On non-Lie ansatzes and new exact solutions of the classical Yang–Mills equations

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We suggest an effective method for reducing Yang–Mills equations to systems of ordinary differential equations. With the use of this method, we construct wide families of new exact solutions of the Yang–Mills equations. Analysis of the solutions obtained shows that they correspond to conditional symmetry of the equations under study.

1 Introduction

The majority of papers devoted to construction of the explicit form of exact solutions of the $SU(2)$ Yang–Mills equations (YME)

$$\begin{aligned}&\partial_\nu\partial^\nu\vec{A}_\mu-\partial^\mu\partial_\nu\vec{A}_\nu+e[(\partial_\nu\vec{A}_\nu)\times\vec{A}_\mu-2(\partial_\nu\vec{A}_\mu)\times\vec{A}_\nu+(\partial^\mu\vec{A}_\nu)\times\vec{A}^\nu]+\\&\quad+e^2\vec{A}_\nu\times(\vec{A}^\nu\times\vec{A}_\mu)=\vec{0}\end{aligned}\tag{1}$$

is based on the ansatzes for the Yang–Mills field $\vec{A}_\mu(x)$ suggested by Wu and Yang, Rosen, 't Hooft, Corrigan and Fairlie, Wilczek, Witten (see [1] and references therein). And what is more, the above ansatzes were obtained in a non-algorithmic way, i.e., there was no regular and systematic method for constructing such ansatzes.

Since one has only a few distinct exact solutions of YME, it is difficult to give their reliable and self-consistent physical interpretation. That is why, the problem of prime importance is the development of an effective regular approach for constructing new exact solutions of the system of nonlinear partial differential equations (PDE) (1) (see also [1]).

A natural approach to construction of particular solutions of YME (1) is to utilize their symmetry properties in the way as it is done in [2–4, 13]. Apparatus of the theory of Lie transformation groups makes it possible to reduce the system of PDE (1) to systems of nonlinear ordinary differential equations (ODE) by using special ansatzes (invariant solutions) [5, 6]. If one succeeds in constructing general or particular solutions of the said ODE (which is extremely difficult problem), then substituting results into the corresponding ansatzes, one gets exact solutions of the initial system of PDE (1).

Another possibility of construction of exact solutions of YME is to use their conditional (non-Lie) symmetry (for more details about conditional symmetry of equations of mathematical physics, see [7, 8] and also [9]). But the prospects of a systematic and exhaustive study of conditional symmetry of the system of twelve second-order nonlinear PDE (1) seem to be rather obscure. It should be said that so far we have no complete description of conditional symmetry of a nonlinear wave equation even in the case of one space variable.

In [9] we suggested an effective approach to study of conditional symmetry of the nonlinear Dirac equation based on its Lie symmetry. We have observed that all

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Poincaré-invariant ansatzes can be represented in the unified form by introducing several arbitrary elements (functions) $u_1(x), u_2(x), \dots, u_N(x)$. As a result, we get an ansatz for the Dirac field which reduces the nonlinear Dirac equation to a system of ODE provided functions $u_i(x)$ satisfy some compatible over-determined system of nonlinear PDE. After integrating it, we have obtained a number of new ansatzes that cannot in principle be obtained within the framework of the classical Lie approach.

In the present paper we construct a number of new exact solutions of YME (1) with the aid of the above described approach.

2 Reduction of YME

In the papers [2, 13] we adduce a complete list of $P(1,3)$-inequivalent ansatzes for the Yang–Mills field which are invariant under three-parameter subgroups of the Poincaré group $P(1,3)$. Analyzing these ansatzes, we come to the conclusion that they can be represented in the following unified form:

$$\vec{A}_\mu(x) = R_{\mu\nu}(x)\vec{B}^\nu(\omega), \tag{2}$$

where $\vec{B}_\nu(\omega)$ are new unknown vector-functions, $\omega = \omega(x)$ is a new independent variable, functions $R_{\mu\nu}(x)$ are given by

$$\begin{aligned} R_{\mu\nu}(x) = {} & (a_\mu a_\nu - d_\mu d_\nu)\operatorname{ch}\theta_0 + (a_\mu d_\nu - d_\mu a_\nu)\operatorname{ch}\theta_0 + \\ & + 2(a_\mu d_\mu)[(\Theta_1\cos\theta_3 + \theta_2\sin\Theta_3)b_\nu + (\theta_2\cos\theta_3 - \theta_1\sin\theta_3)c_\nu + \\ & + (\theta_1^2 + \theta_2^2)e^{-\theta_0}(a_\nu + d_\nu)] - (c_\mu c_\nu + b_\mu b_\nu)\cos\theta_3 - \\ & - (c_\mu b_\nu - b_\mu c_\nu)\sin\theta_3 - 2e^{-\theta_0}(\theta_1 b_\mu + \theta_2 c_\mu)(a_\nu + d_\nu). \end{aligned} \tag{3}$$

In (3) $\theta_\mu(x)$ are some smooth functions and what is more, $\theta_a = \theta_a(\xi, b_\mu x^\mu, c_\mu x^\mu)$, $a = 1, 2$, $\xi = \frac{1}{2}k_\mu x^\mu = \frac{1}{2}(a_\mu x^\mu + d_\mu x^\mu)$; a_μ, b_μ, c_μ, d_μ are arbitrary constants satisfying the following relations:

$$\begin{aligned} & a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1, \\ & a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0. \end{aligned}$$

Hereafter, summation over the repeated indices from 0 to 3 is understood. Rising and lowering of the indices is performed with the help of the tensor $g_{\mu\nu} = \operatorname{diag}(1, -1, -1, -1)$, i.e. $R^\alpha_\mu = g_{\alpha\beta}R_{\beta\mu}$.

The choice of the functions $\omega(x)$, $\theta_\mu(x)$ is determined by the requirement that substitution of the ansatz (2) into YME yields a system of ordinary differential equations for the vector function $\vec{B}_\mu(\omega)$.

By a direct check, one can become convinced of that the following assertion holds true.

Lemma. *Ansatz (2), (3) reduces YME (1) to a system of ODE if the functions $\omega(x)$, $\theta_\mu(x)$ satisfy the system of PDE*

$$\begin{aligned} & 1. \quad \omega_{x_\mu}\omega_{x^\mu} = F_1(\omega), \\ & 2. \quad \Box\omega = F_2(\omega), \\ & 3. \quad R_{\alpha\mu}\omega_{x_\alpha} = G_\mu(\omega), \\ & 4. \quad R_{\alpha\mu x_\alpha} = H_\mu(\omega), \end{aligned} \tag{4}$$

5. $R^\alpha_\mu R_{\alpha\nu x_\beta}\omega_{x\beta} = Q_{\mu\nu}(\omega)$,
6. $R^\alpha_\mu \Box R_{\alpha\nu} = S_{\mu\nu}(\omega)$,
7. $R^\alpha_\mu R_{\alpha\nu x_\beta} R_{\beta\gamma} + R^\alpha_\nu R_{\alpha\gamma x_\beta} R_{\beta\mu} + R^\alpha_\gamma R_{\alpha\mu x_\beta} R_{\beta\nu} = T_{\mu\nu\gamma}(\omega)$,

where $F_1, F_2, G_\mu, \ldots, T_{\mu\nu\gamma}$ are some smooth functions, $\mu,\nu,\gamma = \overline{0,3}$. And what is more, the reduced equation has the form

$$\begin{aligned} & k_{\mu\gamma}\ddot{\vec{B}}^\gamma + l_{\mu\gamma}\dot{\vec{B}}^\gamma + m_{\mu\gamma}\vec{B}^\gamma + eq_{\mu\nu\gamma}\dot{\vec{B}}^\nu \times \vec{B}^\gamma + \\ & \qquad + eh_{\mu\nu\gamma}\vec{B}^\nu \times \vec{B}^\gamma + e^2\vec{B}_\gamma \times (\vec{B}^\gamma \times \vec{B}_\mu) = \vec{0}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} & k_{\mu\gamma} = g_{\mu\gamma}F_1 - G_\mu g_\gamma, \\ & l_{\mu\gamma} = g_{\mu\gamma}F_2 + 2Q_{\mu\gamma} - G_\mu H_\gamma - G_\mu \dot{G}_\gamma, \\ & m_{\mu\gamma} = S_{\mu\gamma} - G_\mu \dot{H}_\gamma, \\ & q_{\mu\nu\gamma} = g_{\mu\gamma}G_\nu + g_{\nu\gamma}G_\mu - 2g\mu\nu G_\gamma, \\ & h_{\mu\nu\gamma} = \frac{1}{2}(g_{\mu\gamma}H_\nu - g_{\mu\nu}H_\gamma) - T_{\mu\nu\gamma}. \end{aligned} \tag{6}$$

Thus, to describe all ansatzes of the form (2), (3) reducing YME to a system of ODE, one has to construct the general solution of the over-determined system of PDE (3), (4). Let us emphasize that system (3), (4) is compatible, since ansatzes invariant under the Poincaré group satisfy equations (3), (4) with some specific choice of the functions $F_1, F_2, \ldots, T_{\mu\nu\gamma}$.

Integration of the system of nonlinear PDE (3), (4) demands a huge amount of computations. That is why, we present here only the principal idea of our approach to solving system (3), (4). When integrating it, we use essentially the fact that the general solution of the system of equations 1, 2 from (4) is known [10]. With already known $\omega(x)$, we proceed to integration of the linear PDE 3, 4 from (4). Next, we substitute the results obtained into the remaining equations and get the final form of the functions $\omega(x)$, $\theta_\mu(x)$.

Before presenting the results of integration of the system of PDE (3), (4), we make a remark. As a direct check shows, the structure of the ansatzes (2), (7) is not altered by the change of variables

$$\begin{aligned} & \omega \to \omega' = T(\omega), \quad \theta_0 \to \theta'_0 = \theta_0 + T_0(\omega), \\ & \theta_1 \to \theta'_1 = \theta_1 + e^{\theta_0}(T_1(\omega)\cos\theta_3 + T_2(\omega)\sin\theta_3), \\ & \theta_2 \to \theta'_2 = \theta_2 + e^{\theta_0}(T_2(\omega)\cos\theta_3 - T_1(\omega)\sin\theta_3), \\ & \theta_3 \to \theta'_3 = \theta_3 + T_3(\omega), \end{aligned} \tag{7}$$

where $T(\omega)$, $T_\mu(\omega)$ are arbitrary smooth functions. That is why, solutions of system (3), (4) connected by the relations (7) are considered as equivalent.

It occurs that new (non-Lie) ansatzes are obtained, if functions $\omega(x)$, $\theta_\mu(x)$ up to the equivalence relations (7) have the form

$$\begin{aligned} & \theta_\mu = \theta_\mu(\xi, b_\nu x^\nu, c_\nu x^\nu), \quad \mu = \overline{0,3}, \\ & \omega = \omega(\xi, b_\nu x^\nu, c_\nu x^\nu), \end{aligned} \tag{8}$$

where $\xi = \frac{1}{2}k_\nu x^\nu$, $k_\nu = a_\nu + d_\nu$.

The list of inequivalent solutions of the system of PDE (3), (4) satisfying (8) is exhausted by the following solutions:

$$
\begin{aligned}
&1.\quad \theta_0=\theta_3=0,\quad \omega=\frac{1}{2}k_\nu x^\nu,\\
&\qquad \theta_1=w_0(\xi)b_\mu x^\mu+w_1(\xi)c_\mu x^\mu,\quad \theta_2=w_2(\xi)b_\mu x^\mu+w_3(\xi)c_\mu x^\mu;\\
&2.\quad \omega=b_\mu x^\mu+w_1(\xi),\quad \theta_0=\alpha\big(c_\mu x^\mu+w_2(\xi)\big),\\
&\qquad \theta_a=-\frac{1}{4}\dot w_a(\xi),\quad a=\overline{1,2},\quad \theta_3=0;\\
&3.\quad \theta_0=T(\xi),\quad \theta_3=w_1(\xi),\\
&\qquad \omega=b_\mu x^\mu\cos w_1+c_\mu x^\mu\sin w_1+w_2(\xi),\\
&\qquad \theta_1=\left[\frac{1}{4}(\varepsilon e^T+\dot T)(b_\mu x^\mu\sin w_1-c_\mu x^\mu\cos w_1)+w_3(\xi)\right]\sin w_1+\\
&\qquad\qquad +\frac{1}{4}\big[\dot w_1(b_\mu x^\mu\sin w_1-c_\mu x^\mu\cos w_1)-\dot w_2\big]\cos w_1,\\
&\qquad \theta_2=-\left[\frac{1}{4}(\varepsilon e^T+\dot T)(b_\mu x^\mu\sin w_1-c_\mu x^\mu\cos w_1)+w_3(\xi)\right]\cos w_1+\\
&\qquad\qquad +\frac{1}{4}\big[\dot w_1(b_\mu x^\mu\sin w_1-c_\mu x^\mu\cos w_1)-\dot w_2\big]\sin w_1;\\
&4.\quad \theta_0=0,\quad \theta_3=\mathrm{arctg}\big[\big(c_\mu x^\mu+w_2(\xi)\big)\big(b_\mu x^\mu+w_1(\xi)\big)^{-1}\big],\\
&\qquad \theta_a=-\frac{1}{4}\dot w_a(\xi),\quad a=1,2,\quad \omega=\big[\big(b_\mu x^\mu+w_1(\xi)\big)^2+\big(c_\mu x^\mu+w_2(\xi)\big)^2\big]^{1/2}.
\end{aligned}
\tag{9}
$$

Here $\alpha\neq 0$, ε are arbitrary constants, w_0, w_1, w_2, w_3 are arbitrary smooth functions of $\xi=\frac{1}{2}k_\mu x^\mu$, $T=T(\xi)$ is a solution of the nonlinear ODE

$$
\big(\dot T+\varepsilon e^T\big)^2+\dot w_1^2=\varkappa e^{2T},\quad \varkappa\in\mathbb{R}^1. \tag{10}
$$

Substitution of the ansatz (2), where $R_{\mu\nu}(x)$ are given by formulae (3), (9), into YME yields systems of nonlinear ODE of the form (5), where

$$
\begin{aligned}
&1.\quad k_{\mu\gamma}=-\frac{1}{4}k_\mu k_\gamma,\quad l_{\mu\gamma}=-(w_0+w_3)k_\mu k_\gamma,\\
&\qquad m_{\mu\gamma}=-4(w_0^2+w_1^2+w_2^2+w_3^2)k_\mu k_\gamma-(\dot w_0+\dot w_3)k_\mu k_\gamma,\\
&\qquad q_{\mu\nu\gamma}=\frac{1}{2}(g_{\mu\gamma}k_\nu+g_{\nu\gamma}k_\mu-2g_{\mu\nu}k_\gamma),\\
&\qquad h_{\mu\nu\gamma}=(w_0+w_3)(g_{\mu\gamma}k_\nu-g_{\mu\nu}k_\gamma)+2\ (w_1-w_2)\big[(k_\mu b_\nu-k_\nu b_\mu)c_\gamma+\\
&\qquad\qquad +(b_\mu c_\nu-b_\nu c_\mu)k_\gamma+(c_\mu k_\nu-c_\nu k_\mu)b_\gamma\big];\\
&2.\quad k_{\mu\gamma}=-g_{\mu\gamma}-b_\mu b_\gamma,\quad l_{\mu\gamma}=0,\quad m_{\mu\gamma}=-\alpha^2(a_\mu a_\gamma-d_\mu d_\gamma),\\
&\qquad q_{\mu\nu\gamma}=g_{\mu\gamma}b_\nu+g_{\nu\gamma}b_\mu-2g_{\mu\nu}b_\gamma,\\
&\qquad h_{\mu\nu\gamma}=\alpha\big[(a_\mu d_\nu-a_\nu d_\mu)c_\gamma+(d_\mu c_\nu-d_\nu c_\mu)a_\gamma+(c_\mu a_\nu c_\nu a_\mu)d_\gamma\big];\\
&3.\quad k_{\mu\gamma}=-g_{\mu\gamma}-b_\mu b_\gamma,\quad l_{\mu\gamma}=-\frac{\varepsilon}{2}b_\mu k_\gamma,\quad m_{\mu\gamma}=-\frac{\varkappa}{4}k_\mu k_\gamma,\\
&\qquad q_{\mu\nu\gamma}=g_{\mu\gamma}b_\nu+g_{\nu\gamma}b_\mu-2g_{\mu\nu}b_\gamma,\quad h_{\mu\nu\gamma}=\frac{\varepsilon}{4}(g_{\mu\gamma}k_\nu-g_{\mu\nu}k_\gamma);\\
&4.\quad k_{\mu\gamma}=-g_{\mu\gamma}-b_\mu b_\gamma,\quad l_{\mu\gamma}=-\omega^{-1}(g_{\mu\gamma}+b_\mu b_\gamma),\quad m_{\mu\gamma}=-\omega^{-2}c_\mu c_\gamma,\\
&\qquad q_{\mu\nu\gamma}=g_{\mu\gamma}b_\nu+g_{\nu\gamma}b_\mu-2g_{\mu\nu}b_\gamma,\quad h_{\mu\nu\gamma}=\frac{1}{2}\omega^{-1}(g_{\mu\gamma}b_\nu-g_{\mu\nu}b_\gamma).
\end{aligned}
\tag{11}
$$

3 Exact solutions of the nonlinear Yang–Mills equations

Systems (5), (11) are systems of twelve nonlinear second-order ODE with variable coefficients. That is why, there is a little hope to construct their general solutions. But it is possible to obtain particular solutions of system (5), which coefficients are given by the formulae 2–4 from (11).

Consider, as an example, the system of ODE (5) with coefficients given by the formulae 2 from (11). We look for its solutions in the form

$$\vec{B}_\mu = k_\mu \vec{e}_1 f(\omega) + b_\mu \vec{e}_2 g(\omega), \quad fg \neq 0, \tag{12}$$

where $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$.

Substituting the expression (12) into the above mentioned system, we get

$$\ddot{f} + (\alpha^2 - e^2 g^2) f = 0, \quad f\dot{g} + 2\dot{f}g = 0. \tag{13}$$

The second ODE from (13) is easily integrated

$$g = \lambda\ f^{-2}, \quad \lambda \in \mathbb{R}^1, \quad \lambda \neq 0. \tag{14}$$

Substitution of the result obtained into the first ODE from (13) yields the Ermakov-type equation for $f(\omega)$

$$\ddot{f} + \alpha^2 f - e^2\lambda^2\ f^{-3} = 0,$$

which is integrated in elementary functions [11]

$$f = \left[\alpha^{-2}C^2 + \alpha^{-2}\left(C^4 - \alpha^2 e^2 \lambda^2\right)^{1/2} \sin 2|\alpha|\omega\right]^{1/2}. \tag{15}$$

Here $C \neq 0$ is an arbitrary constant.

Substituting (12), (14), (15) into the corresponding ansatz for $\vec{A}_\mu(x)$, we get the following class of exact solutions of YME (1):

$$\begin{aligned}\vec{A}_\mu = {} & \vec{e}_1 k_\mu \exp\left(-\alpha cx - \alpha w_2\right)\left[\alpha^{-2}C^2 + \alpha^{-2}\left(C^4 - \alpha^2 e^2\lambda^2\right)^{1/2} \times \right. \\ & \left. \times \sin 2|\alpha|(bx + w_1)\right]^{1/2} + \vec{e}_2 \lambda\left[\alpha^{-2}C^2 + \alpha^{-2}\left(C^4 - \alpha^2 e^2 \lambda^2\right) \times \right. \\ & \left. \times \sin 2|\alpha|(bx + w_1)\right]^{-1}\left(b_\mu + \frac{1}{2}\ k_\mu \dot{w}_1\right).\end{aligned}$$

In a similar way, we have obtained five other classes of exact solutions of the Yang–Mills equations

$$\begin{aligned}\vec{A}_\mu = {} & \vec{e}_1 k_\mu e^{-T}\left(bx\cos w_1 + cx\sin w_1 + w_2\right)^{1/2} Z_{1/4}\Big(\frac{ie\lambda}{2}(bx\cos w_1 + \\ & + cx\sin w_1 + w_2)^2\Big) + \vec{e}_2\lambda(bx\cos w_1 + cx\sin w_1 + w_2) \times \\ & \times\left[c_\mu \cos w_1 - b_\mu \sin w_1 + 2k_\mu\Big(\frac{1}{4}\left(\varepsilon e^T + \dot{T}\right)\left(bx\sin w_1 - cx\cos w_1\right) + w_3\Big)\right];\end{aligned}$$

$$\begin{aligned}\vec{A}_\mu = {} & \vec{e}_1 k_\mu e^{-T}\left[C_1 \operatorname{ch} e\lambda\left(bx\cos w_1 + cx\sin w_1 + w_2\right) + C_2 \operatorname{sh} e\lambda\big(bx\cos w_1 + \right. \\ & \left. + cx\sin w_1 + w_2\big)\right] + \vec{e}_2\lambda\big[C_\mu \cos w_1 - b_\mu \sin w_1 + \\ & + 2k_\mu\Big(\frac{1}{4}\left(\varepsilon e^T + \dot{T}\right)(bx\sin w_1 - cx\cos w_1) + w_3\Big)\big];\end{aligned}$$

$$\begin{aligned}
\vec{A}_\mu &= \vec{e}_1 k_\mu e^{-T}\bigl[C^2\bigl(bx\cos w_1 + cx\sin w_1 + w_2\bigr)^2 + \lambda^2 e^2 C^{-2}\bigr]^{1/2} + \\
&\quad + \vec{e}_2\lambda\bigl[C^2(bx\cos w_1 + cx\sin w_1 + w_2)^2 + \lambda^2 e^2 C^{-2}\bigr]^{-1} \times \\
&\quad \times \Bigl\{b_\mu \cos w_1 + C_\mu \sin w_1 - \frac{1}{2}k_\mu\bigl[\dot{w}_1(bx\sin w_1 - cx\cos w_1) - \dot{w}_2\bigr]\Bigr\}; \\
\vec{A}_\mu &= \vec{e}_1 k_\mu Z_0\Bigl(\frac{ie\lambda}{2}\bigl[(bx+w_1)^2 + (cx+w_2)^2\bigr]\Bigr) + \vec{e}_2\lambda \times \\
&\quad \times \Bigl\{c_\mu(bx+w_1) - b_\mu(cx+w_2) - \frac{1}{2}k_\mu\bigl[\dot{w}_1(cx+w_2) - \dot{w}_2(bx+w_1)\bigr]\Bigr\}; \\
\vec{A}_\mu &= \vec{e}_1 k_\mu\bigl[C_1\bigl((bx+w_1)^2 + (cx+w_2)^2\bigr)^{e\lambda/2} + \\
&\quad + C_2\bigl((bx+w_1)^2 + (cx+w_2)^2\bigr)^{-e\lambda/2} + \vec{e}_2\lambda\bigl[(bx+w_1)^2 + (cx+w_2)^2\bigr]^{-1} \times \\
&\quad \times \Bigl\{c_\mu(bx+w_1) - b_\mu(cx+w_2) - \frac{1}{2}\, k_\mu\bigl[\dot{w}_1(cx+w_2) - \dot{w}_2(bx+w_1)\bigr]\Bigr\}.
\end{aligned}$$

Here $C_1, C_2, C \neq 0$, ε, λ are arbitrary parameters; w_1, w_2, w_3 are arbitrary smooth functions of $\xi = \frac{1}{2}kx$, $T = T(\xi)$ is a solution of ODE (10).

Besides that, we use the following notations:

$$\begin{aligned}
&kx = k_\mu x^\mu, \quad bx = b_\mu x^\mu, \quad cx = c_\mu x^\mu, \\
&Z_s(\omega) = C_1 J_s(\omega) + C_2 Y_s(\omega), \quad \vec{e}_1 = (1,0,0), \quad \vec{e}_2 = (0,1,0),
\end{aligned}$$

where J_s, Y_s are Bessel functions. Thus, we have obtained the wide families of exact non-Abelian solutions of YME (1).

In conclusion we say a few words about a symmetry interpretation of the ansatzes (2), (7), (10). Let us consider, as an example, the ansatz determined by the formulae 1 from (9). As a direct computation shows, generators of the three-parameter Lie group leaving it invariant are of the form

$$\begin{aligned}
&Q_1 = k_\alpha \partial_\alpha, \\
&Q_2 = b_\alpha \partial_\alpha - \Bigl\{\bigl[w_0(k_\mu b_\nu - k_\nu b_\mu) + w_2(k_\mu c_\nu - k_\nu c_\mu)\bigr]\sum_{a=1}^{3} A^{a\nu}\Bigr\}\partial_{A^{a\mu}}, \\
&Q_3 = c_\alpha \partial_\alpha - 2\Bigl\{\bigl[w_1(k_\mu b_\nu - k_\nu b_\mu) + w_3(k_\mu c_\nu - k_\nu c_\mu)\bigr]\sum_{a=1}^{3} A^{a\nu}\Bigr\}\partial_{A^{a\mu}}.
\end{aligned} \tag{16}$$

Evidently, the system of PDE (1) is invariant under the one-parameter group having the generator Q_1. But it is not invariant under the groups having the generators Q_2, Q_3. At the same time, the system of PDE

$$\begin{aligned}
&\partial_\nu\partial^\nu \vec{A}_\mu - \partial^\mu\partial_\nu \vec{A}_\nu + e\bigl[(\partial_\nu \vec{A}_\nu)\times\vec{A}_\mu - 2(\partial_\nu \vec{A}_\mu)\times\vec{A}_\nu + (\partial^\mu \vec{A}_\nu)\times\vec{A}^\nu\bigr] + \\
&\qquad + e^2 \vec{A}_\nu \times (\vec{A}^\nu \times \vec{A}_\mu) = \vec{0}, \\
&Q_0\vec{A}_\mu \equiv k_\alpha\partial_\alpha \vec{A}_\mu = \vec{0}, \\
&Q_1\vec{A}_\mu \equiv b_\alpha\partial_\alpha \vec{A}_\mu + 2\bigl[w_0(k_\mu b_\nu - k_\nu b_\mu) + w_2(k_\mu c_\nu - k_\nu c_\mu)\bigr]\vec{A}^\nu = \vec{0}, \\
&Q_2\vec{A}_\mu \equiv c_\alpha\partial_\alpha \vec{A}_\mu + 2\bigl[w_1(k_\mu b_\nu - k_\nu b_\mu) + w_3(k_\mu c_\nu - k_\nu c_\mu)\bigr]\vec{A}^\nu = \vec{0}
\end{aligned}$$

is invariant under the said group. Consequently, YME (1) are conditionally-invariant under the Lie algebra (16). It means that the solutions of YME obtained with the help of the ansatz invariant under the group with the generators (16) can not be found by the classical symmetry reduction procedure.

As rather tedious computations show, the ansatzes determined by the formulae 2–4 from (9) also correspond to conditional symmetry of YME. Hence it follows, in particular, that YME should be included into the long list of mathematical and theoretical physics equations possessing a nontrivial conditional symmetry [12].

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