Symmetry and reduction of nonlinear Dirac equations

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We present results of symmetry classification of the nonlinear Dirac equations with respect to the conformal group $C(1,3)$ and its principal subgroups. Next we briefly consider the problem of classical and non-classical symmetry reduction and construction of exact solutions for the nonlinear Poincaré-invariant Dirac equations. In particular, a class of exact solutions is constructed which can not be in principle obtained within the framework of the classical Lie approach.

The Dirac equation is a system of four complex partial differential equations of the form

$$(i\gamma_\mu \partial_{x_\mu} - m)\psi(x) = 0, \tag{1}$$

where $\psi = \psi(x_0, \vec{x})$ is a four-component function-column, γ_μ are 4×4 Dirac matrices

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma_a = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix}$$

and σ_a are usual 2×2 Pauli matrices.

In fact in the following we do not use an explicit representation of the Dirac matrices, we use the commutational relations

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} = 2\begin{cases} 1, & \mu=\nu=0, \\ -1, & \mu=\nu=1,2,3, \\ 0, & \mu\neq\nu \end{cases}$$

only.

Nonlinear generalizations of the Dirac equation were suggested by Ivanenko [1]

$$[i\gamma_\mu \partial_{x_\mu} - m + \lambda(\bar\psi\psi)]\psi = 0 \tag{2}$$

and by Heisenberg [2]

$$[i\gamma_\mu \partial_{x_\mu} + \lambda(\bar\psi\gamma_\mu\gamma_4\psi)\gamma^\mu\gamma_4]\psi = 0. \tag{3}$$

Here $\bar\psi = (\psi_0^*, \psi_1^*, -\psi_2^*, -\psi_3^*)$ is a four-component function-row, $\gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3$, $\lambda = \text{const}$.

The above equations can be obtained in a unified way within the framework of symmetry approach. For the equation of the form

$$[i\gamma_\mu \partial_{x_\mu} + F(\bar\psi, \psi)]\psi = 0 \tag{4}$$

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to be physically acceptable generalization of the linear Dirac equation (1) it must obey the Einstein relativity principle. From mathematical point of view, it means that on the set of solutions of Eq. (4) some representation of the Poincaré group $P(1,3)$ is to be realized. Consequently, one has to describe all the matrices $F(\bar\psi,\psi)$ such that Eq. (4) is invariant under the Poincaré group. Furthermore, it is known that the massless Dirac equation is invariant under the 15-parameter conformal group $C(1,3)$. Therefore it is of interest to describe nonlinear equations (4) admitting conformal group. Such procedure is usually called symmetry or group-theoretical classification of nonlinear equations (4).

First, we give the results of symmetry classification and then turn to the problem of constructing exact solutions of the nonlinear Dirac equations (4).

Theorem 1. *System of partial differential equations (4) is Poincaré invariant iff*

$$F(\bar\psi,\psi) = F_1(\bar\psi\psi, \bar\psi\gamma_4\psi) + F_2(\bar\psi\psi, \bar\psi\gamma_4\psi)\gamma_4, \tag{5}$$

where F_1, F_2 are arbitrary complex functions.

Theorem 2. *System of PDE (4) is invariant under the extended Poincaré group $\tilde P(1,3) = P(1,3) \otimes D(1)$, where $D(1)$ is a one-parameter group of scale transformations generated by the following infinitesimal operator:*

$$D = x_\mu\partial_\mu + k, \quad k \in \mathbb{R}^1, \tag{6}$$

iff the matrix-function $F(\bar\psi\psi)$ is of the form (5), F_i being determined by the formulae

$$F_i = (\bar\psi\psi)^{1/2k}\tilde F_i(\bar\psi\psi/\bar\psi\gamma_4\psi), \quad i = 1, 2, \tag{7}$$

with arbitrary complex functions $\tilde F_i$.

Theorem 3. *System of PDE (4) is invariant under the 15-parameter conformal group $C(1,3) = \tilde P(1,3) \otimes K(4)$, where $K(4)$ is a 4-parameter group of special conformal transformations which is generated by the following infinitesimal operators:*

$$K_\mu = 2x_\mu D - x_\nu x^\nu \partial^\mu + \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)x^\nu, \quad \mu = 0, \ldots, 3, \tag{8}$$

iff $F(\bar\psi,\psi)$ is of the form (5), (7) with $k = 3/2$. In formula (8) D is the operator (6) with $k = 3/2$, $x^\mu = g_{\mu\nu}x_\nu$, $\partial^\mu = g_{\mu\nu}\partial_\nu$, $\mu,\nu = 0,\ldots,3$.

Proof of the Theorems 1–3 is carried out with the help of the infinitesimal Lie algorithm [3, 4].

Thus, there exists rather narrow class of Poincaré invariant equations of the form (4)

$$[i\gamma_\mu\partial_{x_\mu} + F_1(\bar\psi\psi, \bar\psi\gamma_4\psi) + F_2(\bar\psi\psi, \bar\psi\gamma_4\psi)\gamma_4]\psi = 0. \tag{9}$$

To construct exact solutions of the nonlinear Dirac equation (9) we apply the symmetry reduction technique.

The general idea of symmetry reduction of PDEs can be formulated in a very simple and natural way. Since coefficients of Eq. (9) do not depend explicitly on the variable x_0, we can look for a particular solution which is also independent of x_0

$$\psi = \varphi(x_1, x_2, x_3). \tag{10}$$

After substituting (10) into Eq. (9) we get system of PDEs with three independent variables

$$[i\gamma_1\partial_{x_1} + i\gamma_2\partial_{x_2} + i\gamma_3\partial_{x_3} + F_1(\bar\varphi\varphi, \bar\varphi\gamma_4\varphi) + F_2(\bar\varphi\varphi, \bar\varphi\gamma_4\varphi)\gamma_4]\varphi = 0. \tag{11}$$

But from the group-theoretical point of view independence of Eq. (9) of the variable x_0 means that this equation is invariant under the one-parameter group of displacements with respect to x_0

$$x_0' = x_0 + \theta, \quad \vec{x}\,' = \vec{x}, \quad \psi' = \psi. \tag{12}$$

Similarly, (10) is a manifold in the space of variables x, ψ invariant under the group of displacements with respect to x_0.

Thus, imposing on the solution to be found requirement of invariance with respect to the one-parameter group (12) which is a subgroup of the invariance group of Eq. (9) we reduce it by one independent variable.

Now we turn to the general case. Let Eq. (9) be invariant under the one-parameter transformation group

$$x_\mu' = f_\mu(x, \theta), \quad \psi' = F(x, \theta)\psi, \tag{13}$$

where f_μ are some real functions and F is a variable 4×4 matrix.

It is known, that there exists such change of variables

$$\omega_\mu = \omega_\mu(x), \quad \varphi = B(x)\psi, \tag{14}$$

where $B(x)$ is some invertible 4×4 matrix, that the group (13) in the space of variables ω_μ, φ takes the form

$$\omega_0' = \omega_0 + \theta, \quad \vec{\omega}' = \vec{\omega}, \quad \varphi' = \varphi. \tag{15}$$

Consequently, if we make in the initial equation (9) the change of variables (14), then the equation obtained will be invariant under the one-parameter group of displacements (15). Therefore, a substitution $\varphi = \varphi(\omega_1, \omega_2, \omega_3)$ reduce it to a system of PDEs with three independent variables ω_1, ω_2, ω_3.

In the initial variables the above said substitution reads

$$\psi(x) = A(x)\varphi(\omega_1(x), \omega_2(x), \omega_3(x)), \tag{16}$$

where $A(x) = B^{-1}(x)$.

And what is more, substitution of the expression (16) into Eq. (9) reduce it to a system of PDEs with three independent variables ω_1, ω_2, ω_3.

In fact, we gave a sketch of the proof of the reduction theorem, which is of utmost importance for applications of Lie transformation groups in mathematical physics. Namely, solution invariant under the one-parameter subgroup of the invariance group of the nonlinear Dirac equation reduce it to a system of PDEs with three independent variables. Obviously, a solution invariant under a three-parameter subgroup of invariance group reduce the nonlinear Dirac equation to a system of ordinary differential equations (ODEs).

So each three-parameter subgroup of the Poincaré group $P(1,3)$ gives rise to an Ansatz

$$\psi(x) = A(x)\varphi(\omega(x)), \tag{17}$$

which reduces the nonlinear Dirac equation (9) to a system of ODEs for a function $\varphi(\omega)$.

In practice it is more convenient to work with Lie algebras. Let the operators

$$Q_a = \xi_{a\mu}(x)\partial_{x_\mu} + \eta_a(x), \quad a = 1, 2, 3 \tag{18}$$

form a three-dimensional Lie algebra corresponding to a given three-parameter subgroup G_3 of the group $P(1,3)$. Then a solution invariant with respect to the group G_3 has the form (18), where function $\omega(x)$ and matrix function $A(x)$ are determined by the following equations:

$$\begin{aligned}&1. \quad \xi_{a\mu}(x)\partial_{x_\mu}\omega(x) = 0, \quad a = 1, 2, 3,\\ &2. \quad (\xi_{a\mu}(x)\partial_{x_\mu} + \eta_a(x))A(x) = 0, \quad a = 1, 2, 3.\end{aligned} \tag{19}$$

Classification of $P(1,3)$-inequivalent subalgebras of the Lie algebra of the Poincaré group $P(1,3)$ has been carried out in [5]. There are, in particular, 27 inequivalent three-dimensional subalgebras. Solving for each of these system of PDEs (19) we obtain 27 Ansätze reducing the nonlinear Dirac equation to systems of ODEs.

Consider, as an example, a subgroup which Lie algebra has the following basis elements:

$$Q_1 = \partial_{x_0}, \quad Q_2 = \partial_{x_3}, \quad Q_3 = x_2\partial_{x_1} - x_1\partial_{x_2} + \frac{1}{2}\gamma_1\gamma_2.$$

Ansatz corresponding to the above algebra has the form

$$\psi(x) = \exp\left(-\frac{1}{2}\gamma_1\gamma_2 \arctan\frac{x_1}{x_2}\right)\varphi(x_1^2 + x_2^2). \tag{20}$$

Substituting the above Ansatz into Eq. (9) after some tedious transformations we get a system of ODEs

$$i\gamma_2\frac{d\varphi}{d\omega} + \frac{i}{2}\omega^{-1/2}\gamma_2\varphi + [F_1(\bar\varphi\varphi, \bar\varphi\gamma_4\varphi) + F_2(\bar\varphi\varphi, \bar\varphi\gamma_4\varphi)\gamma_4]\varphi = 0.$$

For the nonlinear equation suggested by Ivanenko $F_1 = m + (\bar\varphi\varphi)$ and $F_2 = 0$. In such a case the above system is integrated. Substituting the result obtained into the Ansatz (20) we get an exact solution of the nonlinear equation (2)

$$\psi(x) = (x_1^2 + x_2^2)^{\frac{\lambda}{2}\tilde\chi\chi - \frac{1}{4}} \exp\left(-\frac{1}{2}\gamma_1\gamma_2\arctan\frac{x_1}{x_2}\right)\exp\left(-m(x_1^2 + x_2^2)^{1/2}\right)\chi,$$

where χ an arbitrary constant four-component column.

Symmetry approach to construction of exact solutions of PDEs is so systematic and algorithmic that one could get an impression that in this way all Ansätze reducing Eq. (9) to systems of ODEs can be obtained. Luckily, it is not so. The source of principally new Ansätze is the *conditional* symmetry of Eq. (9).

To study conditional symmetry of PDEs one can apply Lie algorithm but the problem is that the determining equations for coefficients of vector field admitted are essentially nonlinear. This is a reason why more or less systematic results on conditional symmetry of PDEs are obtained only for two-dimensional equations.

But we suggested a method making it possible to obtain rich information about conditional symmetry of such a complex nonlinear model as Eq. (9).

The principal idea is based on the following observation: all Ansätze invariant under three-parameter subgroups of the group $P(1,3)$ can be represented in the following unified form [6]:

$$\psi(x) = \exp\left((\gamma_1\theta_1(x) + \gamma_2\theta_2(x))(\gamma_0 + \gamma_3)\right) \times \times \exp\left(\theta_0(x)\gamma_0\gamma_3 + \theta_3(x)\gamma_1\gamma_2\right)\varphi(\omega(x)). \tag{21}$$

Specifying the functions $\theta_\mu(x)$, $\omega(x)$ we get from (21) all Poincaré invariant Ansätze mentioned above.

The idea is not to impose ad hoc conditions on the functions θ_μ, ω. The only condition is a requirement that substitution of expression (21) into Eq. (9) yields a system of ODEs for a four-component function $\varphi(\omega)$.

As a result, one gets a system of twelve nonlinear PDEs for five functions. From the first sight it looks even more complicated than the initial Eq. (9). But the fact that said system is strongly over-determined enabled us to construct its general solution.

In this way we have obtained not only all Poincaré invariant Ansätze (which is quite predictable) but also six principally new classes of Ansätze which correspond to conditional symmetry of the equation under study.

We adduce, as an example, the following Ansatz

$$\psi(x) = \exp\Big(\frac{1}{2}(w_1'\gamma_1 + w_2'\gamma_2)(\gamma_0 + \gamma_3) + C(y_1^2 + y_2^2)^{-1/2}(y_2\gamma_1 - y_1\gamma_2) \times \times (\gamma_0 + \gamma_3)\Big)\exp\Big(-\frac{1}{2}\gamma_1\gamma_2 \arctan\frac{y_1}{y_2}\Big)\varphi(y_1^2 + y_2^2), \tag{22}$$

where $y_a = x_a + w_a$, $a = 1, 2$, $w_a = w_a(x_0 + x_3)$ are arbitrary functions, C is an arbitrary constant.

It is readily seen that provided $C = 0, w_1 = w_2 = 0$ formula (22) gives the Ansatz (20) which has been obtained with the use of the invariance group of Eq. (9). This example demonstrates that invariant solutions are very special cases of conditionally invariant solutions.

It is important to emphasize a principal difference between invariant and conditionally-invariant Ansätze. Ansatz (20) invariant under the three-parameter subgroup of the Poincaré group can be used to reduce any Poincaré invariant system of PDE. But conditionally-invariant Ansatz (22) can be used for Eq. (9) only. It means that the last Ansatz contains more precise information about structure of solutions of the equation under study.

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