On the new approach to variable separation in the time-dependent Schrödinger equation with two space dimensions

R.Z. ZHDANOV, I.V. REVENKO, W.I. FUSHCHYCH

We suggest an effective approach to separation of variables in the Schrödinger equation with two space variables. Using it we classify inequivalent potentials $V(x_1, x_2)$ such that the corresponding Schrödinger equations admit separation of variables. Besides that, we carry out separation of variables in the Schrödinger equation with the anisotropic harmonic oscillator potential $V = k_1x_1^2 + k_2x_2^2$ and obtain a complete list of coordinate systems providing its separability. Most of these coordinate systems depend essentially on the form of the potential and do not provide separation of variables in the free Schrödinger equation ($V = 0$).

1 Introduction

The problem of separation of variables (SV) in the two-dimensional Schrödinger equation

$$iu_t + u_{x_1x_1} + u_{x_2x_2} = V(x_1, x_2)u \tag{1}$$

as well as the most of classical problems of mathematical physics can be formulated in a very simple way (but this simplicity does not, of course, imply an existence of easy way to its solution). To separate variables in Eq. (1) one has to construct such functions $R(t, \boldsymbol{x})$, $\omega_1(t, \boldsymbol{x})$, $\omega_2(t, \boldsymbol{x})$ that the Schrödinger equation (1) after being rewritten in the new variables

$$\begin{aligned} &z_0 = t, \quad z_1 = \omega_1(t, \boldsymbol{x}), \quad z_2 = \omega_2(t, \boldsymbol{x}), \\ &v(z_0, \boldsymbol{z}) = R(t, \boldsymbol{x})u(t, \boldsymbol{x}) \end{aligned} \tag{2}$$

separates into three ordinary differential equations (ODEs). From this point of view the problem of SV in Eq. (1) is studied in [1–4].

But no less of an important problem is the one of description of potentials $V(x_1, x_2)$ such that the Schrödinger equation admits variable separation. That is why saying about SV in Eq. (1) we imply two mutually connected problems. The first one is to describe all such functions $V(x_1, x_2)$ that the corresponding Schrödinger equation (1) can be separated into three ODEs in some coordinate system of the form (2) (classification problem). The second problem is to construct for each function $V(x_1, x_2)$ obtained in this way all coordinate systems (2) enabling us to carry out SV in Eq. (1).

Up to our knowledge, the second problem has been solved provided $V = 0$ [2, 3] and $V = \alpha x_1^{-2} + \beta x_2^{-2}$ [1]. The first one was considered in a restricted sense in [4]. Authors using symmetry approach to classification problem obtained some potentials providing separability of Eq. (1) and carried out SV in the corresponding

J. Math. Phys., 1995, **36**, № 10, P. 5506–5521.

Schrödinger equation. But their results are far from being complete and systematic. The necessary and sufficient conditions imposed on the potential $V(x_1, x_2)$ by the requirement that the Schrödinger equation admits symmetry operators of an arbitrary order are obtained in [5]. But so far there is no systematic and exhaustive description of potentials $V(x_1, x_2)$ providing SV in Eq. (1).

To be able to discuss the description of *all* potentials and *all* coordinate systems making it possible to separate the Schrödinger equation one has to give a definition of SV. One of the possible definitions of SV in partial differential equations (PDEs) is proposed in our article [6]. It is based on the concept of Ansatz suggested by Fushchych [7] and on ideas contained in the article by Koornwinder [8]. The said definition is quite algorithmic in the sense that it contains a regular algorithm of variable separation in partial differential equations which can be easily adapted to handle both linear [6, 9] and nonlinear [10] PDEs. In the present article we apply the said algorithm to solve the problem of SV in Eq. (1).

Consider the following system of ODEs:

$$\begin{gathered} i\frac{d\varphi_0}{dt} = U_0(t, \varphi_0; \lambda_1, \lambda_2), \\ \frac{d^2\varphi_1}{d\omega_1^2} = U_1\left(\omega_1, \varphi_1, \frac{d\varphi_1}{d\omega_1}; \lambda_1, \lambda_2\right), \quad \frac{d^2\varphi_2}{d\omega_2^2} = U_2\left(\omega_2, \varphi_2, \frac{d\varphi_2}{d\omega_2}; \lambda_1, \lambda_2\right), \end{gathered} \tag{3}$$

where U_0, U_1, U_2 are some smooth functions of the corresponding arguments, $\lambda_1, \lambda_2 \subset \mathbb{R}^1$ are arbitrary parameters (separation constants) and what is more

$$\operatorname{rank} \left\| \frac{\partial U_\mu}{\partial \lambda_a} \right\|_{\mu=0}^{2}\,{}_{a=1}^{2} = 2 \tag{4}$$

(the last condition ensures essential dependence of the corresponding solution with separated variables on λ_1, λ_2, see [8]).

Definition 1. *We say that Eq. (1) admits SV in the system of coordinates* t, $\omega_1(t, \boldsymbol{x})$, $\omega_2(t, \boldsymbol{x})$ *if substitution of the Ansatz*

$$u = Q(t, \boldsymbol{x})\varphi_0(t)\varphi_1\big(\omega_1(t, \boldsymbol{x})\big)\varphi_2\big(\omega_2(t, \boldsymbol{x})\big) \tag{5}$$

into Eq. (1) with subsequent exclusion of the derivatives $d\varphi_0/dt$, $d^2\varphi_1/d\omega_1^2$, $d^2\varphi_2/d\omega_2^2$ *according to Eqs. (3) yields an identity with respect to* φ_0, φ_1, φ_2, $d\varphi_1/d\omega_1$, $d\varphi_2/d\omega_2$, λ_1, λ_2.

Thus, according to the above definition to separate variables in Eq. (1) one has

(i) to substitute the expression (5) into (1),

(ii) to exclude derivatives $d\varphi_0/dt$, $\frac{d^2\varphi_1}{d\omega_1^2}$, $d^2\varphi_2/d\omega_2^2$ with the help of Eqs. (3),

(iii) to split the obtained equality with respect to the variables φ_0, φ_1, φ_2, $d\varphi_1/d\omega_1$, $d\varphi_2/d\omega_2$, λ_1, λ_2 considered as independent.

As a result one gets some over-determined system of PDEs for the functions $Q(t, \boldsymbol{x})$, $\omega_1(t, \boldsymbol{x})$, $\omega_2(t, \boldsymbol{x})$. On solving it one obtains a complete description of all coordinate systems and potentials providing SV in the Schrödinger equation. Naturally, an expression *complete description* makes sense only within the framework of our definition. So if one uses a more general definition it may be possible to construct new coordinate systems and potentials providing separability of Eq. (1). But all solutions of the Schrödinger equation with separated variables known to us fit into the scheme suggested by us and can be obtained in the above described way.

2 Classification of potentials $V(x_1, x_2)$

We do not adduce in full detail computations needed because they are very cumbersome. We shall restrict ourselves to pointing out main steps of the realization of the above suggested algorithm.

First of all we make a remark, which makes life a little bit easier. It is readily seen that a substitution of the form

$$
\begin{aligned}
&Q \to Q' = Q\Psi_1(\omega_1)\Psi_2(\omega_2),\\
&\omega_a \to \omega'_a = \Omega_a(\omega_a), \quad a = 1, 2, \quad \lambda_a \to \lambda'_a = \Lambda_a(\lambda_1, \lambda_2), \quad a = 1, 2,
\end{aligned}
\tag{6}
$$

does not alter the structure of relations (3), (4), and (5). That is why, we can introduce the following equivalence relation:

$$
(\omega_1, \omega_2, Q) \;\sim\; (\omega'_1, \omega'_2, Q')
$$

provided Eq. (6) holds with some Ψ_a, Ω_a, Λ_a.

Substituting Eq. (5) into Eq. (1) and excluding the derivatives $d\varphi_0/dt$, $d^2\varphi_1/d\omega_1^2$, $d^2\varphi_2/d\omega_2^2$ with the use of equations (3) we get

$$
\begin{aligned}
&i(Q_t\varphi_0\varphi_1\varphi_2 + QU_0\varphi_1\varphi_2 + Q\omega_{1t}\varphi_0\dot\varphi_1\varphi_2 + Q\omega_{2t}\varphi_0\varphi_1\dot\varphi_2) + (\triangle Q)\varphi_0\varphi_1\varphi_2 +\\
&\qquad + 2Q_{x_a}\omega_{1x_a}\varphi_0\dot\varphi_1\varphi_2 + 2Q_{x_a}\omega_{2x_a}\varphi_0\varphi_1\dot\varphi_2 + Q\big((\triangle\omega_1)\varphi_0\dot\varphi_1\varphi_2 +\\
&\qquad + (\triangle\omega_2)\varphi_0\varphi_1\dot\varphi_2 + \omega_{1x_a}\omega_{1x_a}\varphi_0U_1\varphi_2 + \omega_{2x_a}\omega_{2x_a}\varphi_0\varphi_1U_2 +\\
&\qquad + 2\omega_{1x_a}\omega_{2x_a}\varphi_0\dot\varphi_1\dot\varphi_2\big) = VQ\varphi_0\varphi_1\varphi_2,
\end{aligned}
$$

where the summation over the repeated index a from 1 to 2 is understood. Hereafter an overdot means differentiation with respect to a corresponding argument and $\triangle = \partial^2_{x_1} + \partial^2_{x_2}$.

Splitting the equality obtained with respect to independent variables φ_1, φ_2, $d\varphi_1/d\omega_1$, $d\varphi_2/d\omega_2$, λ_1, λ_2 we conclude that ODEs (3) are linear and up to the equivalence relation (6) can be written in the form

$$
\begin{aligned}
&i\frac{d\varphi_0}{dt} = \big(\lambda_1R_1(t) + \lambda_2R_2(t) + R_0(t)\big)\varphi_0,\\
&\frac{d^2\varphi_1}{d\omega_1^2} = \big(\lambda_1B_{11}(\omega_1) + \lambda_2B_{12}(\omega_1) + B_{01}(\omega_1)\big)\varphi_1,\\
&\frac{d^2\varphi_2}{d\omega_2^2} = \big(\lambda_1B_{21}(\omega_2) + \lambda_2B_{22}(\omega_2) + B_{02}(\omega_2)\big)\varphi_2
\end{aligned}
$$

and what is more, functions ω_1, ω_2, Q satisfy an over-determined system of nonlinear

PDEs

$$
\begin{aligned}
&(1)\quad \omega_{1x_b}\omega_{2x_b}=0,\\
&(2)\quad B_{1a}(\omega_1)\omega_{1x_b}\omega_{1x_b}+B_{2a}(\omega_2)\omega_{2x_b}\omega_{2x_b}+R_a(t)=0,\quad a=1,2,\\
&(3)\quad 2\omega_{ax_b}Q_{x_b}+Q(i\omega_{at}+\triangle\omega_a),\quad a=1,2,\\
&(4)\quad \big(B_{01}(\omega_1)\omega_{1x_b}\omega_{1x_b}+B_{02}(\omega_1)\omega_{2x_b}\omega_{2x_b}\big)Q+iQ_t+\triangle Q+R_0(t)Q-\\
&\qquad -V(x_1,x_2)Q=0.
\end{aligned}
\tag{7}
$$

Thus, to solve the problem of SV for the linear Schrödinger equation it is necessary to construct general solution of system of nonlinear PDEs (7). Roughly speaking, to solve a linear equation one has to solve a system of *nonlinear equations*! This is the reason why so far there is no complete description of all coordinate systems providing separability of the four-dimensional wave equation [3].

But in the case involved we have succeeded in integrating system of nonlinear PDEs (7). Our approach to integration of it is based on the following change of variables (hodograph transformation)

$$z_0=t,\quad z_1=Z_1(t,\omega_1,\omega_2),\quad z_2=Z_2(t,\omega_1,\omega_2),\quad v_1=x_1,\quad v_2=x_2,$$

where z_0, z_1, z_2 are new independent and v_1, v_2 are new dependent variables correspondingly.

Using the hodograph transformation determined above we have constructed the general solution of Eqs. (1)–(3) from Eq. (7). It is given up to the equivalence relation (6) by one of the following formulas:

$$
\begin{aligned}
&(1)\quad \omega_1=A(t)x_1+W_1(t),\quad \omega_2=B(t)x_2+W_2(t),\\
&\qquad Q(t,\boldsymbol{x})=\exp\left\{-\frac{i}{4}\left(\frac{\dot A}{A}x_1^2+\frac{\dot B}{B}x_2^2\right)-\frac{i}{2}\left(\frac{\dot W_1}{A}x_1+\frac{\dot W_2}{B}x_2\right)\right\};\\
&(2)\quad \omega_1=\frac{1}{2}\ln(x_1^2+x_2^2)+W(t),\quad \omega_2=\arctan\frac{x_1}{x_2},\\
&\qquad Q(t,\boldsymbol{x})=\exp\left\{-\frac{i\dot W}{4}(x_1^2+x_2^2)\right\};\\
&(3)\quad x_1=\frac{1}{2}W(t)(\omega_1^2-\omega_2^2)+W_1(t),\quad x_2=W(t)\omega_1\omega_2+W_2(t),\\
&\qquad Q(t,\boldsymbol{x})=\exp\left\{\frac{i\dot W}{4W}\big((x_1-W_1)^2+(x_2-W_2)^2\big)+\frac{i}{2}(\dot W_1x_1+\dot W_2x_2)\right\};\\
&(4)\quad x_1=W(t)\cosh\omega_1\cos\omega_2+W_1(t),\quad x_2=W(t)\sinh\omega_1\sin\omega_2+W_2(t),\\
&\qquad Q(t,\boldsymbol{x})=\exp\left\{\frac{i\dot W}{4W}\big((x_1-W_1)^2+(x_2-W_2)^2\big)+\frac{i}{2}(\dot W_1x_1+\dot W_2x_2)\right\};
\end{aligned}
\tag{8}
$$

Here A, B, W, W_1, W_2 are arbitrary smooth functions on t.

Substituting the obtained expressions for the functions Q, ω_1, ω_2 into the last equation from the system (7) and splitting with respect to variables x_1, x_2 we get explicit forms of potentials $V(x_1,x_2)$ and systems of nonlinear ODEs for unknown functions $A(t)$, $B(t)$, $W(t)$, $W_1(t)$, $W_2(t)$. We have succeeded in integrating these and in constructing all coordinate systems providing SV in the initial equation (1).

Here we consider in detail integration of the fourth equation of system (7) for the case 2 from Eq. (8), since computations needed are not so lengthy as for other cases.

First, we make several important remarks which introduce an equivalence relation on the set of potentials $V(x_1, x_2)$.

Remark 1. The Schrödinger equation with the potential

$$V(x_1, x_2) = k_1x_1 + k_2x_2 + k_3 + V_1(k_2x_1 - k_1x_2), \tag{9}$$

where k_1, k_2, k_3 are constants, is transformed to the Schrödinger equation with the potential

$$V'(x_1', x_2') = V_1(k_2x_1' - k_1x_2') \tag{10}$$

by the following change of variables:

$$\begin{aligned} &t' = t, \quad \boldsymbol{x}' = \boldsymbol{x} + t^2\boldsymbol{k}, \\ &u' = u\exp\left\{\frac{i}{3}(k_1^2 + k_2^2)t^3 + it(k_1x_1 + k_2x_2) + ik_3t\right\}. \end{aligned} \tag{11}$$

It is readily seen that the class of Ansätze (5) is transformed into itself by the above change of variables. That is why, potentials (9) and (10) are considered as equivalent.

Remark 2. The Schrödinger equation with the potential

$$V(x_1, x_2) = k(x_1^2 + x_2^2) + V_1\left(\frac{x_1}{x_2}\right)(x_1^2 + x_2^2)^{-1} \tag{12}$$

with $k = \mathrm{const}$ is reduced to the Schrödinger equation with the potential

$$V'(x_1, x_2) = V_1\left(\frac{x_1'}{x_2'}\right)(x_1'^2 + x_1'^2)^{-1} \tag{13}$$

by the change of variables

$$t' = \alpha(t), \quad \boldsymbol{x}' = \beta(t)\boldsymbol{x}, \quad u' = u\exp\bigl\{i\gamma(t)(x_1^2 + x_2^2) + \delta(t)\bigr\},$$

where $\bigl(\alpha(t), \beta(t), \gamma(t), \delta(t)\bigr)$ is an arbitrary solution of the system of ODEs

$$\dot{\gamma} - 4\gamma^2 = k, \quad \dot{\beta} - 4\gamma\beta = 0, \quad \dot{\alpha} - \beta^2 = 0, \quad \dot{\delta} + 4\gamma = 0$$

such that $\beta \neq 0$.

Since the above change of variables does not alter the structure of the Ansatz (5), when classifying potentials $V(x_1, x_2)$ providing separability of the corresponding Schrödinger equation, we consider potentials (12), (13) as equivalent.

Remark 3. It is well-known (see e.g. [11, 12]) that the general form of the invariance group admitted by Eq. (1) is as follows

$$t' = F(t, \boldsymbol{\theta}), \quad x_a' = g_a(t, \boldsymbol{x}, \boldsymbol{\theta}), \quad a = 1, 2, \quad u' = h(t, \boldsymbol{x}, \boldsymbol{\theta})u + U(t, \boldsymbol{x}),$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ are group parameters and $U(t, \boldsymbol{x})$ is an arbitrary solution of Eq. (1).

The above transformations also do not alter the structure of the Ansatz (5). That is why, systems of coordinates t', x_1', x_2' and t, x_1, x_2 are considered as equivalent.

Now we turn to the integration of the fourth equation of system (7). Substituting into it the expressions for the functions ω_1, ω_2, Q given by formulas (2) from Eq. (8) we get

$$V(x_1,x_2) = \big(B_{01}(\omega_1) + B_{02}(\omega_2)\big)\exp\{-2(\omega_1 - W)\} + \frac{1}{4}(\ddot{W} - \dot{W}^2) \times \\ \times \exp\{2(\omega_1 - W)\} + R_0(t) - i\dot{W}. \tag{14}$$

In the above equality B_{01}, B_{02}, $R_0(t)$, $W(t)$ are unknown functions to be determined from the requirement that the right-hand side of (14) does not depend on t.

Differentiating Eq. (14) with respect to t and taking into account the equalities

$$\omega_{1t} = \dot{W}, \quad \omega_{2t} = 0$$

we have

$$\dot{W}\exp\{-2(\omega_1 - W)\}\dot{B}_{01} + \dot{\alpha}(t)\exp\{2(\omega_1 - W)\} + \dot{\beta}(t) = 0, \tag{15}$$

where $\alpha(t) = \frac{1}{4}(\ddot{W} - \dot{W}^2)$, $\beta(t) = R_0 - i\dot{W}$.

Cases $\dot{W} = 0$ and $\dot{W} \neq 0$ have to be considered separately.

Case 1. $\dot{W} = 0$. In this case $W = C = \mathrm{const}$, $R_0 = 0$. Since coordinate systems ω_1, ω_2 and $\omega_1 + C_1$, $\omega_2 + C_2$ are equivalent with arbitrary constants C_1, C_2, choosing $C_1 = -C$, $C_2 = 0$ we can put $C = 0$. Hence it immediately follows that

$$V(x_1,x_2) = \left[B_{01}\left(\frac{1}{2}\ln(x_1^2 + x_2^2)\right) + B_{02}\left(\arctan\frac{x_1}{x_2}\right)\right](x_1^2 + x_2^2)^{-1},$$

where B_{01}, B_{02} are arbitrary functions. And what is more, the Schrödinger equation (1) with such potential separates only in one coordinate system

$$\omega_1 = \frac{1}{2}\ln(x_1^2 + x_2^2), \quad \omega_2 = \arctan\frac{x_1}{x_2}. \tag{16}$$

Case 2. $\dot{W} \neq 0$. Dividing Eq. (14) into $\dot{W}\exp\{-2(\omega_1 - W)\}$ and differentiating the equality obtained with respect to t we get

$$\exp\{4\omega_1\}\frac{d}{dt}\big(\dot{\alpha}(\dot{W})^{-1}\exp\{-4W\}\big) + \exp\{2\omega_1\}\frac{d}{dt}\big(\dot{\beta}(\dot{W})^{-1}\exp\{-2W\}\big) = 0,$$

whence

$$\frac{d}{dt}\big(\dot{\alpha}(\dot{W})^{-1}\exp\{-4W\}\big) = 0, \quad \frac{d}{dt}\big(\dot{\beta}(\dot{W})^{-1}\exp\{-2W\}\big) = 0.$$

Integration of the above ODEs yields the following result:

$$\alpha = C_1\exp\{4W\} + C_2, \quad \beta = C_3\exp\{2W\} + C_4,$$

where C_j, $j = \overline{1,4}$ are arbitrary real constants.

Inserting the result obtained into Eq. (15) we get an equation for B_{01}

$$\dot{B}_{01} = -4C_1\exp\{4\omega_1\} - 2C_3\exp\{2\omega_1\},$$

which general solution reads

$$B_{01} = -C_1 \exp\{4\omega_1\} - C_3 \exp\{2\omega_1\} + C_5.$$

In the above equality C_5 is an arbitrary real constant.

Substituting the expressions for α, β, B_{01} into Eq. (14) we have the explicit form of the potential $V(x_1, x_2)$

$$V(x_1, x_2) = \left[B_{02}\left(\arctan\frac{x_1}{x_2}\right) + C_5\right](x_1^2 + x_2^2)^{-1} + C_2(x_1^2 + x_2^2) + C_4,$$

where B_{02} is an arbitrary function.

By force of the Remarks 1, 2 we can choose $C_2 = C_4 = 0$. Furthermore, due to arbitrariness of the function B_{02} we can put $C_5 = 0$.

Thus, the case $\dot{W} \neq 0$ leads to the following potential:

$$V(x_1, x_2) = B_{02}\left(\arctan\frac{x_1}{x_2}\right)(x_1^2 + x_2^2)^{-1}. \tag{17}$$

Substitution of the above expression into Eq. (14) yields second-order nonlinear ODE for the function $W = W(t)$

$$\ddot{W} - \dot{W}^2 = 4C_1 \exp\{4W\}, \tag{18}$$

while the function R_0 is given by the formula

$$R_0 = i\dot{W} + C_3 \exp\{2W\}.$$

Integration of ODE (18) is considered in detail in the Appendix A. Its general solution has the form
under $C_1 \neq 0$

$$W = -\frac{1}{2}\ln\big((at - b)^2 - 4C_1\big) + \frac{1}{2}\ln a,$$

under $C_1 = 0$

$$W = a - \ln(t + b).$$

Substituting obtained expressions for W into formulas (2) from (8) and taking into account the Remark 3 we arrive at the conclusion that the Schrödinger equation (1) with the potential (17) admits SV in two coordinate systems. One of them is the polar coordinate system (16) and another one is the following:

$$\omega_1 = \frac{1}{2}\ln(x_1^2 + x_2^2) - \frac{1}{2}\ln(t^2 \pm 1), \quad \omega_2 = \arctan\frac{x_1}{x_2}. \tag{19}$$

Consequently, the case 2 from Eq. (8) gives rise to two classes of the separable Schrödinger equations (1).

Cases 1, 3, 4 from Eq. (8) are considered in an analogous way but computations involved are much more cumbersome. As a result, we obtain the following list of inequivalent potentials $V(x_1, x_2)$ providing separability of the Schrödinger equation.

(1) $V(x_1,x_2) = V_1(x_1) + V_2(x_2)$;
(a) $V(x_1,x_2) = k_1x_1^2 + k_2x_1^{-2} + V_2(x_2)$, $k_2 \neq 0$;

(i) $V(x_1,x_2) = k_1x_1^2 + k_2x_2^2 + k_3x_1^{-2} + k_4x_2^{-2}, \quad k_3k_4 \neq 0,$
$k_1^2 + k_2^2 \neq 0,\ k_1 \neq k_2$;
(ii) $V(x_1,x_2) = k_1x_1^2 + k_2x_1^{-2}, \quad k_1k_2 \neq 0$;
(iii) $V(x_1,x_2) = k_1x_1^{-2} + k_2x_2^{-2}$;

(b) $V(x_1,x_2) = k_1x_1^2 + V_2(x_2)$;

(i) $V(x_1,x_2) = k_1x_1^2 + k_2x_2^2 + k_3x_2^{-2}, \quad k_1k_3 \neq 0,\ k_1 \neq k_2$;
(ii) $V(x_1,x_2) = k_1x_1^2 + k_2x_2^2, \quad k_1k_2 \neq 0, \quad k_1 \neq k_2$;
(iii) $V(x_1,x_2) = k_1x_1^2 + k_2x_2^{-2}, \quad k_1 \neq 0$;

(2) $V(x_1,x_2) = V_1(x_1^2+x_2^2) + V_2(x_1/x_2)(x_1^2+x_2^2)^{-1}$;
(a) $V(x_1,x_2) = V_2(x_1/x_2)(x_1^2+x_2^2)^{-1}$;
(b) $V(x_1,x_2) = k_1(x_1^2+x_2^2)^{-1/2}$, $k_1 \neq 0$;
(3) $V(x_1,x_2) = (V_1(\omega_1)+V_2(\omega_2))(\omega_1^2+\omega_2^2)^{-1}$, where $\omega_1^2 - \omega_2^2 = 2x_1$, $\omega_1\omega_2 = x_2$;
(4) $V(x_1,x_2) = (V_1(\omega_1)+V_2(\omega_2))(\sinh^2\omega_1 + \sin^2\omega_2)^{-1}$, where $\cosh\omega_1\cos\omega_2 = x_1$, $\sinh\omega_1\sin\omega_2 = x_2$;
(5) $V(x_1,x_2) = 0$.

In the above formulas V_1, V_2 are arbitrary smooth functions, k_1, k_2, k_3, k_4 are arbitrary constants.

It should be emphasized that the above potentials are not inequivalent in a usual sense. These potentials differ from each other by the fact that the coordinate systems providing separability of the corresponding Schrödinger equations are different. As an illustration, we give the Fig. 1, where $r = (x_1^2+x_2^2)^{1/2}$ and by the symbol $V^{(j)}$, $j = 1,4$ we denote the potential given in the above list under the number j. Down arrows in the Fig. 1 indicate specifications of the potential $V(x_1,x_2)$ providing new possibilities to separate the corresponding Schrödinger equation (1).

The Schrödinger equation (1) with arbitrary function $V(x_1,x_2)$ (level 1 of the Fig. 1) admits no separation of variables. Next, Eq. (1) with the "root" potentials $V^{(j)}$ (level 2), V_1, V_2 being arbitrary smooth functions, separates in the Cartesian ($j=1$), polar ($j=2$), parabolic ($j=3$) and elliptic ($j=4$) coordinate systems, correspondingly. Specifying the functions V_1, V_2 (i.e. going down to the lower levels) new possibilities to separate variables in the Schrödinger equation (1) arise. For example, Eq. (1) with the potential $V_2(x_1/x_2)r^{-2}$, which is a particular case of the potential $V^{(2)}$, separates not only in the polar coordinate system (16) but also in the coordinate systems (19). The Schrödinger equation with the Coulomb potential k_1r^{-1}, which is a particular case of the potentials $V^{(2)}$, $V^{(3)}$, separates in two coordinate systems (namely, in the polar and parabolic coordinate systems, see below the Theorem 4). An another characteristic example is a transition from the potential $V^{(1)}$ to the potential $k_1x_1^2 + V_2(x_2)$. The Schrödinger equation with the potential $V^{(1)}$ admits SV in the Cartesian coordinate system $\omega_0 = t$, $\omega_1 = x_1$, $\omega_2 = x_2$ only, while the one with the potential $k_1x_1^2 + V_2(x_2)$ separates in seven ($k_1 < 0$) or in three ($k_1 > 0$) coordinate systems.

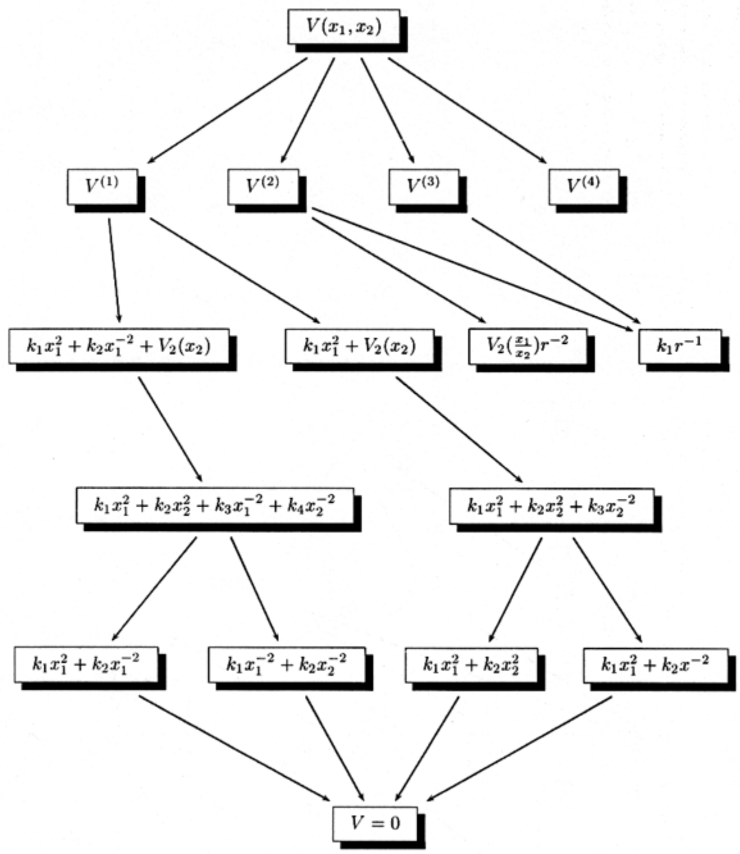

Figure 1.

A complete list of coordinate systems providing SV in the Schrödinger equations with the above given potentials takes two dozen pages. Therefore, we restrict ourself to considering the Schrödinger equation with anisotropic harmonic oscillator potential $V(x_1, x_2) = k_1x_1^2 + k_2x_2^2$, $k_1 \neq k_2$ and Coulomb potential $V(x_1, x_2) = k_1(x_1^2 + x_2^2)^{-1/2}$.

3 Separation of variables in the Schrödinger equation with the anisotropic harmonic oscillator and the Coulomb potentials

Here we will obtain all coordinate systems providing separability of the Schrödinger equation with the potential $V(x_1, x_2) = k_1x_1^2 + k_2x_2^2$

$$iu_t + u_{x_1x_1} + u_{x_2x_2} = (k_1x_1^2 + k_2x_2^2)u. \tag{20}$$

In the following, we consider the case $k_1 \neq k_2$, because otherwise Eq. (1) is reduced to the free Schrödinger equation (see the Remark 2) which has been studied in detail in [1–3].

Explicit forms of the coordinate systems to be found depend essentially on the signs of the parameters k_1, k_2. We consider in detail the case, when $k_1 < 0$, $k_2 > 0$ (the cases $k_1 > 0$, $k_2 > 0$ and $k_1 < 0$, $k_2 < 0$ are handled in an analogous way). It means that Eq. (20) can be written in the form

$$iu_t + u_{x_1x_1} + u_{x_2x_2} + \frac{1}{4}(a^2x_1^2 - b^2x_2^2)u = 0, \tag{21}$$

where a, b are arbitrary non-null real constants (the factor $\frac{1}{4}$ is introduced for further convenience).

As stated above to describe all coordinate systems t, $\omega_1(t,\boldsymbol{x})$, $\omega_2(t,\boldsymbol{x})$ providing separability of Eq. (20) one has to construct the general solution of system (8) with $V(x_1,x_2) = -\frac{1}{4}(a^2x_1^2 - b^2x_2^2)$. The general solution of Eqs. (1)–(3) from Eq. (7) splits into four inequivalent classes listed in Eq. (8). Analysis shows that only solutions belonging to the first class can satisfy the fourth equation of (7).

Substituting the expressions for ω_1, ω_2, Q given by the formulas (1) from (8) into the equation 4 from (7) with $V(x_1,x_2) = -\frac{1}{4}(a^2x_1^2 - b^2x_2^2)$ and splitting with respect to x_1, x_2 one gets

$$B_{01}(\omega_1) = \alpha_1\omega_1^2 + \alpha_2\omega_1, \quad B_{02}(\omega_2) = \beta_1\omega_2^2 + \beta_2\omega_2,$$

$$\left(\frac{\dot{A}}{A}\right)^{\cdot} - \left(\frac{\dot{A}}{A}\right)^2 - 4\alpha_1A^4 + a^2 = 0, \tag{22}$$

$$\left(\frac{\dot{B}}{B}\right)^{\cdot} - \left(\frac{\dot{B}}{B}\right)^2 - 4\beta_1B^4 - b^2 = 0, \tag{23}$$

$$\ddot{\theta}_1 - 2\dot{\theta}_1\frac{\dot{A}}{A} - 2(2\alpha_1\theta_1 + \alpha_2)A^4 = 0, \tag{24}$$

$$\ddot{\theta}_2 - 2\dot{\theta}_2\frac{\dot{B}}{B} - 2(2\beta_1\theta_2 + \beta_2)B^4 = 0. \tag{25}$$

Here α_1, α_2, β_1, β_2 are arbitrary real constants.

Integration of the system of nonlinear ODEs (22)–(25) is carried out in the Appendix A. Substitution of the formulas (A.2), (A.4)–(A.6), (A.8)–(A.11) into the corresponding expressions 1 from (8) yields a complete list of coordinate systems providing separability of the Schrödinger equation (21). These systems can be transformed to canonical form if we use the Remark 3.

The invariance group of Eq. (21) is generated by the following basis operators [11]:

$$\begin{aligned}
&P_0 = \partial_t, \quad I = u\partial_u, \quad M = iu\partial_u, \quad Q_\infty = U(t,\boldsymbol{x})\partial_u,\\
&P_1 = \cosh at\partial_{x_1} + \frac{ia}{2}(x_1\sinh at)u\partial_u,\\
&P_2 = \cos bt\partial_{x_2} - \frac{ib}{2}(x_2\sin bt)u\partial_u,\\
&G_1 = \sinh at\partial_{x_1} + \frac{ia}{2}(x_1\cosh at)u\partial_u,\\
&G_2 = \sin bt\partial_{x_2} + \frac{ib}{2}(x_2\cos bt)u\partial_u,
\end{aligned} \tag{26}$$

where $U(t,\boldsymbol{x})$ is an arbitrary solution of Eq. (21).

Using the finite transformations generated by the infinitesimal operators (26) and the Remark 3 we can choose in the formulas (A.4)–(A.6), (A.8), (A.10), (A.11) $C_3 = C_4 = D_1 = 0$, $D_3 = D_4 = 0$, $C_2 = D_2 = 1$. As a result, we come to the following assertion.

Theorem 1. *The Schrödinger equation (21) admits SV in 21 inequivalent coordinate systems of the form*

$$\omega_0 = t, \quad \omega_1 = \omega_1(t, \boldsymbol{x}), \quad \omega_2 = \omega_2(t, \boldsymbol{x}), \tag{27}$$

where ω_1 is given by one of the formulas from the first and ω_2 by one of the formulas from the second column of the Table 1.

Table 1. Coordinate systems proving SV in Eq. (21).

$\omega_1(t, \boldsymbol{x})$	$\omega_2(t, \boldsymbol{x})$
$x_1\big(\sinh a(t+C)\big)^{-1} + \alpha\big(\sinh a(t+C)\big)^{-2}$	$x_2(\sin bt)^{-1} + \beta(\sin bt)^{-2}$
$x_1\big(\cosh a(t+C)\big)^{-1} + \alpha\big(\cosh a(t+C)\big)^{-2}$	$x_2(\beta + \sin 2bt)^{-1/2}$
$x_1 \exp(\pm at) + \alpha \exp(\pm 4at)$	x_2
$x_1\big(\alpha + \sinh 2a(t+C)\big)^{-1/2}$	
$x_1\big(\alpha + \cosh 2a(t+C)\big)^{-1/2}$	
$x_1\big(\alpha + \exp(\pm 2at)\big)^{-1/2}$	
x_1	

Here C, α, β are arbitrary real constants.

There is no necessity to consider specially the case when in Eq. (20) $k_1 > 0$, $k_2 < 0$, since such an equation by the change of independent variables $u(t, x_1, x_2) \to u(t, x_2, x_1)$ is reduced to Eq. (21).

Below we adduce without proof the assertions describing coordinate systems providing SV in Eq. (20) with $k_1 < 0$, $k_2 < 0$ and $k_1 > 0$, $k_2 > 0$.

Theorem 2. *The Schrödinger equation*

$$iu_t + u_{x_1x_1} + u_{x_2x_2} + \frac{1}{4}(a^2x_1^2 + b^2x_2^2)u = 0 \tag{28}$$

with $a^2 \neq 4b^2$ admits SV in 49 inequivalent coordinate systems of the form (27), where ω_1 is given by one of the formulas from the first and ω_2 by one of the formulas from the second column of the Table 2. Provided $a^2 = 4b^2$ one more coordinate system should be included into the above list, namely

$$\omega_0 = t, \quad \omega_1^2 - \omega_2^2 = 2x_1, \quad \omega_1\omega_2 = x_2. \tag{29}$$

Table 2. Coordinate systems proving SV in Eq. (28).

$\omega_1(t,\boldsymbol{x})$	$\omega_2(t,\boldsymbol{x})$
$x_1\bigl(\sinh a(t+C)\bigr)^{-1}+\alpha\bigl(\sinh a(t+C)\bigr)^{-2}$	$x_2(\sinh bt)^{-1}+\beta(\sinh bt)^{-2}$
$x_1\bigl(\cosh a(t+C)\bigr)^{-1}+\alpha\bigl(\cosh a(t+C)\bigr)^{-2}$	$x_2(\cosh bt)^{-1}+\beta(\cosh bt)^{-2}$
$x_1\exp(\pm at)+\alpha\exp(\pm 4at)$	$x_2\exp(\pm bt)+\beta\exp(\pm 4bt)$
$x_1\bigl(\alpha+\sinh 2a(t+C)\bigr)^{-1/2}$	$x_2(\beta+\sinh 2bt)^{-1/2}$
$x_1\bigl(\alpha+\cosh 2a(t+C)\bigr)^{-1/2}$	$x_2(\beta+\cosh 2bt)^{-1/2}$
$x_1\bigl(\alpha+\exp(\pm 2at)\bigr)^{-1/2}$	$x_2\bigl(\beta+\exp(\pm 2bt)\bigr)^{-1/2}$
x_1	x_2

Here C, α, β are arbitrary constants.

Table 3. Coordinate systems proving SV in Eq. (30).

$\omega_1(t,\boldsymbol{x})$	$\omega_2(t,\boldsymbol{x})$
$x_1\bigl(\sin a(t+C)\bigr)^{-1}+\alpha\bigl(\sin a(t+C)\bigr)^{-2}$	$x_2(\sin bt)^{-1}+\beta(\sin bt)^{-2}$
$x_1\bigl(\beta+\sin 2a(t+C)\bigr)^{-1/2}$	$x_2(\beta+\sin 2bt)^{-1/2}$
x_1	x_2

Here C, α, β are arbitrary constants.

Theorem 3. *The Schrödinger equation*

$$iu_t+u_{x_1x_1}+u_{x_2x_2}-\frac{1}{4}(a^2x_1^2+b^2x_2^2)u=0 \tag{30}$$

with $a^2\neq 4b^2$ admits SV in 9 inequivalent coordinate systems of the form (27), where ω_1 is given by one of the formulas from the first and ω_2 by one of the formulas from the second column of the Table 3. Provided $a^2=4b^2$, the above list should be supplemented by the coordinate system (29).

Remark 4. If we consider Eq. (1) as an equation for a complex-valued function u of three complex variables t, x_1, x_2, then the cases considered in the Theorems 1–3 are equivalent. Really, replacing, when necessary, a with ia and b by ib we can always reduce Eqs. (21), (28) to the form (30). It means that coordinate systems presented in the Tables 1, 2 are complex equivalent to those listed in the Table 3. But if u is a complex-valued function of real variables t, x_1, x_2 it is not the case.

Theorem 4. *The Schrödinger equation with the Coulomb potential*

$$iu_t+u_{x_1x_1}+u_{x_2x_2}-k_1(x_1^2+x_2^2)^{-1/2}u=0$$

admits SV in two coordinate systems (16), (29).

It is important to note that explicit forms of coordinate systems providing separability of Eqs. (21), (28), (30) depend essentially on the parameters a, b contained in the potential $V(x_1,x_2)$. It means that the free Schrödinger equation ($V=0$) does not admit SV in such coordinate systems. Consequently, they are essentially new.

4 Conclusion

In the present paper we have studied the case when the Schrödinger equation (1) separates into one first-order and two second-order ODEs. It is not difficult to prove that there are no functions $Q(t,\boldsymbol{x})$, $\omega_\mu(t,\boldsymbol{x})$, $\mu = 0,1,2$ such that the Ansatz

$$u = Q(t,\boldsymbol{x})\varphi_0(\omega_0(t,\boldsymbol{x}))\varphi_1\big(\omega_1(t,\boldsymbol{x})\big)\varphi_2\big(\omega_2(t,\boldsymbol{x})\big)$$

separates Eq. (1) into three second-order *ODEs* (see Appendix B). Nevertheless, there exists a possibility for Eq. (1) to be separated into two first-order and one second-order ODEs or into three first-order ODEs. This is a probable source of new potentials and new coordinate systems providing separability of the Schrödinger equation. It should be said that separation of the two-dimensional wave equation

$$u_{tt} - u_{xx} = V(x)u$$

into one first-order and one second-order ODEs gives no new potentials as compared with separation of it into two second-order ODEs. But for some already known potentials new coordinate system providing separability of the above equation are obtained [9].

Let us briefly analyze a connection between separability of Eq. (1) and its symmetry properties. It is well-known that each solution of the free Schrödinger equation with separated variables is a common eigenfunction of two mutually commuting second-order symmetry operators of the said equation [2, 3]. And what is more, separation constants λ_1, λ_2 are eigenvalues of these symmetry operators.

We will establish that the same assertion holds for the Schrödinger equation (1). Let us make in Eq. (1) the following change of variables:

$$u = Q(t,\boldsymbol{x})U\big(t,\omega_1(t,\boldsymbol{x}),\omega_2(t,\boldsymbol{x})\big), \tag{31}$$

where (Q,ω_1,ω_2) is an arbitrary solution of the system of PDEs (7).

Substituting the expression (31) into (1) and taking into account equations (7) we get

$$Q\big(iU_t + \big(U_{\omega_1\omega_1} - B_{01}(\omega_1)U\big)\omega_{1x_a}\omega_{1x_a} + \big(U_{\omega_2\omega_2} - B_{02}(\omega_2)U\big)\omega_{2x_a}\omega_{2x_a}\big) = 0. \tag{32}$$

Resolving Eqs. (2) from the system (7) with respect to $\omega_{1x_a}\omega_{1x_a}$ and $\omega_{2x_a}\omega_{2x_a}$ we have

$$\begin{aligned}
\omega_{1x_a}\omega_{1x_a} &= \frac{1}{\delta}\big(R_2(t)B_{21}(\omega_2) - R_1(t)B_{22}(\omega_2)\big),\\
\omega_{2x_a}\omega_{2x_a} &= \frac{1}{\delta}\big(R_1(t)B_{12}(\omega_1) - R_2(t)B_{11}(\omega_1)\big),
\end{aligned}$$

where $\delta = B_{11}(\omega_1)B_{22}(\omega_2) - B_{12}(\omega_1)B_{21}(\omega_2)$ ($\delta \neq 0$ by force of the condition (4)).

Substitution of the above equalities into Eq. (32) with subsequent division by $Q \neq 0$ yields the following PDE:

$$\begin{aligned}
iU_t &+ \frac{R_1(t)}{\delta}\big(B_{12}(\omega_1)\big(U_{\omega_2\omega_2} - B_{02}(\omega_2)U\big) - B_{22}(\omega_2)\big(U_{\omega_1\omega_1} - B_{01}(\omega_1)U\big)\big) + \\
&+ \frac{R_2(t)}{\delta}\big(B_{21}(\omega_2)\big(U_{\omega_1\omega_1} - B_{01}(\omega_1)U\big) - B_{11}(\omega_1)\big(U_{\omega_2\omega_2} - B_{02}(\omega_2)U\big)\big) = 0.
\end{aligned} \tag{33}$$

Thus, in the new coordinates t, ω_1, ω_2, $U(t,\omega_1,\omega_2)$ Eq. (1) takes the form (33).

By direct (and very cumbersome) computation one can check that the following second-order differential operators:

$$X_1 = \frac{B_{22}(\omega_2)}{\delta}\big(\partial^2_{\omega_1} - B_{01}(\omega_1)\big) - \frac{B_{12}(\omega_1)}{\delta}\big(\partial^2_{\omega_2} - B_{02}(\omega_2)\big),$$
$$X_2 = -\frac{B_{21}(\omega_2)}{\delta}\big(\partial^2_{\omega_1} - B_{01}(\omega_1)\big) + \frac{B_{11}(\omega_1)}{\delta}\big(\partial^2_{\omega_2} - B_{02}(\omega_2)\big),$$

commute under arbitrary B_{0a}, B_{ab}, $a,b=1,2$, i.e.

$$[X_1, X_2] \equiv X_1X_2 - X_2X_1 = 0. \tag{34}$$

After being rewritten in terms of the operators X_1, X_2 Eq. (33) reads

$$\big(i\partial_t - R_1(t)X_1 - R_2(t)X_2\big)U = 0.$$

Since the relations

$$\big[i\partial_t - R_1(t)X_1 - R_2(t)X_2, X_a\big] = 0, \quad a = 1,2 \tag{35}$$

hold, operators X_1, X_2 are mutually commuting symmetry operators of Eq. (33). Furthermore, solution of Eq. (33) with separated variables $U = \varphi_0(t)\varphi_1(\omega_1)\varphi_2(\omega_2)$ satisfies the identities

$$X_aU = \lambda_aU, \quad a = 1,2. \tag{36}$$

Consequently, if we designate by X_1', X_2' the operators X_1, X_2 written in the initial variables t, $\boldsymbol{x}$, u, then we get from (34)–(36) the following equalities:

$$\big[i\partial_t + \triangle - V(x_1,x_2), X_a'\big] = 0, \quad a = 1,2,$$
$$\big[X_1', X_2'\big] = 0, \quad X_a'u = \lambda_a u, \quad a = 1,2.$$

where $u = Q(t,\boldsymbol{x})\varphi_0(t)\varphi_1(\omega_1)\varphi_2(\omega_2)$.

It means that each solution with separated variables is a common eigenfunction of two mutually commuting symmetry operators X_1', X_2' of the Schrödinger equation (1), separation constants λ_1, λ_2 being their eigenvalues.

Detailed study of the said operators as well as analysis of separated ODEs for functions φ_μ, $\mu = \overline{0,2}$ (in the way as it is done for the free Schrödinger equation in [2, 3]) is in progress and will be a topic of our future publications.

Acknowledgments. When the paper was in the last stage of preparation, one of the authors (R. Zhdanov) was supported by the Alexander von Humboldt Foundation. Taking an opportunity he wants to express his gratitude to Director of Arnold Sommerfeld Institute for Mathematical Physics Professor H.-D. Doebner for hospitality.

Appendix A. Integration of nonlinear ODEs (22)–(25)

Evidently, equations (22)–(25) can be rewritten in the following unified form:

$$\left(\frac{\dot y}{y}\right)^{\cdot} - \left(\frac{\dot y}{y}\right)^2 - 4\alpha y^4 = k, \quad \ddot z - 2\dot z\frac{\dot y}{y} - 2(2\alpha z + \beta)y^4 = 0. \tag{A1}$$

Provided $k = -a^2 < 0$, system (A.1) coincides with Eqs. (22), (24) and under $k = b^2 > 0$ – with Eqs. (23), (25).

First of all, we note that the function $z = z(t)$ is determined up to addition of an arbitrary constant. Really, the coordinate functions ω_a have the following structure:

$$\omega_a = yx_a + z, \quad a = 1, 2.$$

But the coordinate system t, ω_1, ω_2 is equivalent to the coordinate system t, $\omega_1 + C_1$, $\omega_2 + C_2$, $C_a \in \mathbb{R}^1$. Hence it follows that the function $z(t)$ is equivalent to the function $z(t) + C$ with arbitrary real constant C. Consequently, provided $\alpha \neq 0$, we can choose in (A.1) $\beta = 0$.

The case 1. $\alpha = 0$. On making in (A.1) the change of variables

$$w = \dot{y}/y, \quad v = z/y \tag{A2}$$

we get

$$\dot{w} = w^2 + k, \quad \ddot{v} + kv = 2\beta y^3. \tag{A3}$$

First, we consider the case $k = -a^2 < 0$. Then the general solution of the first equation from (A.3) is given by one of the formulas

$$w = -a \coth a(t + C_1), \quad w = -a \tanh a(t + C_1), \quad w = \pm a, \quad C_1 \in \mathbb{R}^1,$$

whence

$$\begin{aligned} &y = C_2 \sinh^{-1} a(t + C_1), \quad y = C_2 \cosh^{-1} a(t + C_1), \\ &y = C_2 \exp(\pm at), \quad C_2 \in \mathbb{R}^1. \end{aligned} \tag{A4}$$

The second equation of system (A.3) is a linear inhomogeneous ODE. Its general solution after being substituted into (A.2) yields the following expression for $z(t)$:

$$\begin{aligned} &(C_3 \cosh at + C_4 \sinh at) \sinh^{-1} a(t + C_1) + \frac{\beta C_2^4}{a^2} \sinh^{-2} a(t + C_1), \\ &(C_3 \cosh at + C_4 \sinh at) \cosh^{-1} a(t + C_1) + \frac{\beta C_2^4}{a^2} \cosh^{-2} a(t + C_1), \\ &(C_3 \cosh at + C_4 \sinh at) \exp(\pm at) + \frac{\beta C_2^4}{4a^2} \exp(\pm 4at), \quad C_3, C_4 \subset \mathbb{R}^1. \end{aligned} \tag{A5}$$

The case $k = b^2 > 0$ is treated in an analogous way, the general solution of (A.1) being given by the formulas

$$\begin{aligned} &y = D_2 \sin^{-1} b(t + D_1), \\ &z = (D_3 \cos bt + D_4 \sin bt) \sin^{-1} b(t + D_1) + \frac{\beta D_2^4}{b^2} \sin^{-2} b(t + D_1), \end{aligned} \tag{A6}$$

where D_1, D_2, D_3, D_4 are arbitrary real constants.

The case 2. $\alpha \neq 0$, $\beta = 0$. On making in Eq. (A.1) the change of variables

$$y = \exp w, \quad v = z/y$$

we have

$$\ddot{w} - \dot{w}^2 = k + \alpha \exp 4w, \quad \ddot{v} + kv = 0. \tag{A7}$$

The first ODE from Eq. (A.7) is reduced to the first-order linear ODE

$$\frac{1}{2}\frac{dp(w)}{dw} - p(w) = k + \alpha \exp 4w$$

by the substitution $\dot{w} = (p(w))^{1/2}$, whence

$$p(w) = \alpha \exp 4w + \gamma \exp 2w - k, \quad \gamma \in \mathbb{R}^1.$$

Equation $\dot{w} = (p(w))^{1/2}$ has a singular solution $w = C = \mathrm{const}$ such that $p(C) = 0$. If $\dot{w} \neq 0$, then integrating the equation $\dot{w} = p(w)$ and returning to the initial variable y we get

$$\int^{y(t)} \frac{d\tau}{\tau(\alpha\tau^4 + \gamma\tau^2 - k)^{1/2}} = t + C_1.$$

Taking the integral in the left-hand side of the above equality we obtain the general solution of the first ODE from Eq. (A.1). It is given by the following formulas:
under $k = -a^2 < 0$

$$\begin{aligned} y &= C_2\big(\alpha + \sinh 2a(t + C_1)\big)^{-1/2}, \\ y &= C_2\big(\alpha + \cosh 2a(t + C_1)\big)^{-1/2}, \\ y &= C_2\big(\alpha + \exp(\pm 2at)\big)^{-1/2}, \end{aligned} \tag{A8}$$

under $k = b^2 > 0$

$$y = D_2\big(\alpha + \sin 2b(t + D_1)\big)^{-1/2}. \tag{A9}$$

Here C_1, C_2, D_1, D_2 are arbitrary real constants.

Integrating the second ODE from Eq. (A.7) and returning to the initial variable z we have
under $k = -a^2 < 0$

$$z = y(t)(C_3 \cosh at + C_4 \sinh at) \tag{A10}$$

under $k = b^2 > 0$

$$z = y(t)(D_3 \cos bt + D_4 \sin bt), \tag{A11}$$

where C_3, C_4, D_3, D_4 are arbitrary real constants.

Thus, we have constructed the general solution of the system of nonlinear ODEs (A.1) which is given by the formulas (A.5)–(A.11).

Appendix B. Separation of Eq. (1) into three second-order ODEs

Suppose that there exists an Ansatz

$$u = Q(t, \boldsymbol{x})\varphi_0(\omega_0(t, \boldsymbol{x}))\varphi_1\big(\omega_1(t, \boldsymbol{x})\big)\varphi_2\big(\omega_2(t, \boldsymbol{x})\big) \tag{A12}$$

which separates the Schrödinger equation into three second-order ODEs

$$\begin{aligned}&\frac{d^2\varphi_0}{d\omega_0^2} = U_0\left(\omega_0, \varphi_0, \frac{d\varphi_0}{d\omega_0}; \lambda_1, \lambda_2\right), \quad \frac{d^2\varphi_1}{d\omega_1^2} = U_1\left(\omega_1, \varphi_1, \frac{d\varphi_1}{d\omega_1}; \lambda_1, \lambda_2\right),\\ &\frac{d^2\varphi_2}{d\omega_2^2} = U_2\left(\omega_2, \varphi_2, \frac{d\varphi_2}{d\omega_2}; \lambda_1, \lambda_2\right)\end{aligned} \tag{A13}$$

according to the Definition 1.

Substituting the Ansatz (A.12) into Eq. (1) and excluding the second derivatives $d^2\varphi_\mu/d\omega_\mu^2$, $\mu = \overline{0,2}$ according to Eqs. (A.13) we get

$$\begin{aligned}&i(Q_t\varphi_0\varphi_1\varphi_2 + Q\omega_{0t}\dot\varphi_0\varphi_1\varphi_2 + Q\omega_{1t}\varphi_0\dot\varphi_1\varphi_2 + Q\omega_{2t}\varphi_0\varphi_1\dot\varphi_2) + (\triangle Q)\varphi_0\varphi_1\varphi_2 +\\ &\quad + 2Q_{x_a}\omega_{0x_a}\dot\varphi_0\varphi_1\varphi_2 + 2Q_{x_a}\omega_{1x_a}\varphi_0\dot\varphi_1\varphi_2 + 2Q_{x_a}\omega_{2x_a}\varphi_0\varphi_1\dot\varphi_2 +\\ &\quad + Q\big((\triangle\omega_0)\dot\varphi_0\varphi_1\varphi_2 + (\triangle\omega_1)\varphi_0\dot\varphi_1\varphi_2 + (\triangle\omega_2)\varphi_0\varphi_1\dot\varphi_2 + \omega_{0x_a}\omega_{0x_a}U_0\varphi_1\varphi_2 +\\ &\quad + \omega_{1x_a}\omega_{1x_a}\varphi_0U_1\varphi_2 + \omega_{2x_a}\omega_{2x_a}\varphi_0\varphi_1U_2 + 2\omega_{0x_a}\omega_{1x_a}\dot\varphi_0\dot\varphi_1\varphi_2 +\\ &\quad + 2\omega_{0x_a}\omega_{2x_a}\dot\varphi_0\varphi_1\dot\varphi_2 + 2\omega_{1x_a}\omega_{2x_a}\varphi_0\dot\varphi_1\dot\varphi_2\big) = VQ\varphi_0\varphi_1\varphi_2.\end{aligned}$$

Splitting the above equality with respect to $\dot\varphi_0\dot\varphi_1$, $\dot\varphi_0\dot\varphi_2$, $\dot\varphi_1\dot\varphi_2$ we obtain the equalities:

$$\omega_{0x_a}\omega_{1x_a} = 0, \quad \omega_{0x_a}\omega_{2x_a} = 0, \quad \omega_{1x_a}\omega_{2x_a} = 0. \tag{A14}$$

Since the functions ω_μ, $\mu = \overline{0,2}$ are real-valued, equalities (A.14) mean that there are three real two-component vectors which are mutually orthogonal. This is possible only if one of them is a null-vector. Without loss of generality we may suppose that $(\omega_{0x_1}, \omega_{0x_2}) = (0,0)$, whence $\omega_0 = f(t) \sim t$.

Consequently, Ansatz (A.12) necessarily takes the form (5). But Ansatz (5) can not separate Eq. (1) into three second-order ODEs, since it contains no second-order derivative with respect to t.

Thus, we have proved that the Schrödinger equation (1) is not separable into three second-order ODEs.

1. Boyer C., *SIAM J. Math. Anal.*, 1976, **7**, 230.
2. Boyer C., Kalnins E., Miller W., *J. Math. Phys.*, 1975, **16**, 499.
3. Miller W., Symmetry and separation of variables, Massachusetts, Addison-Wesley, 1977.
4. Shapovalov V.N., Sukhomlin N.B., *Izvestiya Vuzov, Fizika*, 1974, № 12, 268.
5. Nikitin A.G., Onufriychuk S.P., Fushchych W.I., *Teoret. Mat. Fiz.*, 1992, **12**, 268.
6. Fushchych W.I., Zhdanov R.Z., Revenko I.V., *Proc. Ukrain. Acad. Sci.*, 1993, № 1, 27.
7. Fushchych W.I., in Symmetry in Mathematical Physics Problems, Kiev, Inst. of Math., 1981, 6.
8. Koornwinder T.H., *Lect. Notes in Math.*, 1980, **810**, 240.
9. Zhdanov R.Z., Revenko I.V., Fushchych W.I., *J. Phys. A: Math. Gen.*, 1993, **26**, 5959.
10. Zhdanov R.Z., *J. Phys. A: Math. Gen.*, 1994, **27**, L291.
11. Boyer C., *Helv. Phys. Acta*, 1974, **47**, 589.
12. Niederer U., *Helv. Phys. Acta*, 1972, **45**, 802.