v^{-N/2}, \quad F(v) = -\lambda + \lambda\frac{N}{2}\left(1 - \frac{N}{2}\right)\sigma^{-4/N}v^{4/N},$$

where $\sigma \neq 0$ is an arbitrary real constant.

With this, we have obtained the following:

Result 1. An exact solution of (1) with nonlinearity

$$F(|\Psi|) = -\lambda + \sigma^{-4/N}\lambda\frac{N}{2}\left(1-\frac{N}{2}\right)|\Psi|^{4/N}$$

is given by

$$\Psi(x) = \sigma v(x)^{-N/2}e^{iv(x)},$$

where $v(x)$ is a solution of the compatible system (4) for $N, \lambda \neq 0$.

Our next step is to perform a local transformation of (4). We do this by setting $w = f(v)$ in (4) (with $\lambda \neq 0$), with f a real, smooth function such that $\dot{f} \neq 0$. With this substitution, we obtain the system:

$$\Box v = \frac{\lambda N}{f(v)\dot{f}(v)} - \frac{\lambda \ddot{f}(v)}{\dot{f}^3(v)}, \tag{9}$$

$$v_\mu v_\mu = \frac{\lambda}{\dot{f}^2(v)}. \tag{10}$$

The system (9), (10) is obviously compatible since it is the local transformation of an already compatible system. However, it should be noted that this does not mean that the exact solutions we obtain by using (9), (10) are equivalent to those obtained from (8), since we have introduced some extra freedom via the function f.

We now equate the right-hand sides of (6), (7) with the right-hand sides of (9), (10), respectively. A little algebraic manipulation gives us

$$g(v) = \sigma\left(\frac{\dot{f}(v)}{f^N(v)}\right)^{1/2}, \tag{11}$$

where σ is an arbitrary non-zero constant. Thus we have a differential relation between f and g which we can integrate. For $N = 1$ we obtain

$$f(v) = C\exp\left(\frac{1}{\sigma^2}\int^v g^2(\xi)d\xi\right) \tag{12}$$

and g has to satisfy the integro-differential equation

$$g\ddot{g} - 2\dot{g}^2 - g^2 - \frac{C^2}{\lambda\sigma^4}g^6\exp\left(\frac{1}{\sigma^2}\int^v g^2(\xi)d\xi\right)F(g) = 0. \tag{13}$$

For $N \neq 1$ we find

$$f(v) = \left(\frac{1-N}{\sigma^2}\int^v g^2(\xi)d\xi + C\right)^{1/(1-N)}, \tag{14}$$

C being an arbitrary real constant, and with the following condition on g:

$$g\ddot{g} - 2\dot{g}^2 - g^2 - \frac{1}{\lambda\sigma^4}g^6\left(\frac{1-N}{\sigma^2}\int^v g^2(\xi)d\xi + C\right)^{2N/(1-N)}F(g) = 0. \tag{15}$$

Our result is summarized in the following:

Result 2. (i) The function

$$\Psi(x) = g(v(x)) \exp[iv(x)]$$

is a solution of (1) whenever g is a solution of (13) and $w(x) = f(v(x))$ is a solution of

$$\Box w = \frac{\lambda}{w}, \quad w_\mu w_\mu = \lambda$$

with f given by (12).

(ii) The function

$$\Psi(x) = g(v(x)) \exp[iv(x)]$$

is a solution of (1) whenever g is a solution of (15) and $w(x) = f(v(x))$ is a solution of

$$\Box w = \frac{\lambda N}{w}, \quad w_\mu w_\mu = \lambda$$

with f given by (14) for $N \neq 1$.

One may treat (13) and (15) in two ways: consider F as given, and then attempt to solve for g, or make an assumption about g and then find the corresponding F. We take this second approach, and in doing so, we determine the function f which appears in (12) and (14), which also relates (4) to the system (6), (7).

This is illustrated in the following example, where we take g as $g(v) = v^\beta$. Then we obtain after some elementary manipulation

$$w = f(v) = Cv^{1/\sigma^2}$$

when $N = 1$, $\beta = -\frac{1}{2}$ In this case we find the corresponding nonlinear version of (1) and an exact solution:

$$\Box\psi + \frac{\lambda\sigma^4}{C^2}\left(\frac{1}{4}|\psi|^{4/\sigma^2} - |\psi|^{4(1-\sigma^2)/\sigma^2}\right)\psi = 0,$$

$$\psi(x) = \left(\frac{1}{C}w(x)\right)^{-\sigma^2/2} \exp\left[i\left(\frac{1}{C}w(x)\right)^{\sigma^2}\right],$$

where w is a solution of

$$\Box w = \frac{\lambda}{w}, \quad w_\mu w_\mu = \lambda.$$

The solutions of this system are given in table 1. We can choose the nonlinearity in the above wave equation by choosing σ. For instance, for $\sigma^2 = \frac{2}{3}$ we obtain the equation

$$\Box\Psi - \left(\frac{2}{3}\right)^2 \frac{\lambda}{C^2}\left(|\Psi|^2 - \frac{1}{4}|\Psi|^6\right)\Psi = 0. \tag{16}$$

Equation (16) is of the type considered by Grundland and Tuczynski [12].

Table 1. Solutions for the system $\Box_n w = \lambda N/w$, $(\nabla_n w)^2 = \lambda$. Inner products are with respect to the Minkowski metric.

λ	N	Solutions w	Conditions on $a_\mu \in \mathbb{R}^n, b_\mu \in \mathbb{R}^n$
±1	$N \in \{1,2,\ldots,n-1\}$	$\left[(a_0\cdot x)^2 \pm (a_1\cdot x)^2 \pm \cdots \pm (a_N\cdot x)^2\right]^{1/2}$	$a_0\cdot a_0 = 1,\ a_0\cdot a_j = 1,\ a_j\cdot a_k = \pm\delta_{jk}$, $j,k = 1,2,\ldots,N$
1	$N \in \{1,2,\ldots,n-1\}$	$\left[(a_0\cdot x)^2 \pm (a_1\cdot x)^2 \pm \cdots \pm (a_N\cdot x)^2\right]^{1/2}$	$a_\mu\cdot a_\mu = -1,\ a_\mu\cdot a_\nu = 0,$, $\nu,\mu = 0,1,\ldots,N$
1	$N \in \{1,2,\ldots,n-3\}$	$\left[\left(b_1\cdot x + h_1\left(\sqrt{k}a_0\cdot x + \sum\limits_{i=1}^{k} a_i\cdot x\right)\right)^2 + \right.$ $+\left(b_2\cdot x + h_2\left(\sqrt{k}a_0\cdot x + \sum\limits_{i=1}^{k} a_i\cdot x\right)\right)^2 + \cdots$ $\left.\cdots + \left(b_{N+1}\cdot x + h_{N+1}\left(\sqrt{k}a_0\cdot x + \sum\limits_{i=1}^{k} a_i\cdot x\right)\right)^2\right]^{1/2}$ $h_1, h_2 \ldots, h_{N+1}$ are arbitrary real functions	$a_0\cdot a_0 = 1$ $a_i\cdot a_i = -1$ $b_j\cdot b_j = -1$ $a_0\cdot a_i = 0,\ a_0\cdot b_j = 0,\ a_i\cdot b_j = 0,\ b_j\cdot b_l = 0$ $i = 1,\ldots,k,\ (k\le n-1),\ j\ne l = 1,\ldots,N+1$
±1	$N = 0$	$b_1\cdot x\cos h_1\left(\sqrt{k}a_0\cdot x + \sum\limits_{i=1}^{k} a_i\cdot x\right) +$ $+ b_2\cdot x\sin h_1\left(\sqrt{k}a_0\cdot x + \sum\limits_{i=1}^{k} a_i\cdot x\right) + h_2\left(\sqrt{k}a_0\cdot x + \sum\limits_{i=1}^{k} a_i\cdot x\right)$ h_1 and h_2 are arbitrary real functions	$a_0\cdot a_0 = \pm1,\ a_i\cdot a_i = \mp1,\ b_j\cdot b_j = \mp1$ $a_0\cdot a_i = 0,\ a_0\cdot b_j = 0,\ a_i\cdot b_j = 0,\ b_j\cdot b_l = 0$ $i = 1,\ldots,k,\ (k\le n-1),\ j\ne l = 1,2$
±1	$N = 0$	$b_1\cdot x\cos h_1\left(\sqrt{k}a_0\cdot x + \sum\limits_{i=1}^{k} a_i\cdot x\right) +$ $+ b_2\cdot x\sin h_1\left(\sqrt{k}a_0\cdot x + \sum\limits_{i=1}^{k} a_i\cdot x\right) + h_2\left(\sqrt{k}a_0\cdot x + \sum\limits_{i=1}^{k} a_i\cdot x\right)$ h_1 and h_2 are arbitrary real functions	$a_0\cdot a_0 = 1,\ a_i\cdot a_i = -1$ $b_1\cdot b_1 = \pm1,\ b_2\cdot b_2 = \mp1$ $a_0\cdot a_i = 0,\ a_0\cdot b_j = 0,\ a_i\cdot b_j = 0,\ b_j\cdot b_l = 0$ $i = 1,\ldots,k,\ (k\le n-1),\ j\ne l = 1,2$
0	$N = 0$	$h_1\left(\sqrt{k}a_0\cdot x + \sum\limits_{i=1}^{k} a_i\cdot x\right)$ h_1 is an arbitrary real function	$a_0\cdot a_0 = 1,\ a_i\cdot a_i = -1,\ a_0\cdot a_i = 0$ $i = 1,\ldots,k\ (k\le n-1)$

For $N = 2$, $\beta = -1$ we obtain the following wave equation and exact solution:

$$\Box\Psi + \frac{1}{\sigma^8}\left(|\Psi|^2 + \frac{1}{\lambda\sigma^4}|\Psi|^6\right)\Psi = 0, \tag{17}$$

$$\Psi(x) = (C - w(x))\sigma^2 \exp\left[i\frac{1}{(C - w(x))\sigma^2}\right], \tag{18}$$

where w is a solution of the compatible system

$$\Box w = \frac{2\lambda}{w}, \quad w_\mu w_\mu = \lambda$$

and exact solutions of this system are given in table 1. Equation (17) is also of a type considered by Grundland and Tuczynski [12]. Our exact solutions are new.

2.2. $\nu = g(u)$. We now assume that the phase is a function of the amplitude: $v = g(u)$. On substituting this in equations (2), (3), we obtain

$$\Box u = \frac{(u^2\ddot{g} + 2u\dot{g})F(u)}{u\ddot{g} + 2\dot{g} + u^2\dot{g}^3} \equiv F_1(u), \tag{19}$$

$$u_\mu u_\mu = \frac{-u^2\dot{F}(u)}{u\ddot{g} + 2\dot{g} + u^2\dot{g}^3} \equiv F_2(u). \tag{20}$$

Here $\dot{g} = dg/du$.

We perform the same analysis as before. First, letting $F_1(u) = \lambda N/u$, $F_2(u) = \lambda$, $\lambda \neq 0$, we find (after some computation)

$$g(u) = -\frac{\sigma u^{N+1}}{N+1} + \sigma_1, \quad F(u) = \frac{\lambda N}{u^2} - \frac{\lambda\sigma_1}{u^{2(N+2)}}.$$

Having determined g and the nonlinearity of the wave equation (19), we have the following:

Result 3. An exact solution of (1) with nonlinearity

$$F(|\Psi|) = \lambda N|\Psi|^{-2} - \lambda\sigma_1|\Psi|^{-2(N+2)}$$

is given by

$$\Psi(x) = Cu(x)\exp\left(\frac{-i\sigma}{(N+1)u(x)^{(N+1)}}\right),$$

where $\lambda \neq 0$ and $C \neq 0$ is an arbitrary real constant, and where $u(x)$ is a solution of the system (4).

Another way of dealing with (19), (20) is to transform (4) locally using the transformation $w = f(u)$ with $\dot{f} \neq 0$, which gives us

$$\Box u = \frac{\lambda N}{f(u)\dot{f}(u)} - \frac{\lambda\ddot{f}(u)}{\dot{f}^3(u)}, \tag{21}$$

$$u_\mu u_\mu = \frac{\lambda}{\dot{f}^2(u)}. \tag{22}$$

Then, equating the right-hand sides of (21), (22) with the right-hand sides of (19), (20), we find (for $\lambda \neq 0$, as before) that

$$u^2\dot g(u) = \sigma\frac{\dot f(u)}{f^N(u)},$$

where $\sigma \neq 0$ is an arbitrary real constant. Again we see that there are two cases to consider: $N = 1$ and $N \neq 1$.

For $N = 1$ we obtain

$$f(u) = C\exp\left(\frac{1}{\sigma}\int^u \xi^2\frac{dg(\xi)}{d\xi}d\xi\right) \tag{23}$$

with C an arbitrary real constant. The condition on g is

$$u\ddot g + 2\dot g + u^2\dot g^3 + \frac{u^4C^2}{\lambda\sigma^2}\dot g^3\exp\left(\frac{1}{\sigma}\int^u \xi^2\frac{dg(\xi)}{d\xi}d\xi\right)F(u) = 0. \tag{24}$$

When $N \neq 1$, f is given by

$$f(u) = \left(C - \frac{N-1}{\sigma}\int^u \xi^2\frac{dg(\xi)}{d\xi}d\xi\right)^{1/(1-N)} \tag{25}$$

with C an arbitrary real constant and with the following condition on g:

$$u\ddot g + 2\dot g + u^2\dot g^3 + \frac{u^4}{\lambda\sigma^2}\left(C - \frac{N-1}{\sigma}\int^u \xi^2\frac{dg(\xi)}{d\xi}d\xi\right)^{2N/(1-N)}F(u) = 0.$$

This reasoning can be summarized in the following:

Result 4. (i) The function

$$\Psi(x) = u(x)\exp(ig(u(x)))$$

is a solution of (1) whenever g is a solution of (24) and $w(x) = f(u(x))$ is a solution of

$$\Box w = \frac{\lambda}{w}, \quad w_\mu w_\mu = \lambda$$

with f given by (23).

(ii) The function

$$\Psi(x) = u(x)\exp(ig(u(x)))$$

is a solution of (1) whenever g is a solution of (26) and $w(x) = f(u(x))$ is a solution of

$$\Box w = \frac{\lambda N}{w}, \quad w_\mu w_\mu = \lambda$$

with f given by (25) for $N \neq 1$.

We treat equations (24) and (26) relating g to the nonlinearity F as before: we assume a form for g and treat the equations as determining F. Taking $g(u) = u^\beta$, we have the following examples of the wave equation, exact solution and relation between u and w:

$N = 1$, $\beta \neq -2$.

$$\Box\Psi + \frac{\lambda\sigma^2}{C^2}|\Psi|^{-2}\left(1 + \frac{\beta+1}{\beta^2}|\Psi|^{-2\beta}\right)\exp\left[\frac{-\beta}{\sigma(\beta+2)}|\Psi|^{\beta+2}\right)\Psi = 0,$$

$$\Psi(x) = u(x)\exp[iu(x)^{\beta}],$$

$$u = \left(\frac{\sigma(\beta+2)}{\beta}\ln\left|\frac{w}{C}\right|\right)^{1/(\beta+2)},$$

where w is a solution (listed in table 1) of the compatible system

$$\Box w = \frac{\lambda}{w}, \quad w_\mu w_\mu = \lambda.$$

$N \neq 1$, $\beta \neq -2$.

$$\Box\Psi + \lambda\sigma^2|\Psi|^{-2}\left(1 + \frac{\beta+1}{\beta^2}|\Psi|^{-2\beta}\right)\left(C - \frac{(N-1)\beta}{\sigma(\beta+2)}|\Psi|^{\beta+2}\right)^{2N/(N-1)}\Psi = 0,$$

$$\Psi(x) = u(x)\exp[iu(x)^{\beta}],$$

$$u = \left(\frac{\sigma(\beta+2)}{(N-1)\beta}(C - w^{1-N})\right)^{1/(\beta+2)},$$

where w is a solution (listed in table 1) of the compatible system

$$\Box w = \frac{\lambda N}{w}, \quad w_\mu w_\mu = \lambda.$$

If we choose $\beta = -1$, $N = 2$, $C = 0$, then we find that the wave equation is

$$\Box\Psi + \lambda\sigma^{-2}|\Psi|^2\Psi = 0 \tag{26}$$

with the exact solution

$$\Psi(x) = u(x)\exp\left(\frac{i}{u(x)}\right)$$

and

$$u(x) = \sigma w(x),$$

where w solves

$$\Box w = \frac{2\lambda}{w}, \quad w_\mu w_\mu = \lambda.$$

Equation (27) is of some interest: of all the possible nonlinearities $F(|\Psi|)$, the nonlinearity $F(|\Psi|) = |\Psi|^2$ gives the widest possible symmetry group, admitting the conformal group. Equation (27) (and indeed equation (1)) can be reduced to the nonlinear Schrödinger equation in $(1+2)$-dimensional time-space (see [1]) with the same nonlinearity. This equation also admits the widest possible symmetry group for nonlinearities of the given type. It can be reduced to the $(1+1)$-dimensional nonlinear Schrödinger equation with the same nonlinearity, and this equation has soliton solutions (the well known Zakharov–Shabat soliton). Using this soliton, we can construct a new type of solution of the hyperbolic wave equation (27). Of course, this does not imply that (27) has soliton solutions located in three-dimensional space.

3. Conclusion

We have demonstrated an approach which can give new exact solutions of some nonlinear wave equations of the same type as (1). The novelty in our approach lies in the fact that we exploit the compatibility conditions for the d'Alembert–Hamilton system to dictate the type of nonlinearity and the exact solution(s). Moreover, some of the equations we obtain appear to be of interest in physics, but we are unable to make any statement about the physical nature of the exact solutions we obtain, as our approach has not used any physical criteria to single out any type of solution.

Of course, this is not the only approach possible; we could, for instance, reduce (1) to the Schrödinger equation (as in [1]) and then apply a similar method to this new equation in the amplitude-phase representation. Also, it is possible to consider a more general connection between the amplitude and phase, such as $u = G(v_\mu v_\mu)$ for some function G. This leads to a system involving the Born–Infeld equation, which has a very wide symmetry group, and we obtain new exact solutions of (1). This differential connection between amplitude and phase will of course be important when we allow nonlinearities dependent on derivatives, such as $F(|\Psi|, \Psi_\mu^* \Psi_\mu)$. We will report on this work in a forthcoming paper.

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