Reduction of the self-dual Yang–Mills equations. I. The Poincaré group

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We have obtained a complete description of ansatzes for the vector-potential of the Yang–Mills field invariant under 3-parameter $P(1,3)$-inequivalent subgroups of the Poincaré group. Using these, we carry out a reduction of the self-dual Yang–Mills equations to system of ordinary differential equations.

Для вектор-потенціалу поля Янга–Міллса побудовано повний набір інваріантних відносно $P(1,3)$-нееквівалентних підгруп групи Пуанкаре анзаців, з використанням яких проведено редукцію самодуальних рівнянь Янга–Мілса до систем звичайних диференціальних рівнянь.

Classical $SU(2)$ Yang–Mills equations form a system of twelve nonlinear second-order partial differential equations (PDE) in the Minkowski space $\mathbb{R}(1,3)$. But one can obtain an important subclass of solutions by considering the following first-order system of PDE:

$$\vec{F}_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}\vec{F}^{\alpha\beta}, \tag{1}$$

where $\vec{F}_{\mu\nu} = \partial^\mu \vec{A}_\nu - \partial^\nu \vec{A}_\mu + e\vec{A}_\mu \times \vec{A}_\nu$ is a tensor of the Yang–Mills field; $\partial_\mu = \partial/\partial x_\mu$, $\varepsilon_{\nu\alpha\beta}$ is the antisymmetric fourth-order tensor; $\mu,\nu,\alpha,\beta = \overline{0,3}$. Hereafter, the summation over the repeated indices from 0 to 3 is understood, rising and lowering of the tensor indices is carried out with the help of the metric tensor $g_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$ of the Minkowski space.

Equations (1) are called self-dual Yang–Mills equations. They are very interesting because of the fact that any solution of equations (1) automatically satisfies Yang–Mills equations (see, e.g. [1]). Moreover, symmetry groups of the Yang–Mills and of the self-dual Yang–Mills equations are the same. Maximal symmetry group admitted by equations (1) is the conformal group $C(1,3)$ supplemented by the gauge group $SU(2)$ [2].

In the present paper, we carry out a symmetry reduction of the self-dual Yang–Mills equations (1) by using ansatzes for the vector-potential of the Yang–Mills $\vec{A}_\mu(x)$ invariant under the three-parameter subgroups of the Poincaré group $P(1,3) \subset C(1,3)$.

It is known that the problem of classification of inequivalent subgroups of a Lie transformation group is equivalent to the one of classification of inequivalent subalgebras of the Lie algebra (see, e.g. [3, 4]). Complete description of $P(1,3)$-inequivalent three-dimensional subalgebras of the Poincaré algebra $AP(1,3)$ had been obtained in [3].

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To establish correspondence between the three-dimensional subalgebra of the symmetry algebra of equations (1) having the basis elements

$$X_a = \xi_{a\mu}(x,A)\partial_\mu + \sum_{b=1}^{3} \eta^b_{a\mu}(x,A)\frac{\partial}{\partial A^b_\mu}, \quad a=\overline{1,3}, \tag{2}$$

where $\{A^a_\mu, a=\overline{1,3}, \mu=\overline{0,3}\}$, and the ansatz for $\vec{A}_\mu(x)$ reducing equations (1) to a system of ordinary differential equations, one has:

(1) to construct a complete system of functionally-different invariants of the operators (2) $\omega=\{\omega_i(x,A), i=\overline{1,13}\}$;

(2) to resolve the relations

$$F_j(\omega_1(x,A),\ldots,\omega_{13}(x,A))=0, \quad j=\overline{1,13} \tag{3}$$

with respect to the functions A^a_μ.

As proved in [5], the above procedure can be significantly simplified if coefficients of operators (2) have the following structure:

$$\xi_{a\mu}=\xi_{a\mu}(x), \quad \eta^b_{a\mu} = \sum_{c=1}^{3} R^{bc}_{a\mu\nu}(x)A^c_\nu. \tag{4}$$

The ansatz for $\vec{A}_\mu$ can be searched for in the form

$$A^a_\mu(x) = \sum_{c=1}^{3} Q^{ab}_{\mu\nu}(x)B^b_\nu(\omega(x)), \tag{5}$$

where $B^b_\nu(\omega)$ are arbitrary smooth and the functions $\omega(x)$, $Q^{ab}_{\mu\nu}(x)$ satisfy the system of PDE

$$\begin{gathered} \xi_{a\mu}(x)\omega_{x_\mu}=0, \\ \sum_{c=1}^{3}(\xi_{a\mu}\delta^{bc}\partial_\mu - R^{bc}_{a\mu\nu})Q^{cd}_{\nu\alpha}=0. \end{gathered} \tag{6}$$

Here, δ^{bc} is the Kronecker symbol, $a,b,d=\overline{1,3}$, $\alpha=\overline{0,3}$.

On the set of solutions of equations (1), the following representation of the Poincaré algebra is realized:

$$P_\mu=\partial^\mu, \quad J_{\mu\nu}=x_\mu P_\nu - x_\nu P_\mu + \sum_{a=1}^{3}\left(A^a_\mu\frac{\partial}{\partial A^{a\nu}} - A^a_\nu\frac{\partial}{\partial A^{a\mu}}\right), \quad \mu,\nu=\overline{0,3}. \tag{7}$$

Consequently, relations (4) hold true. Moreover, expression for $\eta^b_{a\mu}$ has the form

$$\eta^b_{a\mu}=R_{a\mu\nu}(x)A^b_\nu, \quad a,b=\overline{1,3}, \quad \mu=\overline{0,3}. \tag{8}$$

That is why formulae (5), (6) can be rewritten in a simpler way. Namely, an ansatz for the vector-potential of the Yang–Mills field $\vec{A}(x)$ invariant under a subalgebra of the algebra $AP(1,3)$ with basis operators (7) should be searched for in the form

$$A^a_\mu(x)=Q_{\mu\nu}(x)B^a_\nu(\omega(x)), \tag{9}$$

where $B^a_\nu(\omega)$ are arbitrary smooth functions and functions $\omega(x)$, $Q_{\mu\nu}(x)$ satisfy the system of PDE

$$\begin{aligned}&\xi_{a\mu}(x)\omega_{x_\mu}=0,\\&\xi_{a\alpha}(x)\partial_\alpha Q_{\mu\nu}-R_{a\mu\alpha}(x)Q_{\alpha\nu}=0,\end{aligned}\tag{10}$$

where $a=\overline{1,3}$, $\mu,\nu=\overline{0,3}$.

Thus, to get a complete description of $P(1,3)$-inequivalent ansatzes invariant under three-dimensional subalgebras of the Poincaré algebra, one has to integrate over-determined system of PDE (10) for each subalgebra. Let us note that compatibility of equations (10) is guaranteed by the fact that the operators X_1, X_2, X_3 form a Lie algebra.

Bellow, we adduce a complete list of $C(1,3)$-inequivalent three-dimensional subalgebras of the Poincaré algebra $AP(1,3)$ following [4]:

$$\begin{array}{ll}
L_1=\langle P_0,P_1,P_2\rangle, & L_2=\langle P_1,P_2,P_3\rangle,\\
L_3=\langle P_0+P_3,P_1,P_2\rangle, & L_4=\langle J_{03}+\alpha J_{12},P_1,P_2\rangle,\\
L_5=\langle J_{03},P_0+P_3,P_1\rangle, & L_6=\langle J_{03}+P_1,P_0,P_3\rangle,\\
L_7=\langle J_{03}+P_1,P_0+P_3,P_1\rangle, & L_8=\langle J_{12}+\alpha J_{03},P_0,P_3\rangle,\\
L_9=\langle J_{12}+P_0,P_1,P_2\rangle, & L_{10}=\langle J_{12}+P_3,P_1,P_2\rangle,\\
L_{11}=\langle J_{12}+P_0-P_3,P_1,P_2\rangle, & L_{12}=\langle G_1,P_0+P_3,P_2+\alpha P_1\rangle,\\
L_{13}=\langle G_1+P_2,P_0+P_3,P_1\rangle, & L_{14}=\langle G_1+P_0-P_3,P_0+P_3,P_2\rangle,\\
L_{15}=\langle G_1+P_0-P_3,P_1+\alpha P_2,P_0+P_3\rangle, & L_{16}=\langle J_{12},J_{03},P_0+P_3\rangle,\\
L_{17}=\langle G_1+P_2,G_2-P_1+\alpha P_2,P_0+P_3\rangle, & L_{18}=\langle G_1,J_{03},P_2\rangle,\\
L_{19}=\langle J_{03},G_1,P_0+P_3\rangle, & L_{20}=\langle J_{03}+P_2,G_1,P_0+P_3\rangle,\\
L_{21}=\langle G_1,J_{03}+P_1+\alpha P_2,P_0+P_3\rangle, & L_{22}=\langle G_1,G_2,J_{03}+\alpha J_{12}\rangle,\\
L_{23}=\langle G_1,P_0+P_3,P_1\rangle, & L_{24}=\langle J_{12},P_1,P_2\rangle,\\
L_{25}=\langle J_{03},P_0,P_3\rangle, & L_{26}=\langle J_{01},J_{02},J_{12}\rangle,\\
L_{27}=\langle J_{12},J_{23},J_{13}\rangle, &
\end{array}$$

Here, $G_i=J_{0i}-J_{i3}$ $(i=1,2)$, $\alpha\in\mathbb{R}$.

Let us consider, as an example, the procedure of construction of ansatz (9) invariant under subalgebra L_4 $(\alpha=0)$. In this case, system (10) reads

$$\omega_{x_1}=\omega_{x_2}=0,\quad x_0\omega_{x_3}+x_3\omega_{x_0}=0,\tag{11a}$$

$$Q_{x_1}=Q_{x_2}=0,\quad x_0Q_{x_3}+x_3Q_{x_0}-SQ=0,\tag{11b}$$

where $Q=\|Q_{\mu\nu}(x)\|^3_{\mu,\nu}=0$,

$$S=\begin{Vmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{Vmatrix}.$$

The first integral of system (11a) has the form $\omega=x_0^2-x_3^2$. Next, from first two equations of system (11b), it follows that $Q=Q(x_0,x_3)$. Since S is a constant matrix, solutions of the third equation from (11b) can be looked for in the form (see, e.g. [6])

$$Q=\exp\{f(x_0,x_3)S\}.$$

By substituting this expression into (11b), we get

$$(x_0 f_{x_3}, x_3 f_{x_0} - 1)\exp\{fS\} = 0,$$

where $f = \ln(x_0 + x_3)$.

Consequently, a particular solution of equations (lib) can be chosen in the following way:

$$Q = \exp\{\ln(x_0 + x_3)S\}.$$

By using evident identity $S = S^3$, we obtain the equality

$$Q = I + S\,\mathrm{sh}(\ln(x_0 + x_3)) + S^2(\mathrm{ch}(\ln(x_0 + x_3)) - 1), \tag{12}$$

where I is a unit (4×4)-matrix.

By substituting the obtained expressions into formula (9), we get an ansatz for $\vec{A}_\mu(x)$ which is invariant under the algebra L_4

$$\begin{aligned}
&A_0^a = B_0^a(x_0^2 - x_3^2)\,\mathrm{ch}(\ln(x_0 + x_3)) + B_3^a(x_0^2 - x_3^2)\,\mathrm{sh}(\ln(x_0 + x_3)),\\
&A_1^a = B_1^a(x_0^2 - x_3^2), \quad A_2^a = B_2^a(x_0^2 - x_3^2),\\
&A_3^a = B_3^a(x_0^2 - x_3^2)\,\mathrm{ch}(\ln(x_0^2 - x_3^2)) + B_0^a(x_0^2 - x_3^2)\,\mathrm{sh}(\ln(x_0^2 - x_3^2)), \quad a = \overline{1,3}.
\end{aligned} \tag{13}$$

The above ansatz has such an unpleasant feature as an asymmetric dependence on independent variables x_μ. To remove this asymmetry, one has to use a solution generation procedure [7]. As a result, we arrive at the following representation of the Poincaré invariant ansatz for the vector-potential of the Yang–Mills field:

$$\begin{aligned}
\vec{A}_\mu(x) = Q_{\mu\nu}(x)\vec{B}^\nu(\omega) = \{&(a_\mu a_\nu - d_\mu d_\nu)\,\mathrm{ch}\,\theta_0 + (d_\mu a_\nu - d_\nu a_\mu)\,\mathrm{sh}\,\theta_0 + \\
&+ 2k_\mu[(\theta_1\cos\theta_3 + \theta_2\sin\theta_3)b_\nu + (\theta_2\cos\theta_3 - \theta_1\sin\theta_3)c_\nu + \\
&+ (\theta_1^2 + \theta_2^2)k_\nu\exp(-\theta_0)] + (b_\mu c_\nu - b_\nu c_\mu)\sin\theta_3 - (c_\mu c_\nu - b_\mu b_\nu)\cos\theta_3 - \\
&- 2(\theta_1 b_\mu + \theta_2 c_\mu)k_\nu\exp(-\theta_0)]\}\vec{B}^\nu(\omega).
\end{aligned}$$

Here, a_μ, b_μ, c_μ, d_μ are arbitrary constants satisfying the following equalities:

$$\begin{aligned}
&a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1,\\
&a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0,
\end{aligned}$$

$k_\mu = a_\mu + d_\mu$, Q_μ, ω are some functionals of x whose explicit form depends on the choice of the algebra $AP(1,3)$, $\mu = \overline{0,3}$. Below, we adduce a complete list of functions Q_μ, $\mu = \overline{0,3}$, ω co corresponding to three-dimensional subalgebras of the Poincaré algebra (7).

$$\begin{aligned}
&L_1: \quad \theta_\mu = 0, \quad \omega = dx;\\
&L_2: \quad \theta_\mu = 0, \quad \omega = ax;\\
&L_3: \quad \theta_\mu = 0, \quad \omega = ax + dx;\\
&L_4: \quad \theta_0 = -\ln|ax + dx|, \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = a\ln|ax + dx|,\\
&\qquad\ \ \omega = (ax)^2 - (dx)^2;\\
&L_5: \quad \theta_0 = -\ln|ax + dx|, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = cx;\\
&L_6: \quad \theta_0 = bx, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = cx;\\
&L_7: \quad \theta_0 = bx, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = -bx + \ln|ax + dx|;
\end{aligned}$$

$$L_8: \quad \theta_0 = \alpha\,\mathrm{arctg}[bx(cx)^{-1}], \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = -\,\mathrm{arctg}[bx(cx)^{-1}],$$
$$\omega = (bx)^2 + (cx)^2;$$
$$L_9: \quad \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = -ax, \quad \omega = dx;$$
$$L_{10}: \quad \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = dx, \quad \omega = ax;$$
$$L_{11}: \quad \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = -\frac{1}{2}(dx + ax), \quad \omega = ax + dx;$$
$$L_{12}: \quad \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = \frac{1}{2}(bx - \alpha cx)(ax + dx)^{-1}, \quad \omega = ax + dx;$$
$$L_{13}: \quad \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = \frac{1}{2}cx, \quad \omega = ax + dx;$$
$$L_{14}: \quad \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\frac{1}{4}(ax + dx), \quad \omega = 4bx - (ax + dx)^2;$$
$$L_{15}: \quad \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\frac{1}{4}(ax + dx),$$
$$\omega = 4(\alpha bx - cx) - \alpha(ax + dx)^2;$$
$$L_{16}: \quad \theta_0 = -\ln|ax + dx|, \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = -\,\mathrm{arctg}[bx(cx)^{-1}],$$
$$\omega = (bx)^2 + (cx)^2;$$
$$L_{17}: \quad \theta_0 = 0, \quad \theta_1 = \frac{1}{2}\frac{cx + (\alpha + ax + dx)bx}{1 + (ax + dx)(\alpha + ax + dx)},$$
$$\theta_2 = -\frac{1}{2}\frac{bx - cx(ax + dx)}{1 + (ax + dx)(\alpha + ax + dx)}, \quad \theta_3 = 0, \quad \omega = ax + dx;$$
$$L_{18}: \quad \theta_0 = -\ln|ax + dx|, \quad \theta_1 = \frac{1}{2}\frac{bx}{ax + dx}, \quad \theta_2 = \theta_3 = 0,$$
$$\omega = (ax)^2 - (bx)^2 - (dx)^2;$$
$$L_{19}: \quad \theta_0 = -\ln|ax + dx|, \quad \theta_1 = \frac{1}{2}\frac{bx}{ax + dx}, \quad \theta_2 = \theta_3 = 0, \quad \omega = cx;$$
$$L_{20}: \quad \theta_0 = -\ln|ax + dx|, \quad \theta_1 = \frac{1}{2}\frac{bx}{ax + dx}, \quad \theta_2 = \theta_3 = 0,$$
$$\omega = cx + \ln|ax + dx|;$$
$$L_{21}: \quad \theta_0 = -\ln|ax + dx|, \quad \theta_1 = -\frac{1}{2}\frac{-bx + \ln|ax - dx|}{ax + dx}, \quad \theta_2 = \theta_3 = 0,$$
$$\omega = cx + \alpha\ln|ax + dx|;$$
$$L_{22}: \quad \theta_0 = -\ln|ax + dx|, \quad \theta_1 = \frac{1}{2}\frac{bx}{ax - dx}, \quad \theta_2 = \frac{1}{2}\frac{cx}{ax - dx},$$
$$\theta_3 = \alpha\ln|ax + dx|, \quad \omega = (ax)^2 - (bx)^2 - (cx)^2 - (dx)^2.$$

Here, $ax = a_\mu x^\mu$, $bx = b_\mu x^\mu$, $cx = c_\mu x^\mu$, $dx = d_\mu x^\mu$, $\mu = \overline{0,3}$.

Note. Ansatzes invariant under subalgebras L_{23}, L_{24}, L_{25}, L_{26}, L_{27} yield so-called partially-invariant solutions (the term was introduced by L.V. Ovsyannikov [8]) which cannot be represented in the form (13) and are not considered here.

Substitution of ansatzes (13), (14) into system of PDE (1) demands very cumbersome computations. This is why we omit these and adduce only the final result-system of ordinary differential equations for $\vec{B}_\mu(\omega)$.

General form of the reduced system is the following:

$$\vec{T}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\alpha\beta}\vec{T}^{\alpha\beta}, \quad \mu,\nu = \overline{0,3}, \tag{14}$$

where

$$\vec{T}_{\mu\nu} = G_\mu(\omega)\vec{B}_\mu - G_\nu\vec{B}_\mu - H_{\mu\nu\gamma}(\omega)\vec{B}^\gamma + e\vec{B}_\mu \times \vec{B}_\nu$$

and functions $G_\mu(\omega), H_{\mu\gamma}(\omega)$ are calculated according to the following formulae:

$$G_\mu(\omega) = Q_{\mu\nu}\omega_{x_\nu},$$
$$H_{\mu\nu\gamma}(\omega) = Q_\mu^\alpha Q_{\alpha\gamma x_\beta}Q_{\beta\nu} - Q_\nu^\alpha - Q_{\alpha\gamma x_\beta}Q_{\beta\mu}.$$

In the above formulae, overdot means differentiation with respect to ω.

Thus, the form of the reduced equations for functions $\vec{B}_\mu(\omega)$ depends on the explicit forms of functions $G_\mu(\omega)$, $H_{\mu\nu\gamma}(\omega)$. Below, we adduce a list of these functions corresponding to ansatzes (13), (14).

$L_1: \quad G_\mu = -d_\mu, \quad H_{\mu\nu\gamma} = 0;$

$L_2: \quad G_\mu = a_\mu, \quad H_{\mu\nu\gamma} = 0;$

$L_3: \quad G_\mu = k_\mu, \quad H_{\mu\nu\gamma} = 0;$

$L_4: \quad G_\mu = \varepsilon[a_\mu - d_\mu + k_\mu\omega],$

$\quad H_{\mu\nu\gamma} = -\varepsilon[(a_\mu d_\nu - d_\mu a_\nu)k_\gamma + \alpha(k_\nu(b_\gamma c_\mu - c_\gamma b_\mu) - k_\mu(b_\gamma c_\mu - c_\gamma b_\nu))];$

$L_5: \quad G_\mu = c_\mu, \quad H_{\mu\nu\gamma} = -\varepsilon(a_\mu d_\nu - d_\mu a_\nu)k_\gamma;$

$L_6: \quad G_\mu = c_\mu, \quad H_{\mu\nu\gamma} = (a_\mu d_\gamma - a_\gamma d_\mu)b_\nu + (a_\nu d_\gamma - a_\gamma d_\nu)b_\mu;$

$L_7: \quad G_\mu = -b_\mu + \varepsilon k_\mu, \quad H_{\mu\nu\gamma} = -(a_\mu d_\gamma - a_\gamma d_\mu)b_\nu + (a_\nu d_\gamma - a_\gamma d_\nu)b_\mu;$

$L_8: \quad G_\mu = 2c_\mu\sqrt{\omega},$

$\quad H_{\mu\nu\gamma} = \dfrac{1}{\sqrt{\omega}}\{(c_\mu b_\nu - c_\nu b_\mu)b_\gamma + \alpha[(d_\mu a_\gamma - a_\mu d_\gamma)b_\nu - (d_\nu a_\gamma - a_\nu d_\gamma)b_\mu]\};$

$L_9: \quad G_\mu = -d_\mu, \quad H_{\mu\nu\gamma} = -a_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + a_\nu(b_\mu c_\gamma - c_\mu b_\gamma);$

$L_{10}: \quad G_\mu = a_\mu, \quad H_{\mu\nu\gamma} = (b_\mu c_\gamma - c_\mu b_\gamma)d_\nu - (b_\nu c_\gamma - c_\nu b_\gamma)d_\mu;$

$L_{11}: \quad G_\mu = a_\mu - d_\mu, \quad H_{\mu\nu\gamma} = \dfrac{1}{2}[(b_\nu c_\gamma - c_\nu b_\gamma)b_\mu - (b_\mu c_\gamma - c_\mu b_\gamma)b_\nu];$

$L_{12}: \quad G_\mu = k_\mu,$

$\quad H_{\mu\nu\gamma} = \dfrac{1}{\omega}\{(k_\mu b_\nu - k_\nu b_\mu)b_\gamma - \alpha[(k_\mu b_\gamma - k_\gamma b_\mu)c_\nu - (k_\nu b_\gamma - k_\gamma b_\nu)c_\mu]\};$

$L_{13}: \quad G_\mu = k_\mu, \quad H_{\mu\nu\gamma} = (k_\mu b_\gamma - k_\gamma b_\mu)c_\nu - (k_\nu c_\gamma - k_\gamma b_\nu)c_\mu;$

$L_{14}: \quad G_\mu = 4b_\mu, \quad H_{\mu\nu\gamma} = \dfrac{1}{2}(b_\mu k_\nu - b_\nu k_\mu)k_\gamma;$

$L_{15}: \quad G_\mu = 4(c_\mu - \alpha b_\mu), \quad H_{\mu\nu\gamma} = \dfrac{1}{2}(b_\mu k_\nu - b_\nu k_\mu)k_\gamma;$

$L_{16}: \quad G_\mu = 2c_\mu\sqrt{\omega},$

$\quad H_{\mu\nu\gamma} = \varepsilon(a_\mu d_\nu - a_\nu d_\mu)k_\gamma - \dfrac{1}{\sqrt{\omega}}k_\gamma - \dfrac{1}{\sqrt{\omega}}(b_\mu c_\nu - c_\mu b_\nu)b_\gamma;$

$L_{17}: \quad G_\mu = k_\mu,$

$\quad H_{\mu\nu\gamma} = \dfrac{1}{1+\omega(\omega+\alpha)}\{2(b_\nu c_\mu - b_\mu c_\nu)k_\gamma + (k_\mu c_\nu - k_\nu c_\mu)b_\gamma +$

$$
\begin{aligned}
& \quad + (k_\nu b_\mu - k_\mu b_\nu)c_\gamma + (\alpha+\omega)(k_\mu b_\nu - k_\nu b_\mu)b_\gamma + \\
& \quad + \omega(k_\mu c_\nu - k_\nu c_\mu)c_\gamma\}; \\
L_{18}: \quad & G_\mu = \varepsilon(k_\mu\omega + a_\mu - d_\mu), \\
& H_{\mu\nu\gamma} = \varepsilon[(k_\mu b_\nu - k_\nu b_\mu)b_\gamma + (a_\mu d_\nu - k_\nu b_\mu)k_\gamma]; \\
L_{19}: \quad & G_\mu = c_\mu, \quad H_{\mu\nu\gamma} = \varepsilon[(k_\mu b_\nu - k_\nu b_\mu)b_\gamma + (a_\mu d_\nu - a_\nu d_\mu)k_\gamma]; \\
L_{20}: \quad & G_\mu = c_\mu + \varepsilon k_\mu, \quad H_{\mu\nu\gamma} = \varepsilon[(a_\mu d_\nu - a_\nu d_\mu)k_\gamma + (k_\mu b_\nu - k_\nu b_\mu)b_\gamma]; \\
L_{21}: \quad & G_\mu = c_\mu + \varepsilon\alpha k_\mu, \\
& H_{\mu\nu\gamma} = \varepsilon[(a_\mu d_\nu - a_\nu d_\mu)k_\gamma + (k_\mu b_\nu - k_\nu b_\mu)b_\gamma - (k_\mu b_\nu - k_\nu b_\mu)k_\gamma]; \\
L_{22}: \quad & G_\mu = \varepsilon(k_\mu\omega + a_\mu - d_\mu), \\
& H_{\mu\nu\gamma} = \varepsilon\{(k_\mu b_\nu - k_\nu b_\mu)b_\gamma + (k_\mu c_\nu - k_\nu c_\mu)c_\gamma + \\
& \quad + \alpha[(b_\mu c_\gamma - c_\mu b_\mu)k_\nu - (b_\nu c_\gamma - c_\nu b_\gamma] + (a_\mu d_\nu - a_\nu d_\mu)k_\gamma\}.
\end{aligned}
$$

Here, $k_\mu = a_\mu + d_\mu$, $\varepsilon = 1$ for $ax + dx > 0$ and $\varepsilon = -1$ for $ax + dx < 0$.

Thus, using symmetry properties of the self-dual Yang–Mills equations and subalgebraic structure of the Poincaré algebra, we reduced system of PDE (1) to the system of ordinary differential equations (15). Let us emphasize that system (15) contains nine equations for twelve functions, which means that it is underdetermined. This fact simplifies essentially finding its particular solutions.

If one constructs a solution of one of equations (15) (general or particular), then substitution of the obtained result into the corresponding ansatz from (13). (14) yields an exact solution of the nonlinear self-dual Yang–Mills equations (1). We intend to study in detail the reduced system of ordinary differential equations (15) and construct new classes of exact solutions of equations (1) but this will be a topic of our future publication.

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