On the general solution of the d'Alembert equation with a nonlinear eikonal constraint and its applications

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We construct the general solutions of the system of nonlinear differential equations $\Box_n u = 0$, $u_\mu u^\mu = 0$ in the four- and five-dimensional complex pseudo-Euclidean spaces. The results obtained are used to reduce the multi-dimensional nonlinear d'Alembert equation $\Box_4 u = F(u)$ to ordinary differential equations and to construct its new exact solutions.

1 Introduction

Kaluza [1] was the first who put forward an idea of extension of the four-dimensional Minkowski space in order to use it as a geometric basis for unification of the electromagnetic and gravitational fields. Nowadays, Kaluza's idea is well-known and there are a lot of papers where further development and various generalizations of this idea are obtained [2].

In [3–5] it was proposed to apply five-dimensional wave equations to describe particles (fields) having variable spins and masses. Such physical interpretation of the five-dimensional equations is based on the fact that the generalized Poincaré group $P(1,4)$ acting in the five-dimensional de Sitter space contains the Poincaré group $P(1,3)$ as a subgroup. It means that the mass and spin Casimir operators have continuous and discrete spectrum, respectively, in the space of irreducible representations of the group $P(1,4)$ [3–6].

The simplest $P(1,4)$-invariant scalar linear equation has the form

$$\Box_5 u + \chi^2 u = 0, \quad \chi = \text{const}, \tag{1}$$

where $\Box_5$ is the d'Alembert operator in the five-dimensional Minkowski space with the signature $(+,-,-,-,-)$.

The problem of construction of exact solutions of the above equation is, in fact, completely open. One can obtain some its particular solutions applying the symmetry reduction procedure or the method of separation of variables (both approaches use essentially symmetry properties of the whole set of solutions of Eq. (1)). In the present paper we suggest a method for construction of solutions of partial differential equation (1) which utilizes implicitly the symmetry of a *subset* of the set of its solutions. Namely, a special subset of its exact solutions obtained by imposing an additional constraint

$$u_{x_0}^2 - u_{x_1}^2 - u_{x_2}^2 - u_{x_3}^2 - u_{x_4}^2 = 0,$$

which is the eikonal equation in the five-dimensional space, will be investigated. As shown in [7, 8], the system obtained is compatible if and only if $\chi = 0$. We

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will construct general solutions of multi-dimensional systems of partial differential equations (PDE)

$$\Box_n u = 0, \quad u_\mu u^\mu = 0 \tag{2}$$

in the four- and five-dimensional complex pseudo-Euclidean spaces.

In (2) $u = u(x_0, x_1, \ldots, x_{n-1}) \in C^2(\mathbb{C}^n, \mathbb{C}^1)$. Hereafter, the summation over the repeated indices in the pseudo-Euclidean space $M(1, n-1)$ with the metric tensor $g_{\mu\nu} = \mathrm{diag}\,(1, \underbrace{-1, \ldots, -1}_{n-1})$ is understood, e.g. $\Box_n \equiv \partial_\mu \partial^\mu = \partial_0^2 - \partial_1^2 - \cdots - \partial_{n-1}^2$, $\partial_\mu = \partial/\partial x_\mu$.

It occurs that solutions of system of PDE (2), being very interesting by itself, can be used to reduce the *nonlinear* d'Alembert equation

$$\Box_4 u = F(u), \quad F(u) \in C(\mathbb{R}^1, \mathbb{R}^1), \tag{3}$$

to ordinary differential equations, thus giving rise to families of principally new exact solutions of (3). More precisely, we will establish that there exists a nonlinear map from the set solutions of the system of PDE (2) into the set of solutions of the nonlinear d'Alembert equation, such that each solution of (2) corresponds to a family of exact solutions of Eq. (3) containing two arbitrary functions of one argument. It will be shown that solutions of the nonlinear d'Alembert equation obtained in this way can be related to its *conditional* symmetry.

The paper is organized as follows. In Section 2 we give assertions describing the general solution of system of PDE (2) in the n-dimensional real and in the four- and five-dimensional complex pseudo-Euclidean spaces. In Section 3 we prove these assertions. Section 4 is devoted to discussion of the connection between exact solutions of system (2) and the problem of reduction of the nonlinear d'Alembert equation (3). In Section 5 we construct principally new exact solutions of Eq. (3).

2 Integration of the system (2): the list of principal results

Below we adduce assertions giving general solutions of the system of PDE (2) with arbitrary $n \in \mathbb{N}$ provided $u(x) \in C^2(\mathbb{R}^n, \mathbb{R}^1)$, and with $n = 4, 5$, provided $u(x) \in C^2(\mathbb{C}^n, \mathbb{C}^1)$.

Theorem 1. *Let $u(x)$ be a sufficiently smooth real function of n real variables $x_0, \ldots, x_{n-1}$. Then, the general solution of the system of nonlinear PDE (2) is given by the following formula:*

$$A_\mu(u) x^\mu + B(u) = 0, \tag{4}$$

where $A_\mu(u)$, $B(u)$ are arbitrary real functions which satisfy the condition

$$A_\mu(u) A^\mu(u) = 0. \tag{5}$$

Note 1. As far as we know, Jacobi, Smirnov and Sobolev were the first who obtained the formulas (4), (5) with $n = 3$ [9, 10]. That is why, it is natural to call (4), (5) the Jacoby–Smirnov–Sobolev formulas (JSSF). Later on, in 1944 Yerugin generalized

JSSF for the case $n = 4$ [11]. Recently, Collins [12] has proved that JSSF give the general solution of system (2) for an arbitrary $n \in \mathbb{N}$. He applied rather complicated differential geometry technique. Below we show that to integrate Eqs. (2) it is quite enough to make use of the classical methods of mathematical physics only.

Theorem 2. *The general solution of the system of nonlinear PDE (2) in the class of functions* $u = u(x_0, x_1, x_2, x_3, x_4) \in C^2(\mathbb{C}^5, \mathbb{C}^1)$ *is given by one of the following formulas:*

$$(1)\ A_\mu(\tau, u)x^\mu + C_1(\tau, u) = 0, \tag{6}$$

where $\tau = \tau(u, x)$ *is a complex function determined by the equation*

$$B_\mu(\tau, u)x^\mu + C_2(\tau, u) = 0, \tag{7}$$

and $A_\mu, B_\mu, C_1, C_2 \in C^2(\mathbb{C}^2, \mathbb{C}^1)$ *are arbitrary functions satisfying the conditions*

$$A_\mu A^\mu = A_\mu B^\mu = B_\mu B^\mu = 0, \quad B^\mu \frac{\partial A_\mu}{\partial \tau} = 0, \tag{8}$$

and what is more,

$$\Delta = \det \left\| \begin{matrix} x^\mu \dfrac{\partial A_\mu}{\partial \tau} + \dfrac{\partial C_1}{\partial \tau} & x^\mu \dfrac{\partial A_\mu}{\partial u} + \dfrac{\partial C_1}{\partial u} \\ x^\mu \dfrac{\partial B_\mu}{\partial \tau} + \dfrac{\partial C_2}{\partial \tau} & x^\mu \dfrac{\partial B_\mu}{\partial u} + \dfrac{\partial C_2}{\partial u} \end{matrix} \right\| \neq 0; \tag{9}$$

$$(2)\ A_\mu(u)x^\mu + C_1(u) = 0, \tag{10}$$

where $A_\mu(u)$, $C_1(u)$ *are arbitrary smooth functions satisfying the relations*

$$A_\mu A^\mu = 0 \tag{11}$$

(in the formulas (6)–(11) the index μ *takes the values* $0, 1, 2, 3, 4$*).*

Theorem 3. *The general solution of the system of nonlinear PDE (2) in the class of functions* $u = u(x_0, x_1, x_2, x_3) \in C^2(\mathbb{C}^4, \mathbb{C}^1)$ *is given by the formulas (6)–(11), where the index* μ *is supposed to take the values* $0, 1, 2, 3$.

Note 2. Investigating particular solutions of the Maxwell equations, Bateman [13] arrived at the problem of integrating the d'Alembert equation $\Box_4 u = 0$ with an additional nonlinear condition (the eikonal equation) $u_{x_\mu} u_{x^\mu} = 0$. He has obtained the following class of exact solutions of the said system of PDE:

$$u(x) = c_\mu(\tau)x^\mu + c_4(\tau), \tag{12}$$

where $\tau = \tau(x)$ is a complex-valued function determined in implicit way

$$\dot{c}_\mu(\tau)x^\mu + \dot{c}_4(\tau) = 0, \tag{13}$$

and $c_\mu(\tau)$, $c_4(\tau)$ are arbitrary smooth functions satisfying conditions

$$c_\mu c^\mu = \dot{c}_\mu \dot{c}^\mu = 0. \tag{14}$$

(hereafter, a dot over a symbol means differentiation with respect to a corresponding argument).

It is not difficult to check that solutions (12)–(14) are complex (see the Lemma 1 below). An another class of complex solutions of the system (2) with $n = 4$ was constructed by Yerugin [11]. But neither the Bateman's formulas (12)–(14) nor the Yerugin's results give the general solution of the system (2) with $n = 4$.

3 Proofs of Theorems 1–3

It is well-known that the maximal symmetry group admitted by equation (1) is finite-dimensional (we neglect a trivial invariance with respect to an infinite-parameter group $u(x) \to u(x) + U(x)$, where $U(x)$ is an arbitrary solution of Eq. (1), which is due to its linearity). But being restricted to a set of solutions of the eikonal equation the set solutions of PDE (1) admits an infinite-dimensional symmetry group [14]! It is this very fact that enables us to construct the general solution of (2).

Proof of the Theorem 1. Let us make in (2) the hodograph transformation

$$z_0 = u(x), \quad z_a = x_a, \quad a = \overline{1, n-1}, \quad w(z) = x_0. \tag{15}$$

Evidently, the transformation (15) is defined for all functions $u(x)$, such that $u_{x_0} \not\equiv 0$. But the system (2) with $u_{x_0} = 0$ takes the form

$$\sum_{a=1}^{n-1} u_{x_a x_a} = 0, \quad \sum_{a=1}^{n-1} u_{x_a}^2 = 0,$$

whence $u_{x_a} \equiv 0$, $a = \overline{1, n-1}$ or $u(x) = \text{const}$.

Consequently, the change of variables (9) is defined on the whole set of solutions of the system (2) with the only exception $u(x) = \text{const}$.

Being rewritten in the new variables z, $w(z)$ the system (2) takes the form

$$\sum_{a=1}^{n-1} w_{z_a z_a} = 0, \quad \sum_{a=1}^{n-1} w_{z_a}^2 = 1. \tag{16}$$

Differentiating the second equation with respect to z_b, z_c we get

$$\sum_{a=1}^{n-1} (w_{z_a z_b z_c} w_{z_a} + w_{z_a z_b} w_{z_a z_c}) = 0.$$

Choosing in the above equality $c = b$ and summing up we have

$$\sum_{a,b=1}^{n-1} (w_{z_a z_b z_b} w_{z_a} + w_{z_a z_b} w_{z_a z_b}) = 0,$$

whence, by force of (16),

$$\sum_{a,b=1}^{n-1} w_{z_a z_b}^2 = 0. \tag{17}$$

Since $u(z)$ is a real-valued function, it follows from (17) that an equality $w_{z_a z_b} = 0$ holds for all $a, b = \overline{1, n-1}$, whence

$$w(z) = \sum_{a=1}^{n-1} \alpha_a(z_0) z_a + \alpha(z_0). \tag{18}$$

In (18) $\alpha_a, \alpha \in C^2(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions.

Substituting (18) into the second equation of system (16), we have

$$\sum_{a,b=1}^{n-1} \alpha_a^2(z_0) = 1. \tag{19}$$

Thus, the formulas (18), (19) give the general solution of the system of nonlinear PDE (16). Rewriting (18), (19) in the initial variables x, $u(x)$, we get

$$x_0 = \sum_{a=1}^{n-1} \alpha_a(u)x_a + \alpha(u), \quad \sum_{a=1}^{n-1} \alpha_a^2(u) = 1. \tag{20}$$

To represent the formulas (20) in a manifestly covariant form (4), (5) we redefine the functions $\alpha_a(u)$ in the following way:

$$\alpha_a(u) = \frac{A_a(u)}{A_0(u)}, \quad \alpha(u) = -\frac{B(u)}{A_0(u)}, \quad a = \overline{1, n-1}.$$

Substituting the above expressions into (20) we arrive at the formulas (4), (5).

Next, as $u = \mathrm{const}$ is contained in the class of functions $u(x)$ determined by the formulas (4), (5) under $A_\mu \equiv 0$, $\mu = \overline{0, n-1}$, $B(u) = u + \mathrm{const}$, JSSF (4), (5) give the general solution of the system of the PDE (2) with an arbitrary $n \in \mathbb{N}$. The theorem is proved.

Let us emphasize that the reasonings used above can be applied to the case of a real-valued function $u(x)$ only. If a solution of the system (2) is looked for in a class of complex-valued functions $u(x)$, then JSSF (4), (5) do not give its general solution with $n > 3$. Each case $n = 4, 5 \ldots$ requires a special consideration.

Proof of the Theorem 2. *Case 1:* $u_{x_0} \neq 0$. In this case the hodograph transformation (15) reducing the system (2) with $n = 5$ to the form

$$\sum_{a=1}^{4} w_{z_a z_a} = 0, \quad \sum_{a=1}^{4} w_{z_a}^2 = 1, \quad w_{z_0} \not\equiv 0 \tag{21}$$

is defined.

The general solution of nonlinear complex Eqs. (21) was constructed in [15]. It is given by one of the following formulas:

$$(1)\ w(z) = \sum_{a=1}^{4} \alpha_a(\tau, z_0) z_a + \gamma_1(\tau, z_0), \tag{22}$$

where $\tau = \tau(z_0, \ldots, z_4)$ is a function determined in implicit way

$$\sum_{a=1}^{4} \beta_a(\tau, z_0) z_a + \gamma_2(\tau, z_0) = 0 \tag{23}$$

and $\alpha_a, \beta_a, \gamma_1, \gamma_2 \in C^2(\mathbb{C}^2, \mathbb{C}^1)$ are arbitrary smooth functions satisfying the relations

$$\sum_{a=1}^{4} \alpha_a^2 = 1, \quad \sum_{a=1}^{4} \alpha_a \beta_a = \sum_{a=1}^{4} \beta_a^2 = 0, \quad \sum_{a=1}^{4} \alpha_a \frac{\partial \beta_a}{\partial \tau} = 0; \tag{24}$$

$$(2)\ w(z) = \sum_{a=1}^{4} \alpha_a(z_0) z_a + \gamma_1(z_0), \tag{25}$$

where $\alpha_a, \gamma_1 \in C^2(\mathbb{C}^1, \mathbb{C}^1)$ are arbitrary functions satisfying the relation

$$\sum_{a=1}^{4} \alpha_a^2 = 1. \tag{26}$$

Rewriting the formulas (23), (24) in the initial variables x, $u(x)$, we have

$$x_0 = \sum_{a=1}^{4} \alpha_a(\tau, u) x_a + \gamma_1(\tau, u), \tag{27}$$

where $\tau = \tau(u, x)$ is a function determined in implicit way

$$\sum_{a=1}^{4} \beta_a(\tau, u) x_a + \gamma_2(\tau, u) = 0, \tag{28}$$

and the relations (24) hold.

Evidently, the formulas (27), (28) are obtained from (6)–(8) with a particular choice of functions A_μ, B_μ, C_1, C_2

$$\begin{aligned} &A_0 = 1, \quad A_a = \alpha_a, \quad C_1 = -\gamma_1, \\ &B_0 = 0, \quad B_a = \beta_a, \quad C_2 = -\gamma_2, \end{aligned} \tag{29}$$

where $a = \overline{1,4}$.

Next, by force of inequality $w_{z_0} \not\equiv 0$ we get from (22)

$$\sum_{a=1}^{4} (\alpha_{az_0} + \alpha_{a\tau}\tau_{z_0}) x_a + \gamma_{1z_0} + \gamma_{1\tau}\tau_{z_0} \neq 0. \tag{30}$$

Differentiation of (23) with respect to z_0 yields the following expression for τ_{z_0}:

$$\tau_{z_0} = -\left(\sum_{a=1}^{4} \beta_{az_0} x_a + \gamma_{2z_0}\right)\left(\sum_{a=1}^{4} \beta_{a\tau} x_a + \gamma_{2\tau}\right)^{-1}$$

Substitution of the above result into (30) yields the relation

$$\left(\sum_{a=1}^{4} \beta_{a\tau} x_a + \gamma_{2\tau}\right)^{-1} \begin{vmatrix} \sum\limits_{a=1}^{4} \alpha_{az_0} x_a + \gamma_{1z_0} & \sum\limits_{a=1}^{4} \alpha_{a\tau} x_a + \gamma_{1\tau} \\ \sum\limits_{a=1}^{4} \beta_{az_0} x_a + \gamma_{2z_0} & \sum\limits_{a=1}^{4} \beta_{a\tau} x_a + \gamma_{2\tau} \end{vmatrix} \neq 0.$$

As the direct check shows, the above inequality is equivalent to (9) provided the conditions (29) hold.

Now we turn to solutions of the system (21) of the form (25). Rewriting the formulas (25), (26) in the initial variables x, $u(x)$ we get

$$x_0 = \sum_{a=1}^{4} \alpha_a(u) x_a + \gamma_1(u), \quad \sum_{a=1}^{4} \alpha_a^2(u) = 1.$$

Making in the equalities obtained the change $\alpha_a = A_a A_0^{-1}$, $a = \overline{1,4}$, $\gamma_1 = -C_1 A_0^{-1}$, we arrive at the formulas (10), (11).

Thus, under $u_{x_0} \not\equiv 0$ the general solution of the system (2) is contained in the class of functions $u(x)$ given by the formulas (6)–(9) or (10), (11).

Case 2: $u_{x_0} \equiv 0$, $u \neq \text{const}$. It is a common knowledge that the system of PDE (2) is invariant under the generalized Poincaré group $P(1, n-1)$ (see, e.g. [16])

$$x'_\mu = \Lambda_{\mu\nu} x^\nu + \Lambda_\mu, \quad u'(x') = u(x),$$

where $\Lambda_{\mu\nu}$, Λ_μ are arbitrary complex parameters satisfying the relations $\Lambda_{\alpha\mu}\Lambda^\alpha{}_\nu = g_{\mu\nu}$, $\mu, \nu = \overline{0, n-1}$. Hence, it follows that the transformation

$$u(x) \to u(x') = u(\Lambda_{\mu\nu} x^\nu) \tag{31}$$

leaves the set of solutions of the system (2) invariant. Consequently, provided $u(x) \neq \text{const}$ we can always transform u to such a form that $u_{x_0} \neq 0$. Thus, in the case 2 the general solution is also given by the formulas (6)–(11) within the transformation (31).

Case 3: $u = \text{const}$. Choosing in (10), (11) $A_\mu = 0$, $\mu = \overline{0,4}$, $C_1 = u = \text{const}$ we come to the conclusion that this solution is described by the formulas (6)–(11).

Thus, we have proved that, within a transformation from the group $P(1,4)$ (31), the general solution of the system of PDE (2) with $n = 5$ is given by the formulas (6)–(11). But these formulas are represented in a manifestly covariant form and are not altered with the transformation (31). Consequently, to complete the proof of the theorem it is enough to demonstrate that each function $u = u(x)$ determined by the equalities (6)–(11) is a solution of the system of equations (2).

Differentiating the relations (6), (7) with respect to x_μ, we have

$$A^\mu + \tau_{x_\mu}(A_{\nu\tau} x^\nu + C_{1\tau}) + u_{x_\mu}(A_{\nu u} x^\nu + C_{1u}) = 0,$$
$$B^\mu + \tau_{x_\mu}(B_{\nu\tau} x^\nu + C_{2\tau}) + u_{x_\mu}(B_{\nu u} x^\nu + C_{2u}) = 0.$$

Resolving the above system of linear algebraic equations with respect to u_{x_μ}, τ_{x_μ}, we get

$$\begin{aligned} u_{x_\mu} &= \frac{1}{\Delta}\big(B_\mu(A_{\nu\tau} x^\nu + C_{1\tau}) - A_\mu(B_{\nu\tau} x^\nu + C_{2\tau})\big), \\ \tau_{x_\mu} &= \frac{1}{\Delta}\big(A_\mu(B_{\nu u} x^\nu + C_{1u}) - B_\mu(A_{\nu u} x^\nu + C_{2u})\big), \end{aligned} \tag{32}$$

where $\Delta \neq 0$ by force of (9). Consequently,

$$\begin{aligned} u_{x_\mu} u_{x^\mu} = \Delta^{-2}\big(&B_\mu B^\mu (A_{\nu\tau} x^\nu + C_{1\tau})^2 - 2A_\mu B^\mu (A_{\nu\tau} x^\nu + C_{1\tau})(B_{\nu\tau} x^\nu + C_{2\tau}) + \\ &+ A_\mu A^\mu (B_{\nu\tau} x^\nu + C_{2\tau})^2\big) = 0. \end{aligned}$$

Analogously, differentiating (32) with respect to x_ν and convoluting the expression obtained with the metric tensor $g_{\mu\nu}$, we get $g^{\mu\nu} u_{x_\mu x_\nu} \equiv \Box_5 u = 0$.

Next, differentiating (10) with respect to x_μ we have

$$u_{x_\mu} = -A_\mu(\dot{A}_\nu x^\nu + \dot{C}_1)^{-1}, \quad \mu = \overline{0,4},$$

whence

$$u_{x_\mu x_\nu} = -(\dot{A}^\mu A^\nu + \dot{A}^\nu A^\mu)(\dot{A}_\alpha x^\alpha + \dot{C}_1)^{-2} + A^\mu A^\nu (\ddot{A}_\alpha x^\alpha + \ddot{C}_1)(\dot{A}_\alpha x^\alpha + \dot{C}_1)^{-2}.$$

Consequently,

$$
\begin{aligned}
&u_{x_\mu}u_{x^\mu} = A_\mu A^\mu(\dot{A}_\nu x^\mu + \dot{C}_1)^{-2} = 0,\\
&\Box_5 u \equiv u_{x_\mu x^\mu} = -2(A_\mu \dot{A}^\mu)(\dot{A}_\nu x^\nu + \dot{C}_1)^{-2} +\\
&\qquad + A_\mu A^\mu(\ddot{A}_\nu x^\nu + \ddot{C}_1)(\dot{A}_\nu x^\nu + \dot{C}_1)^{-2} = 0.
\end{aligned}
$$

The Theorem 2 is proved.

The Theorem 3 is a direct consequence of the Theorem 2. Really, solutions of the system of PDE (2) with $n = 4$ are obtained from solutions of the system of PDE (2) with $n = 5$ provided $u_{x_4} \equiv 0$. Imposing on functions $u(x)$ determined by the formulas (6)–(11) a condition $u_{x_4} \equiv 0$ we arrive at the following restrictions on the functions A_μ, B_μ, C_1, C_2:

$$
A_4 = 0, \quad B_4 = 0
$$

the same as what was to be proved.

4 Applications: reduction of the nonlinear d'Alembert equation

Following [8, 15, 16], we look for a solution of the nonlinear d'Alembert equation

$$
\Box_4 w = F(w), \quad F \in C^1(\mathbb{R}^1, \mathbb{R}^1) \tag{33}
$$

in the form

$$
w = \varphi(\omega_1, \omega_2), \tag{34}
$$

where $\omega_i = \omega_i(x) \in C^2(\mathbb{R}^4, \mathbb{R}^1)$ are supposed to be functionally-independent. The functions $\omega_1(x)$, $\omega_2(x)$ are determined by the requirement that the substitution of (34) into (33) yields two-dimensional PDE for a function $\varphi = \varphi(\omega_1, \omega_2)$. As a result, we obtain an over-determined system of PDE [16]

$$
\begin{aligned}
&\Box_4 \omega_1 = f_1(\omega_1, \omega_2), \quad \Box_4 \omega_2 = f_2(\omega_1, \omega_2),\\
&\omega_{1x_\mu}\omega_{1x^\mu} = g_1(\omega_1, \omega_2), \quad \omega_{2x_\mu}\omega_{2x^\mu} = g_2(\omega_1, \omega_2),\\
&\omega_{1x_\mu}\omega_{2x^\mu} = g_3(\omega_1, \omega_2), \quad \operatorname{rank} \left\| \frac{\partial \omega_i}{\partial x_\mu} \right\|_{i=1}^{2} {}_{\mu=0}^{3} = 2,
\end{aligned} \tag{35}
$$

and besides, the function $\varphi(\omega_1, \omega_2)$ satisfies a two-dimensional PDE,

$$
g_1 \varphi_{\omega_1\omega_1} + g_2 \varphi_{\omega_2\omega_2} + 2g_3 \varphi_{\omega_1\omega_2} + f_1 \varphi_{\omega_1} + f_2 \varphi_{\omega_2} = F(\varphi). \tag{36}
$$

Consider the following problem: to describe all smooth real functions $\omega_1(x)$, $\omega_2(x)$ such that the Ansatz (34) reduces Eq. (33) to an ordinary differential equation (ODE) with respect to the variable ω_1. It means that one has to put coefficients g_2, g_3, f_2 in (36) equal to zero. In other words, it is necessary to construct a general solution of the system of nonlinear PDE

$$
\begin{aligned}
&\Box_4 \omega_1 = f_1(\omega_1, \omega_2), \quad \omega_{1x_\mu}\omega_{1x^\mu} = g_1(\omega_1, \omega_2),\\
&\omega_{1x_\mu}\omega_{2x^\mu} = 0, \quad \omega_{2x^\mu}\omega_{2x_\mu} = 0, \quad \Box_4 \omega_2 = 0.
\end{aligned} \tag{37}
$$

The above system includes Eqs. (2) as a subsystem. So, the d'Alembert-eikonal system (2) arises in a natural way when solving the problem of reduction of Eq. (33) to PDE having a smaller dimension (see, also [15, 17]).

With an appropriate choice of a function $G(\omega_1, \omega_2)$ the change of variables

$$v = G(\omega_1, \omega_2), \quad u = \omega_2$$

reduces the system (37) to the form

$$\Box_4 v = f(u, v), \quad v_{x_\mu} v_{x^\mu} = \lambda, \tag{38}$$

$$u_{x_\mu} v_{x^\mu} = 0, \quad u_{x_\mu} u_{x^\mu} = 0, \quad \Box_4 u = 0, \tag{39}$$

$$\operatorname{rank} \left\| \begin{matrix} v_{x_0} v_{x_1} v_{x_2} v_{x_3} \\ u_{x_0} u_{x_1} u_{x_2} u_{x_3} \end{matrix} \right\| = 2, \tag{40}$$

where λ is a real parameter taking the values -1, 0, 1.

Before formulating the principal assertion, we will prove an auxiliary lemma.

Lemma 1. *Let $a = (a_0, a_1, a_2, a_3)$, $b = (b_0, b_1, b_2, b_3)$ be four-vectors defined in the real Minkowski space $M(1,3)$. Suppose they satisfy the relations*

$$a_\mu b^\mu = b_\mu b^\mu = 0, \quad \sum_{\mu=0}^{3} b_\mu^2 \neq 0. \tag{41}$$

Then, an inequality $a_\mu a^\mu \leq 0$ holds.

Proof. It is known that any isotropic non-null vector b in the space $M(1,3)$ can be reduced to the form $b' = (\alpha, \alpha, 0, 0)$, $\alpha \neq 0$ by means of a transformation from the group $P(1,3)$. Substituting $b' = (\alpha, \alpha, 0, 0)$ into the first equality from (41), we get

$$\alpha(a'_0 - a'_2) = 0 \;\Leftrightarrow\; a'_0 = a'_3.$$

Consequently, the vector a' has the following components: a'_0, a'_1, a'_2, a'_0. That is why, $a'_\mu a'^\mu = a'^2_0 - a'^2_1 - a'^2_2 - a'^2_0 = -(a'^2_1 + a'^2_2) \leq 0$. As the quadratic form $a_\mu a^\mu$ is invariant with respect to the group $P(1,3)$, hence it follows that $a_\mu a^\mu \leq 0$.

Let us note that $a_\mu a^\mu = 0$ if and only if $a_2 = a_3$, i.e. $a_\mu a^\mu = 0$ if and only if the vectors a and b are parallel.

Theorem 4. *Eqs. (38)–(40) are compatible if and only if*

$$\lambda = -1, \quad f = -N\big(v + h(u)\big)^{-1}, \tag{42}$$

where $h \in C^1(\mathbb{R}^1, \mathbb{R}^1)$ is an arbitrary function, $N = 0, 1, 2, 3$.

Theorem 4. *The general solution of the system of Eqs. (38)–(40) being determined within a transformation from the group $P(1,3)$ is given by the following formulas:*

a) under $f = -3\big(v + h(u)\big)^{-1}$, $\lambda = -1$

$$\begin{aligned} \big(v + h(u)\big)^2 &= (-\dot{A}_\nu \dot{A}^\nu)^{-1}(\dot{A}_\mu x^\mu + \dot{B})^2 + \\ &\quad + (-\dot{A}_\nu \dot{A}^\nu)^{-3}(\varepsilon^{\mu\nu\alpha\beta} A_\mu \dot{A}_\nu \ddot{A}_\alpha x_\beta + C)^2, \end{aligned} \tag{43}$$
$$A_\mu x^\mu + B = 0;$$

b) under $f = -2\big(v + h(u)\big)^{-1}$, $\lambda = -1$

$$\big(v + h(u)\big)^2 = (-\dot A_\nu \dot A^\nu)^{-1}(\dot A_\mu x^\mu + \dot B)^2, \quad A_\mu x^\mu + B = 0, \tag{44}$$

where $A_\mu = A_\mu(u)$, $B = B(u)$, $C = C(u)$ *are arbitrary smooth functions satisfying the relations*

$$A_\mu A^\mu = 0, \quad \dot A_\mu \dot A^\mu \neq 0, \tag{45}$$

c) under $f = -\big(v + h(u)\big)^{-1}$, $\lambda = -1$

$$\begin{gathered} \big(v + h(x_0 - x_3)\big)^2 = \big(x_1 + C_1(x_0 - x_3)\big)^2 + \big(x_2 + C_2(x_0 - x_3)\big)^2, \\ u = C_0(x_0 - x_3), \end{gathered} \tag{46}$$

where C_0, C_1, C_2 *are arbitrary smooth functions;*
d) under $f = 0$, $\lambda = -1$

$$(1) \quad v = (-\dot A_\nu \dot A^\nu)^{-3/2} \varepsilon^{\mu\nu\alpha\beta} A_\mu \dot A_\nu \ddot A_\alpha x_\beta + C, \quad A_\mu x^\mu + B = 0, \tag{47}$$

where $A_\mu = A_\mu(u)$, $B = B(u)$, $C = C(u)$ *are arbitrary smooth functions satisfying the relations (45);*

$$\begin{aligned} (2) \quad & v = x_1 \cos\big(C_1(x_0 - x_3)\big) + x_2 \sin\big(C_1(x_0 - x_3)\big) + C_2(x_0 - x_3), \\ & u = C_0(x_0 - x_3), \end{aligned} \tag{48}$$

where C_0, C_1, C_2 *are arbitrary smooth functions.*

In the above formulas (43), (47) we denote by $\varepsilon_{\mu\nu\alpha\beta}$ *the completely anti-symmetric fourth-order tensor (the Levi-Civita tensor), i.e.*

$$\varepsilon_{\mu\nu\alpha\beta} = \begin{cases} 1, & (\mu,\nu,\alpha,\beta) = \text{cycle}\,(0,1,2,3), \\ -1, & (\mu,\nu,\alpha,\beta) = \text{cycle}\,(1,0,2,3), \\ 0, & \text{in the remaining cases.} \end{cases}$$

Proof of the Theorems 4, 5. By force of (40) $u \neq \text{const}$. Consequently, within a transformation from the group $P(1,3)$ $u_{x_0} \neq 0$. That is why, one can apply to Eqs. (38)–(40) the hodograph transformation

$$z_0 = u(x), \quad z_a = x_a, \quad a = \overline{1,3}, \quad w(z) = x_0, \quad v = v(z_0, z_a).$$

As a result, the system (38), (39) reads

$$\sum_{a=1}^{3} w_{z_a}^2 = 1, \quad \sum_{a=1}^{3} w_{z_a z_a} = 0, \tag{49}$$

$$\sum_{a=1}^{3} v_{z_a} w_{z_a} = 0, \tag{50}$$

$$\sum_{a=1}^{3} v_{z_a}^2 = -\lambda, \quad \sum_{a=1}^{3} (v_{z_a z_a} + 2 w_{z_0}^{-1} v_{z_a} w_{z_a z_0}) = -f(v, z_0). \tag{51}$$

As $v(z)$ is a real-valued function, $\lambda \le 0$. Scaling, if necessary, the function v we can put $\lambda = -1$ or $\lambda = 0$.

Case 1: $\lambda = -1$. As it is shown in the Section 2, the general solution of the system (49) in the class of real-valued functions $w(z)$ is given by the formulas (18), (19) with $n = 4$. Substituting (18) into (50), we obtain a first-order linear PDE

$$\sum_{a=1}^{3} \alpha_a(z_0) v_{z_a} = 0, \tag{52}$$

whose general solution is represented in the form

$$v = v(z_0, \rho_1, \rho_2). \tag{53}$$

In (53),

$$z_0, \quad \rho_1 = \left(\sum_{a=1}^{3} \dot{\alpha}_a^2\right)^{-1/2} \left(\sum_{a=1}^{3} \dot{\alpha}_a z_a + \dot{\alpha}\right),$$
$$\rho_2 = \left(\sum_{a=1}^{3} \dot{\alpha}_a^2\right)^{-3/2} \sum_{a,b,c=1}^{3} \varepsilon_{abc} z_a \alpha_b \dot{\alpha}_c$$

are the first integrals of Eq. (52) and what is more, $\sum\limits_{a=1}^{3} \dot{\alpha}_a^2 \ne 0$ (the case $\alpha_a = \mathrm{const}$, $a = \overline{1,3}$ will be treated separately), ε_{abc} is the third-order anti-symmetric tensor with $\varepsilon_{123} = 1$.

Substitution the expression (53) into (51) yields the system of two PDE for a function $v = v(z_0, \rho_1, \rho_2)$

$$v_{\rho_1\rho_1} + v_{\rho_2\rho_2} + 2\rho_1^{-1} v_{\rho_1} = -f(v, z_0), \tag{54}$$

$$v_{\rho_1}^2 + v_{\rho_2}^2 = 1. \tag{55}$$

To get rid of an arbitrary element (function) $f(v, z_0)$ from (54) we consider instead of system (54), (55) its differential consequence

$$v_{\rho_2}(v_{\rho_1\rho_1} + v_{\rho_2\rho_2} + 2\rho_1^{-1} v_{\rho_1})_{\rho_1} - v_{\rho_1}(v_{\rho_1\rho_1} + v_{\rho_2\rho_2} + 2\rho_1^{-1} v_{\rho_1})_{\rho_1} = 0, \tag{56}$$

$$v_{\rho_1}^2 + v_{\rho_2}^2 = 1, \tag{57}$$

that is obtained by differentiating the first equation with respect to ρ_1, ρ_2, multiplying the expressions obtained by v_{ρ_2} and $-v_{\rho_1}$, respectively, and summing.

Further, we will consider the subcases $v_{\rho_2\rho_2} = 0$ and $v_{\rho_2\rho_2} \ne 0$ separately.

Subcase 1.A: $v_{\rho_2\rho_2} = 0$. Then,

$$v = g_1(z_0, \rho_1)\rho_2 + g_2(z_0, \rho_1), \tag{58}$$

where $g_1, g_2 \in C^2(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions.

Substituting (58) into (57) and splitting an equality obtained by the powers of ρ_2, we have

$$g_{1\rho_1} = 0, \quad g_1^2 + (g_{2\rho_2})^2 = 1,$$

whence

$$v = \alpha\rho_1 \pm \sqrt{1-\alpha^2}\rho_2 - h(z_0). \tag{59}$$

Here $\alpha \in \mathbb{R}^1$, h is an arbitrary smooth function.

Inserting (59) into (56) we get an algebraic equation $\alpha\sqrt{1-\alpha^2} = 0$, whence $\alpha = 0, \pm 1$.

Finally, substitution of (59) into (54) yields the equation for $f(v, z_0)$

$$2\alpha\rho_1^{-1} = -f\left(\alpha\rho_1 \pm \sqrt{1-\alpha^2}\rho_2 - h(z_0), z_0\right). \tag{60}$$

From Eq. (60) it follows that, under $\alpha = 0$,

$$f = 0, \quad v = \pm\rho_2 - h(z_0) \tag{61}$$

and under $\alpha = \pm 1$,

$$f = -2\bigl(v + h(z_0)\bigr)^{-1}, \quad v = \pm\rho_1 - h(z_0). \tag{62}$$

Subcase 1.B: $v_{\rho_2\rho_2} \neq 0$. In this case one can apply to Eqs. (56), (57) the Euler–Ampére transformation

$$\begin{aligned} &z_0 = y_0, \quad \rho_1 = y_1, \quad \rho_2 = G_{y_2}, \quad v + G = \rho_2 y_2, \quad v_{\rho_1} = -G_{y_1}, \quad v_{\rho_2} = y_2, \\ &v_{\rho_2\rho_2} = (G_{y_2y_2})^{-1}, \quad v_{\rho_1\rho_2} = -G_{y_1y_2}(G_{y_2y_2})^{-1}, \\ &v_{\rho_1\rho_1} = (G^2_{y_1y_2} - G_{y_1y_1}G_{y_2y_2})(G_{y_2y_2})^{-1}. \end{aligned} \tag{63}$$

Here y_0, y_1, y_2 are new independent variables, $G = G(y_0, y_1, y_2)$ is a new function.

Being rewritten in the new variables y, $G(y)$ the Eq. (57), becomes linear

$$G_{y_1} = \pm\sqrt{1 - y_2^2},$$

whence

$$G = \pm y_1\sqrt{1-y_2^2} + H(y_0, y_2), \quad H \in C^2(\mathbb{R}^2, \mathbb{R}^1). \tag{64}$$

Making in the Eq. (56) the change of variables (63) and inserting the expression (64), we transform it as follows

$$\bigl(y_2 - (1-y_2^2)^{3/2}H_{y_2y_2}\bigr)^{-2}\bigl(3y_2H_{y_2y_2} + (y_2^2-1)H_{y_2y_2y_2}\bigr) + 2y_1^{-2}y_2H_{y_2y_2} = 0. \tag{65}$$

Splitting (65) by the powers of y_1 and integrating the equations obtained, we get

$$H = h_1(y_0)y_2 + h_2(y_0).$$

Substituting the above result into (64) and returning to the initial variables z_0, ρ_1, ρ_2, $v(z_0, \rho_1, \rho_2)$ we obtain the general solution of the system of PDE (56), (57)

$$v + h_2(z_0) = \pm\bigl([\rho_2 - h_1(z_0)]^2 + \rho_1^2\bigr)^{1/2}. \tag{66}$$

At last, inserting (66) into the equation (54), we arrive at the conclusion that the function f is determined by the formula (42) with $N = 3$.

If $\alpha_a = \text{const}$, $a = \overline{1,3}$, then the equality $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ holds. Applying, if necessary, a transformation from the group $P(1,3)$ one can put $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$, i.e. $u = C_0(x_0 - x_3)$, $C_0 \in C^2(\mathbb{R}^1, \mathbb{R}^2)$.

As a consequence of Eqs. (39) we get $v = v(\xi, x_1, x_2)$, where $\xi = x_0 - x_3$, and what is more, Eqs. (38) take the form

$$v_{x_1}^2 + v_{x_2}^2 = 1, \quad v_{x_1x_1} + v_{x_2x_2} = -f\big(v, C_0(\xi)\big). \tag{67}$$

It is known [15, 18] that Eqs. (67) are compatible if and only if $f = 0$ or $f = -(v + h(u))^{-1}$, $h \in C^1(\mathbb{R}^1, \mathbb{R}^1)$. And besides, the general solution of (67) is given by the formulas (48) and (46), respectively.

Thus, we have completely investigated the case $\lambda = -1$.

Case 2: $\lambda = 0$. By force of the fact that the function v is a real one, it follows from (51) that $v = v(z_0)$. Consequently, an equality $v = v(u)$ holds that breaks the condition (40) which means that under $\lambda = 0$ the system (38)–(40) is incompatible.

Thus, we have proved that the system of nonlinear PDE (38)–(40) is compatible if and only if the relations (42) hold and that its general solution is given by one of the formulas (46), (48), (61), (62), and (66). To complete the proof, one has to rewrite the expressions (61), (62), (66) in the manifestly covariant from (43), (44), (47).

Consider, as an example, the formula (62)

$$v = \pm\rho_1 - h(z_0) \equiv \pm\left(\sum_{a=1}^{3}\dot\alpha_a^2(u)\right)^{-1/2}\left(\sum_{a=1}^{3}x_a\dot\alpha_a(u) + \dot\alpha(u)\right) - h(u), \tag{68}$$

the function $u(x)$ being determined by the formula (20),

$$\sum_{a=1}^{3}\alpha_a(u)x_a + \alpha(u) = x_0, \quad \sum_{a=1}^{3}\alpha_a^2(u) = 1. \tag{69}$$

Let us make in (68), (69) a substitution $\alpha_a = A_aA_0^{-1}$, $\alpha = -BA_0^{-1}$, whence

$$\begin{aligned}
&A_\mu(u)x^\mu + B(u) = 0, \quad A_\mu A^\mu = 0,\\
&v = \pm\left(\sum_{a=1}^{3}(\dot A_aA_0^{-1} - A_a\dot A_0A_0^{-2})^2\right)^{-1/2}\times\\
&\quad\times\left(\sum_{a=1}^{3}x_a(\dot A_aA_0^{-1} - A_a\dot A_0A_0^{-2}) + B\dot A_0A_0^{-2} - \dot BA_0^{-1}\right) - h(u) =\\
&\ = \pm\left(\sum_{a=1}^{3}(\dot A_a^2A_0^{-2} + A_a^2\dot A_0^2A_0^{-4} - 2\dot A_aA_a\dot A_0A_0^{-3})^{-1/2}\right)\times\\
&\quad\times\left(\sum_{a=1}^{3}x_a(\dot A_aA_0^{-1} - A_a\dot A_0A_0^{-2}) + B\dot A_0A_0^{-2} - \dot BA_0\right) - h(u) =\\
&\ = \pm\left(-\dot A_\mu\dot A^\mu A_0^{-2} - A_\mu A^\mu\dot A_0^2A_0^{-4} + 2\dot A_\mu A^\mu\dot A_0A_0^{-3}\right)^{-1/2}\times\\
&\quad\times\left(-A_0^{-1}(x_\mu\dot A^\mu + \dot B) + A_0^{-2}\dot A_0(x_\mu A^\mu + B)\right) - h(u) =\\
&\ = \mp(-\dot A_\mu\dot A^\mu)^{-1/2}(x_\mu\dot A^\mu + \dot B) - h(u).
\end{aligned}$$

The only thing left is to prove that $\dot{A}_\mu \dot{A}^\mu < 0$. Since $A_\mu A^\mu = 0$, the equality $\dot{A}_\mu A^\mu = 0$ holds. Consequently, by force of the Lemma $-\dot{A}_\mu \dot{A}^\mu \geq 0$, and what is more, the equality $\dot{A}_\mu \dot{A}^\mu = 0$ holds if and only if $\dot{A}_\mu = k(u)A_\mu$. General solution of the above system of ordinary differential equations reads $A_\mu = l(u)\theta_\mu$, where $l(u)$ is an arbitrary function, θ_μ are arbitrary real parameters obeying the equality $\theta_\mu\theta^\mu = 0$. Hence it follows that $\alpha_a = A_a A_0^{-1} = \theta_a \theta_0^{-1} = \text{const}$, and the condition $\sum\limits_{a=1}^{3} \dot{\alpha}_a^2 \neq 0$ does not hold. We come to the contradiction, whence it follows that $\dot{A}_\mu \dot{A}^\mu < 0$.

Thus, we have obtained the formula (44). Derivation of the remaining formulas from (43), (47) is carried out in the same way. The theorems are proved.

Substitution of the results obtained above into the formula (34) yields the following collection of Ansätze for the nonlinear d'Alembert equation (33):

$$
\begin{aligned}
&(1)\ w(x) = \varphi\Big(\big[\big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{-1}\big(\dot{A}_\mu(u)x^\mu + \dot{B}(u)\big)^2 + \\
&\qquad + \big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{-3}\big(\varepsilon^{\mu\nu\alpha\beta}A_\mu(u)\dot{A}_\nu(u)\ddot{A}_\alpha(u)x_\beta + C(u)\big)^2\big]^{1/2}, u\Big);\\
&(2)\ w(x) = \varphi\Big(\big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{1/2}\big(\dot{A}_\mu(u)x^\mu + \dot{B}(u)\big), u\Big);\\
&(3)\ w(x) = \varphi\Big(\big[\big(x_1 + C_1(x_0 - x_3)\big)^2 + \big(x_2 + C_2(x_0 - x_3)\big)^2\big]^{1/2}, x_0 - x_3\Big);\\
&(4)\ w(x) = \varphi\Big(\big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{-3/2}\big(\varepsilon^{\mu\nu\alpha\beta}A_\mu(u)\dot{A}_\nu(u)\ddot{A}_\alpha(u)x_\beta + C(u)\big), u\Big);\\
&(5)\ w(x) = \varphi\big(x_1 \cos C_1(x_0 - x_3) + x_2 \sin C_1(x_0 - x_3) + C_2(x_0 - x_3), x_0 - x_3\big).
\end{aligned}
\tag{70}
$$

Here B, C, C_1, C_2 are arbitrary smooth functions of the corresponding arguments, $A_\mu(u)$ are arbitrary smooth functions satisfying the condition $A_\mu A^\mu = 0$ and the function $u = u(x)$ is determined by JSSF (10) with $C_1(u) = B(u)$, $n = 4$. Note that arbitrary functions h contained in the functions $v(x)$ (see above the formulas (43), (44), (46)) are absorbed by the function $\varphi(v, u)$ at the expense of the second argument.

Substitution of the expressions (70) into (33) gives the following equations for $\varphi = \varphi(u, v)$:

$$(1)\ \varphi_{vv} + \frac{3}{v}\varphi_v = -F(\varphi), \tag{71}$$

$$(2)\ \varphi_{vv} + \frac{2}{v}\varphi_v = -F(\varphi), \tag{72}$$

$$(3)\ \varphi_{vv} + \frac{1}{v}\varphi_v = -F(\varphi), \tag{73}$$

$$(4)\ \varphi_{vv} = -F(\varphi), \tag{74}$$

$$(5)\ \varphi_{vv} = -F(\varphi), \tag{75}$$

Equations (4), (5) from (71)–(75) are known to be integrable in quadratures. Therefore, any solution of the d'Alembert-eikonal system (2) corresponds to some class of exact solutions of the nonlinear wave equation (33) that contains arbitrary functions. Saying it in another way, the formulas (70) make it possible to construct wide families of exact solutions of the nonlinear PDE (33) using exact solutions of the linear d'Alembert equation $\Box_4 u = 0$ satisfying an additional constraint $u_{x_\mu} u_{x^\mu} = 0$.

It is interesting to compare our approach to the problem of reduction of Eq. (33) with the classical Lie approach. Within the framework of the Lie approach functions $\omega_1(x)$, $\omega_2(x)$ from (34) are looked for as invariants of the symmetry group of the equation under study (in the case involved it is the Poincaré group $P(1,3)$). Since the group $P(1,3)$ is a finite-parameter group, its invariants cannot contain an arbitrary function (a complete description of invariants of the group $P(1,3)$ had been carried out in [19]). Therefore, the Ansätze (70) cannot, in principle, be obtained by means of the Lie symmetry of the PDE (33).

All Ansätze listed in (70) correspond to a *conditional invariance* of the nonlinear d'Alembert equation (33). It means that for each Ansatz from (70) there exist two differential operators $Q_a = \xi_{a\mu}(x)\partial_{x_\mu}$, $a = 1,2$ such that

$$Q_a w(x) \equiv Q_a \varphi(\omega_1, \omega_2) = 0, \quad a = \overline{1,2}$$

and besides, the system of PDE

$$\Box_4 w - F(w) = 0, \quad Q_a w = 0, \quad a = 1,2$$

is invariant in Lie's sense under the one-parameter groups with the generators Q_1, Q_2. For example, the fourth Ansatz from (16) is invariant with respect to the operators: $Q_1 = A_\mu(u)\partial_\mu$, $Q_2 = \dot{A}_\mu(u)\partial_\mu$. A direct computation shows that the following relations hold:

$$\underset{2}{Q}_i(\Box_4 \omega) = -(\dot{A}^\alpha x_\alpha + \dot{B})^{-1} A^\mu \partial_\mu Q_i w, \quad i = 1,2,$$
$$[Q_1, Q_2] = 0,$$

where $\underset{2}{Q}_i$ stands for the second prolongation of the operator Q_i. Hence it follows that the nonlinear d'Alembert equation (33) is conditionally-invariant under the two-dimensional commutative Lie algebra having the basis elements Q_1, Q_2 (for more details about conditional symmetry of PDE see [20, 21]). It should be said that the notion of conditional symmetry of PDE is closely connected with the "non-classical reduction" [22–24] and "direct reduction" [25] methods.

5 On the new exact solutions of the nonlinear d'Alembert equation

According to [26], general solutions of Eqs. (74), (75) are given by the following quadrature:

$$v + D(u) = \int_0^{\varphi(u,v)} \left(-2\int_0^\tau F(z)dz + C(u)\right)^{-1/2} d\tau, \tag{76}$$

where $D(u), C(u) \in C^2(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions.

Substituting the expressions for $u(x)$, $v(x)$ given by the formulas (4), (5) from (70) into (76) we obtain two classes of exact solutions of the nonlinear d'Alembert equation (33) that contain several arbitrary functions of one variable.

Equations (71) and (72) with $F(\varphi) = \lambda\varphi^k$ are Emden–Fowler type equations. They were investigated by many authors (see, e.g. [26]). In particular, it is known that the equations

$$\varphi_{vv} + 2v^{-1}\varphi_v = -\lambda\varphi^5, \tag{77}$$

$$\varphi_{vv} + 3v^{-1}\varphi_v = -\lambda\varphi^3 \tag{78}$$

are integrated in quadratures. In the paper [27] it has been established that Eqs. (77), (78) possess a Painlevé property. This fact makes it possible to integrate these by applying rather complicated technique. In [28] we have integrated Eqs. (77), (78) using a standard technique based on their Lie symmetry. Substituting the results obtained into the corresponding Ansätze from (70) we get exact solutions of the nonlinear PDE (33) with $F(w) = \lambda w^3$, λw^5, which include an arbitrary solution of the system (2) with $n = 4$. Consequently, we have constructed principally new exact solutions of the nonlinear d'Alembert equation (33) depending on several arbitrary functions. Let us stress that following the classical Lie symmetry reduction procedure one can not in principle obtain solutions with arbitrary functions since the maximal symmetry group of Eq. (33) is finite-dimensional (see, e.g. [16]).

Below we give new exact solutions of the nonlinear d'Alembert equation (33) obtained with the use of the technique described above. We adduce only those ones that can be written down explicitly

1. $F(w) = \lambda w^3$

(1) $$w(x) = \frac{1}{a\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 - x_0^2)^{-1/2}\times$$
$$\times \tan\left\{-\frac{\sqrt{2}}{4}\ln\bigl(C(u)(x_1^2 + x_2^2 + x_3^2 - x_0^2)\bigr)\right\},$$
where $\lambda = -2a^2 < 0$,

(2) $$w(x) = \frac{2\sqrt{2}}{a}C(u)\bigl(1 \pm C^2(u)(x_1^2 + x_2^2 + x_3^2 - x_0^2)\bigr)^{-1},$$
where $\lambda = \pm a^2$;

2. $F(w) = \lambda w^5$

(1) $$w(x) = a^{-1}(x_1^2 + x_2^2 - x_0^2)^{-1/4}\left\{\sin\ln\bigl(C(u)(x_1^2 + x_2^2 - x_0^2)^{-1/2}\bigr) + 1\right\}^{1/2}\times$$
$$\times\left\{2\sin\ln\bigl(C(u)(x_1^2 + x_2^2 - x_0^2)^{-1/2}\bigr) - 4\right\}^{-1/2},$$
where $\lambda = a^4 > 0$,

(2) $$w(x) = \frac{3^{1/4}}{\sqrt{a}}C(u)\bigl(1 \pm C^4(u)(x_1^2 + x_2^2 - x_0^2)\bigr)^{-1/2},$$
where $\lambda = \pm a^2$.

In the above formulas $C(u)$ is an arbitrary twice continuously differentiable function on

$$u(x) = \frac{x_0x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2}}{x_1^2 + x_2^2},$$

$a \neq 0$ is an arbitrary real parameter.

6 Conclusion

The present paper demonstrates once more that possibilities to construct in explicit form new exact solutions of the nonlinear d'Alembert equation (33) (as compared with those obtainable by the standard symmetry reduction technique [16, 19, 27]) are far from being exhausted. A source of new (non-Lie) reductions is the conditional symmetry of Eq. (33).

Roughly speaking, a principal idea of the method of conditional symmetries is the following: to be able to reduce given PDE it is enough to require an invariance of a *subset* of its solutions with respect to some Lie transformation group. And what is more, this subset is not obliged to coincide with the whole set. This specific subsets can be chosen in different ways: one can fix in some way an Ansatz for a solution to be found (the method of Ansätze [16, 17] or the direct reduction method [25]) or one can impose an additional differential constraint (the method of non-classical [22–24] or conditional symmetries [20, 21]). But all the above approaches have a common feature: to find new (non-Lie) reduction of a given PDE one has to solve some nonlinear over-determined system of differential equations. For example, to describe Ansätze of the form (34) reducing Eq. (33) to ODE one has to integrate system of five nonlinear PDE (37). This is a "price" to be paid for the new possibilities to reduce a given nonlinear PDE to equations with less number of independent variables and to construct its explicit solutions.

As mentioned in the Introduction, the Ansatz (34) can also be interpreted as a map (more exactly, a family of maps) from the set of solutions of the linear d'Alembert equation,

$$\Box_4 u = 0 \tag{79}$$

into the set of solutions of the nonlinear d'Alembert equation (33).

Really, we started with a subset of solutions of Eq. (79) which was chosen by an additional eikonal constraint $u_{x_\mu} u_{x^\mu} = 0$. Then, we constructed the functions $v(x)$ and $\varphi(v, u)$ in such a way that the function $w(x)$ determined by the equality $w = \varphi(v(x), u(x))$ satisfied the nonlinear d'Alembert equation (33) (see below the Fig. 1).

There is an analogy between the map described above and Bäcklund transformations of partial differential equations. System of PDE (38)–(40) and the Ansatz (34) (level 2 of the Fig. 1) can be interpreted as a Bäcklund transformation of a set of solutions of linear PDE (level 1 of the Fig. 1) into a set of solutions of nonlinear PDE (level 3). A principal difference is that a classical Bäcklund transformation acts on the whole spaces of solutions of equations under study and the above map acts on subsets of solutions of the linear and nonlinear d'Alembert equations. It is known that technique of linearization of PDE with the use of Bäcklund transformations can be effectively applied to two-dimensional equations only. The results obtained in the present paper imply the following way of extension of applicability of Bäcklund transformations: one should consider Bäcklund transformations connecting subsets of solutions of linear and nonlinear equations. And these subsets may not coincide with the whole sets of solutions.

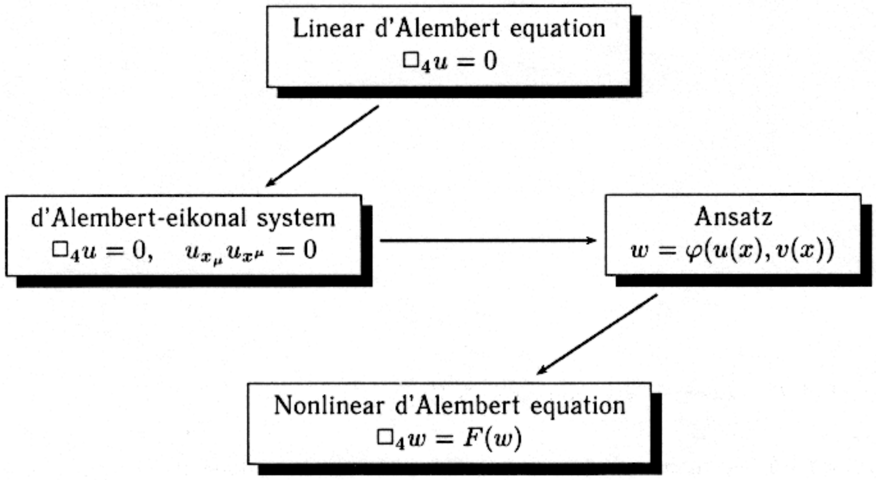

Figure 1.

As an illustration we consider the case when in (33) $F(w) = 0$, i.e. the case when the map constructed above transforms a subset of solutions of the linear d'Alembert equation into another subset of solutions of the same equation. Integrating ODE (71)–(75) we obtain explicit forms of functions $\varphi(v, u)$

(1) $\varphi(v,u) = f_1(u)v^{-2} + f_2(u)$,
(2) $\varphi(v,u) = f_1(u)v^{-1} + f_2(u)$,
(3) $\varphi(v,u) = f_1(u)\ln v + f_2(u)$,
(4) $\varphi(v,u) = f_1(u)v + f_2(u)$,
(5) $\varphi(v,u) = f_1(u)v + f_2(u)$,

where f_1, f_2 are arbitrary smooth enough functions. Consequently, we have the following maps transforming subsets of solutions of the linear d'Alembert equation (79) into another subsets of its solutions:

$$
\begin{aligned}
(1)\quad & u \to f_1(u)\big[\big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{-1}\big(\dot{A}_\mu(u)x^\mu + \dot{B}(u)\big)^2 + \\
& \quad + \big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{-3}\big(\varepsilon^{\mu\nu\alpha\beta}A_\mu(u)\dot{A}_\nu(u)\ddot{A}_\alpha(u)x_\beta + C(u)\big)^2\big]^{-1} + f_2(u),\\
(2)\quad & u \to f_1(u)\big[\big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{1/2}\big(\dot{A}_\mu(u)x^\mu + \dot{B}(u)\big)\big]^{-1} + f_2(u),\\
(3)\quad & x_0 - x_3 u \to f_1(x_0 - x_3)\ln\big[\big(x_1 + C_1(x_0 - x_3)\big)^2 + \\
& \quad + \big(x_2 + C_2(x_0 - x_3)\big)^2\big]^{-1/2} + f_2(x_0 - x_3),\\
(4)\quad & u \to \big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{-3/2}\big(\varepsilon^{\mu\nu\alpha\beta}A_\mu(u)\dot{A}_\nu(u)\ddot{A}_\alpha(u)x_\beta + C(u)\big),\\
(5)\quad & x_0 - x_3 \to f_1(x_0 - x_3)(x_1\cos C_1(x_0 - x_3) + \\
& \quad + x_2 \sin C_1(x_0 - x_3) + C_2(x_0 - x_3).
\end{aligned}
$$

Note that in the cases 4, 5 function f_2 is absorbed by arbitrary functions C, C_2.

And one more remark seems to be noteworthy. If one takes as a particular solution of the system (2) the function $u(x) = (x_0 x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2})/(x_1^2 + x_2^2)$ and substitutes it into the first, second and fourth Ansätze from (70), then the following Ansätze are obtained:

$$(1) \quad w(x) = \varphi\left(x_1^2 + x_2^2 + x_3^2 - x_0^2, \frac{x_0 x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2}}{x_1^2 + x_2^2}\right),$$

$$(2) \quad w(x) = \varphi\left(x_1^2 + x_2^2 - x_0^2, \frac{x_0 x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2}}{x_1^2 + x_2^2}\right),$$

$$(4) \quad w(x) = \varphi\left(x_3, \frac{x_0 x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2}}{x_1^2 + x_2^2}\right).$$

Provided the above Ansätze do not depend on the second argument, the usual Lie Ansätze are obtained which are invariant under some subgroups of the Poincaré group $P(1,3)$ [19]. Consequently, if we imagine invariant solutions as dots in a solution space of the nonlinear d'Alembert equation, then through some of them one can draw curves which are conditionally-invariant solutions. In this respect a number of interesting questions arise, let us mention two of these:

(1) Is any invariant solution of the nonlinear d'Alembert equation (33) a particular case of some more general conditionally-invariant solution?

(2) Does there exist such conditionally-invariant solution of Eq. (33) that all invariant solutions of Eq. (33) are its particular cases? (saying about invariant solutions we mean solutions invariant under some subgroup of the symmetry group of Eq. (33)).

An answer to the first question seems to be positive. A positive answer to the second one would provide us with a concept of a "general invariant solution". But so far this problem is completely open and needs further investigation.

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1. Kaluza Th., *Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl.*, 1921, **996**.
2. Khodos A., *Uspekhi Fiz. Nauk*, 1985, **146**, 647.
3. Fushchych W., Krivsky I., *Nuclear Physics*, 1968, **17**, 79.
4. Fushchych W., Krivsky I., *Nuclear Physics*, 1969, **18**, 573.
5. Fushchych W., *Theor. Math. Phys.*, 1970, **4**, 360.
6. Fushchych W., Nikitin A., *J. Phys. A: Math. Gen.*, 1980, **13**, 2319.
7. Cieciura G., Grundland A., *J. Math. Phys.*, 1984, **25**, 3460.
8. Fushchych W., Zhdanov R., Yegorchenko I., *J. Math. Anal. Appl.*, 1991, **161**, 352.
9. Smirnov V., Sobolev S., *Trudy Seismol. Instituta AN SSSR*, 1932, **20**, 37.
10. Smirnov V., Sobolev S., *Trudy Seismol. Instituta AN SSSR*, 1933, **29**, 43.
11. Yerugin N.P., *Doklady AN SSSR*, 1944, **20**, 385.
12. Collins C.B., *J. Math. Phys.*, 1983, **24**, 22.
13. Bateman H., The mathematical analysis of electrical and optical wave motion of the basis of Maxwell's equations, Cambridge, Univ. Press, 1915.
14. Shulga M., in Group-Theoretical Studies of the Mathematical Physics Equations, Kiev, Inst. of Mathematics of Ukrain. Acad. of Sci., Kiev, 1985, 36.

15. Fushchych W., Zhdanov R., Revenko I., *Ukr. Math. J.*, 1991, **43**, 1364.
16. Fushchych W., Shtelen W., Serov M., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer Academic Publisher, 1992.
17. Fushchych W., Zhdanov R., *Phys. Reports*, 1989, **172** 123.
18. Collins C.B., *Math. Proc. Cambr. Phyl. Soc.*, 1976, **80**, 165.
19. Patera J., Sharp R.T., Winternitz P., Zassenhaus H., *J. Math. Phys.*, 1976, **17**, 977.
20. Fushchych W., *Ukr. Math. J.*, 1991, **43**, 1456.
21. Fushchych W., Zhdanov R., *Ukr. Math. J.*, 1992, **44**, 970.
22. Bluman G., Cole J., *J. Math. Mech.*, 1969, **18**, 1025.
23. Olver P., Rosenau Ph., *Phys. Lett. A*, 1987, **112**, 107.
24. Levi D., Winternitz P., *J. Phys. A: Math. Gen.*, 1989, **22**, 2922.
25. Clarkson P., Kruskal M., *J. Math. Phys.*, 1989, **30**, 2201.
26. Kamke E., Differentialgleichungen Lösungmethoden und Lösungen, Leipzig, Akademische Verlagsgesellschaft, 1976.
27. Winternitz P., Grundland A., Tuszyński J., *J. Math. Phys.*, 1987, **28**, 2194.
28. Fushchych W., Zhdanov R., Revenko I., in Symmetry Analysis of Equations of Mathematical Physics, Kiev, Inst. of Mathematics of Ukrain. Acad. of Sci., 1992, 68.