# Symmetry reduction and exact solutions of the Yang–Mills equations

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We present a detailed account of symmetry properties of $SU(2)$ Yang–Mills equations. Using a subgroup structure of the Poincaré group $P(1,3)$ we have constructed all $P(1,3)$-inequivalent ansatzes for the Yang-Mills field which are invariant under the three-dimensional subgroups of the Poincaré group. With the aid of these ansatzes reduction of Yang-Mills equations to systems of ordinary differential equations is carried out and wide families of their exact solutions are constructed.

## 1 Introduction

Since Newton's and Euler's works, exact solutions of differential equations describing physical processes were highly estimated. Green, Lame, Liouville, Cayley, Donkin, Stokes, Kirchhoff, Poincaré, Stieltjes, Forsyth, Volterra, Appel, Macdonald, Weber, Bateman, Whittaker, Sommerfeld and many other famous researchers constructed different classes of exact solutions of linear Laplace, d'Alembert, heat, and Maxwell equations.

Nowadays, this constructive branch of mathematical physics is not so popular as earlier. But if one wants to have some nontrivial information on solutions of basic motion equations in quantum mechanics, field theory, gravitation theory, acoustics, and hydrodynamics, then the more intensive research work should be carried out in order to develop analytical methods of solution of partial differential equations (PDE). And what is more, unlike the mathematical physics of the 19th century, modern mathematical physics is essentially nonlinear. It means that all principal equations of modern physics, biology and chemistry are nonlinear. This fact complicates very much the problem of constructing their exact solutions (see, e.g. [1] and references therein).

Up to now, we have comparatively few papers devoted to construction of exact solutions of nonlinear multi-dimensional d'Alembert, Maxwell, Schrödinger, Dirac, Maxwell–Dirac, Yang–Mills equations. Whereas, a huge amount of papers and monographs are devoted to construction of exact solutions of equations for gravitational field. It is difficult even to estimate the number of papers and monographs, where the soliton solutions of the one-dimensional nonlinear KdV, Schrödinger and Sine-Gordon equations are studied. We are sure that the above mentioned equations should deserve much more attention of researchers in mathematical physics.

With the present paper we start a series of papers devoted to construction of new classes of exact solutions of the classical Yang–Mills equations (YME) with the use of their Lie and non-Lie symmetry. Here we study in detail symmetry reduction of YME by Poincaré-invariant ansatzes and obtain wide families of its exact Poincaré-invariant solutions.

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By the classical YME, we mean the following nonlinear system of twelve second-order PDE:

$$\partial_\nu\partial^\nu \vec{A}_\mu - \partial^\mu\partial_\nu \vec{A}_\nu + e[(\partial_\nu \vec{A}_\nu)\times\vec{A}_\mu - 2(\partial_\nu\vec{A}_\mu)\times\vec{A}_\nu + (\partial^\mu\vec{A}_\nu)\times\vec{A}^\nu] + \\ + e^2\vec{A}_\nu\times(\vec{A}^\nu\times\vec{A}_\mu) = \vec{0}. \tag{1.1}$$

Here $\partial_\nu = \frac{\partial}{\partial x_\nu}$, $\mu,\nu=\overline{0,3}$, $e=\mathrm{const}$, $\vec{A}_\mu = \vec{A}_\mu(x_0,x_1,x_2,x_3)$ is the three-component vector-potential of the Yang–Mills field (called, for bravity, the Yang–Mills field). Hereafter, the summation over the repeated indices $\mu,\nu$ from 0 to 3 is understood. Raising and lowering the vector indices is performed with the aid of the metric tensor

$$g_{\mu\nu} = \begin{cases} 1, & \mu=\nu=0, \\ -1, & \mu=\nu=1,2,3, \\ 0, & \mu\neq\nu \end{cases}$$

(i.e. $\partial^\mu = g_{\mu\nu}\partial_\nu$).

It should be said that there were several reviews devoted to classical solutions of YME (see [2] and the literature cited there). But, in fact, symmetry properties of YME were not used. The solutions were obtained with the help of ad hoc substitutions suggested by Wu and Yang, Rosen, 't Hooft, Corrigan and Fairlie, Wilczek, Witten (for more detail, see [2]).

The structure of our paper is as follows. In the second Section we give all necessary information about symmetry properties of YME and about a solution generation procedure by virtue of the finite transformations of the symmetry group admitted by YME. In Section 3 we construct $P(1,3)$-inequivalent ansatzes for the Yang–Mills field invariant under the three-parameter subgroups of the Poincaré group. Section 4 is devoted to reduction of YME to systems of ordinary differential equations (ODE). Integrating these in Section 5 we construct multi-parameter families of exact solutions of YME. In Section 6 we consider some generalizations of the solutions obtained and, in particular, construct the generalization of Coleman's solution.

## 2 Symmetry and solution generation for the Yang–Mills equations

It was known long ago that YME are invariant with respect to the group $C(1,3)\otimes SU(2)$, where $C(1,3)$ is the 15-parameter conformal group having the following generators:

$$\begin{aligned} &P_\mu = \partial_\mu, \\ &J_{\alpha\beta} = x^\alpha\partial_\beta - x^\beta\partial_\alpha + A^{a\alpha}\partial_{A^a_\beta} - A^{a\beta}\partial_{A^a_\alpha}, \\ &D = x_\mu\partial_\mu - A^a_\mu\partial_{A^a_\mu}, \\ &K_\mu = 2x^\mu D - (x_\nu x^\nu)\partial_\mu + 2A^{a\mu}x_\nu\partial_{A^a_\nu} - 2A^a_\nu x^\nu\partial_{A^a_\mu}, \end{aligned} \tag{2.1}$$

and $SU(2)$ is the infinite-parameter special unitary group with the following basis generator:

$$Q = (\varepsilon_{abc}A^b_\mu w^c(x) + e^{-1}\partial_\mu w^a(x))\partial_{A^a_\mu}. \tag{2.2}$$

In (2.1), (2.2) $\partial_{A^a_\mu} = \frac{\partial}{\partial A^a_\mu}$, $w^c(x)$ are arbitrary smooth functions, $\varepsilon_{abc}$ is the third-order anti-symmetrical tensor with $\varepsilon_{123} = 1$. Hereafter, summation over the repeated indices $a$, $b$, $c$ from 1 to 3 is understood.

But the fact that the group with generators (2.1), (2.2) is a maximal (in Lie's sense) invariance group admitted by YME was established only recently [3] with the use of a symbolic computation technique. The only explanation for this situation is a very cumbersome structure of the system of PDE (1.1). As a consequence, realization of the Lie algorithm of finding the maximal invariance group admitted by YME demands a huge amount of computations. This difficulty had been overcome with the aid of computer facilities.

One of the remarkable possibilities provided by the fact that the considered equation admits a nontrivial symmetry group gives the possibility of getting new solutions from the known ones by the solution generation technique [1, 4]. This technique is based on the following assertion.

**Lemma.** *Let*

$$x'_\mu = f_\mu(x, u, \tau), \quad \mu = \overline{0, n-1},$$
$$u'_a = g_a(x, u, \tau), \quad a = \overline{1, N},$$

*where* $\tau = (\tau_1, \tau_2, \ldots, \tau_r)$ *be the* $r$*-parameter invariance group of some system of PDE and* $U_a(x)$*,* $a = \overline{1, N}$ *be its particular solution. Then the* $N$*-component function* $u_a(x)$ *determined by implicit formulae*

$$U_a(f(x, u, \tau)) = g_a(x, u, \tau), \quad a = \overline{1, N} \tag{2.3}$$

*is also a solution of the same system of PDE.*

To make use of the above assertion we need formulae for finite transformations generated by infinitesimal operators (2.1), (2.2). We adduce these formulae following [1, 2].

1. The group of translations (generator $X = \tau_\mu P_\mu$)

$$x'_\mu = x_\mu + \tau_\mu, A^{d\,\prime}_\mu = A^d_\mu.$$

2. The Lorentz group $O(1, 3)$

a) the group of rotations (generator $X = \tau J_{ab}$)

$$x'_0 = 0, \quad x'_c = x_c, \quad c \neq a, \quad c \neq b,$$
$$x'_a = x_a \cos\tau + x_b \sin\tau,$$
$$x'_b = x_b \cos\tau - x_a \sin\tau,$$
$$A^{d\,\prime}_0 = A^d_0, \quad A^{d\,\prime}_c = A^d_c, \quad c \neq a, \quad c \neq b,$$
$$A^{d\,\prime}_a = A^d_a \cos\tau + A^d_b \sin\tau,$$
$$A^{d\,\prime}_b = A^d_b \cos\tau - A^d_a \sin\tau;$$

b) the group of Lorentz transformations (generator $X = \tau J_{0a}$)

$$x'_0 = x_0 \cosh\tau + x_a \sinh\tau,$$
$$x'_a = x_a \cosh\tau + x_0 \sinh\tau, \quad x'_b = x_b, \quad b \neq a,$$
$$A^{d\,\prime}_0 = A^d_0 \cosh\tau + A^d_a \sinh\tau,$$
$$A^{d\,\prime}_a = A^d_a \cosh\tau + A^d_0 \sinh\tau, \quad A^{d\,\prime}_b = A^d_b, \quad b \neq a.$$

3. The group of scale transformations (generator $X = \tau D$)

$$x'_\mu = x_\mu e^\tau, \quad A^{d\prime}_\mu = A^d_\mu e^{-\tau}.$$

4. The group of conformal transformations (generator $X = \tau_\mu K^\mu$)

$$x'_\mu = (x_\mu - \tau_\mu x_\nu x^\nu)\sigma^{-1}(x),$$
$$A^{d\prime}_\mu = [g_{\mu\nu}\sigma(x) + 2(x_\mu\tau_\nu - x_\nu\tau_\mu + 2\tau_\alpha x^\alpha \tau_\mu x_\nu - x_\alpha x^\alpha \tau_\mu \tau_\nu - \tau_\alpha\tau^\alpha x_\mu x_\nu]A^{d\nu}.$$

5. The group of gauge transformations (generator $X = Q$)

$$x'_\mu = x_\mu,$$
$$A^{d\prime}_\mu = A^d_\mu \cos w + \varepsilon_{dbc} A^b_\mu n^c \sin w + 2n^d n^b A^b_\mu \sin^2\frac{w}{2} +$$
$$+ e^{-1}\left[\frac{1}{2}n^d \partial_\mu w + \frac{1}{2}(\partial_\mu n^d)\sin w + \varepsilon_{dbc}(\partial_\mu n^b)n^c\right].$$

In the above formulae $\sigma(x) = 1 - \tau_\alpha x^\alpha + (\tau_\alpha\tau^\alpha)(x_\beta x^\beta)$, $n^a = n^a(x)$ is a unit vector determined by the equality $w^a(x) = w(x)n^a(x)$, $a = \overline{1,3}$.

Using the Lemma it is not difficult to obtain formulae for generating solutions of YME by the above transformation groups. We adduce them omitting derivation (see also [3]).

1. The group of translations

$$A^a_\mu(x) = u^a_\mu(x + \tau).$$

2. The Lorentz group

$$A^d_\mu(x) = a_\mu u^d_0(ax, bx, cx, dx) + b_\mu u^d_1(ax, bx, cx, dx) +$$
$$+ c_\mu u^d_2(ax, bx, cx, dx) + d_\mu u^d_3(ax, bx, cx, dx).$$

3. The group of scale transformations

$$A^d_\mu(x) = e^\tau u^d_\mu(xe^\tau).$$

4. The group of conformal transformations

$$A^d_\mu(x) = [g_{\mu\nu}\sigma^{-1}(x) + 2\sigma^{-2}(x)(x_\mu\tau_\nu - x_\nu\tau_\mu + 2\tau_\alpha x^\alpha\tau_\mu x_\nu -$$
$$- x_\alpha x^\alpha \tau_\mu\tau_\nu - \tau_\alpha\tau^\alpha x_\mu x_\nu)]u^{d\nu}((x - \tau(x_\alpha x^\alpha))\sigma^{-1}(x)).$$

5. The group of gauge transformations

$$A^d_\mu(x) = u^d_\mu \cos w + \varepsilon_{dbc} u^b_\mu n^c \sin w + 2n^d n^b u^b_\mu \sin^2\frac{w}{2} +$$
$$+ e^{-1}\left[\frac{1}{2}n^d\partial_\mu w + \frac{1}{2}(\partial_\mu n^d)\sin w + \varepsilon_{dbc}(\partial_\mu n^b)n^c\right].$$

Here $u^d_\mu(x)$ is an arbitrary given solution of YME; $A^d_\mu(x)$ is a new solution of YME; $\tau$, $\tau_\mu$ are arbitrary parameters; $a_\mu$, $b_\mu$, $c_\mu$, $d_\mu$ are arbitrary parameters satisfying the equalities

$$a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1,$$
$$a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0.$$

Besides that, we use the following designations: $x + \tau = \{x_\mu + \tau_\mu, \mu = \overline{0,3}\}$, $ax = a_\mu x^\mu$.

Thus, each particular solution of YME gives rise to a multi-parameter family of exact solutions by virtue of the above solution generation formulae.

# 3 Ansatzes for the Yang–Mills field

A key idea of the symmetry approach to the problem of reduction of PDE is a special choice of the form of a solution. This choice is dictated by a structure of the symmetry group admitted by the equation under study.

In the case involved, to reduce YME by $N$ variables one has to construct ansatzes for the Yang–Mills field $A^a_\mu(x)$ invariant under $N$-dimensional subalgebras of the algebra with the basis elements (2.1), (2.2) [1, 5]. Since we are looking for Poincaré-invariant ansatzes reducing YME to systems of ODE, $N$ is equal to 3. Due to invariance of YME under the Poincaré group $P(1,3)$, it is enough to consider only subalgebras which can not be transformed one into another by group transformation, i.e. $P(1,3)$-inequivalent subalgebras. Complete description of $P(1,3)$-inequivalent subalgebras of the Poincaré algebra was obtained in [6] (see also [7]).

According to the classical symmetry approach, to construct the ansatz invariant under the invariance algebra having the basis elements

$$X_a = \xi_{a\mu}(x,A)\partial_\mu + \eta^b_{a\mu}(x,A)\partial_{A^b_\mu}, \quad a=\overline{1,3}, \tag{3.1}$$

where $A=\{A^a_\mu, a=\overline{1,3}, \mu=\overline{0,3}\}$, one has

1) to construct a complete system of functionally-independent invariants of the operators (3.1) $\Omega=\{w_i(x,A), i=\overline{1,13}\}$;

2) to resolve relations

$$F_j(w_1(x,A),\dots,w_{13}(x,A))=0, \quad j=\overline{1,13} \tag{3.2}$$

with respect to the function $A^a_\mu$.

As a result, one gets the ansatz for the field $A^a_\mu(x)$ which reduces YME to the system of twelve nonlinear ODE.

**Note.** Equalities (3.2) can be resolved with respect to $A^a_\mu$, $a=\overline{1,3}$, $\mu=\overline{0,3}$ if the condition

$$\operatorname{rank}\|\xi_{a\mu}(x,A)\|^{3}_{a=1}\,{}^{3}_{\mu=0}=3 \tag{3.3}$$

holds. If (3.3) does not hold, the above procedure leads to partially-invariant solutions [5], which are not considered in the present paper.

In [1, 4] we established that the procedure of construction of invariant ansatzes could be essentially simplified if coefficients of operators $X_a$ have the following structure:

$$\xi_{a\mu}=\xi_{a\mu}(x), \quad \eta^b_{a\mu}=\rho^{bc}_{a\mu\nu}(x)A^c_\nu \tag{3.4}$$

(i.e. basis elements of the invariance algebra realize the linear representation). In this case, the invariant ansatz for the field $A^a_\mu(x)$ is searched for in the form

$$A^a_\mu(x)=Q^{ab}_{\mu\nu}(x)B^b_\nu(w(x)). \tag{3.5}$$

Here $B^b_\nu(w)$ are arbitrary smooth functions and $w(x)$, $Q^{ab}_{\mu\nu}(x)$ are particular solutions of the system of PDE

$$\begin{gathered}\xi_{a\mu}w_{x_\mu}=0, \quad a=\overline{1,3},\\ (\xi_{a\nu}\partial_\nu-\rho^{bc}_{a\mu\alpha})Q^{cd}_{\alpha\beta}=0, \quad \mu=\overline{0,3}, \quad a,b,d=\overline{1,3}.\end{gathered} \tag{3.6}$$

Basis elements of the Poincaré algebra $P_\mu$, $J_{\alpha\beta}$ from (2.1) evidently satisfy the conditions (3.4) and besides the equalities

$$\eta^b_{a\mu} = \rho_{a\mu\nu}(x)A^b_\nu, \quad a, b = \overline{1,3}, \quad \mu = \overline{0,3} \tag{3.7}$$

hold.

This fact permits further simplification of formulae (3.5), (3.6). Namely, the ansatz for the Yang–Mills field invariant under the 3-dimensional subalgebra of the Poincaré algebra with basis elements of the form (3.1), (3.7) should be looked for in the form

$$A^a_\mu = Q_{\mu\nu}(x)B^a_\nu(w(x)), \tag{3.8}$$

where $B^a_\nu(w)$ are arbitrary smooth functions and $w(x)$, $Q_{\mu\nu}(x)$ are particular solutions of the following system of PDE:

$$\xi_{a\mu}w_{x_\mu} = 0, \qquad a = \overline{1,3}, \tag{3.9}$$

$$\xi_{a\alpha}\partial_\alpha Q_{\mu\nu} - \rho_{a\mu\alpha}Q_{\alpha\nu} = 0, \quad a = \overline{1,3}, \quad \mu, \nu = \overline{0,3}. \tag{3.10}$$

Thus, to obtain the complete description of $P(1,3)$-inequivalent ansatzes for the field $A^a_\mu(x)$ invariant under 3-dimensional subalgebras of the Poincaré algebra, one has to integrate the over-determined system of PDE (3.9), (3.10) for each $P(1,3)$-inequivalent subalgebra. Let us note that compatibility of (3.9), (3.10) is guaranteed by the fact that operators $X_1$, $X_2$, $X_3$ form a Lie algebra.

Consider, as an example, the procedure of constructing ansatz (3.8) invariant under the subalgebra $\langle P_1, P_2, J_{03}\rangle$. In this case system (3.9) reads

$$w_{x_1} = 0, \quad w_{x_2} = 0, \quad x_0 w_{x_3} + x_3 w_{x_0} = 0,$$

whence $w = x_0^2 - x_3^2$.

Next, we note that coefficients $\rho_{1\mu\nu}$, $\rho_{2\mu\nu}$ of the operators $P_1$, $P_2$ are equal to zero, while coefficients $\rho_{3\mu\nu}$ form the following $(4\times 4)$ matrix

$$\|\rho_{3\mu\nu}\|^3_{\mu,\nu=0} = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix}$$

(we designate this constant matrix by the symbol $S$).

With account of the above fact, equations (3.10) take the form

$$Q_{x_1} = 0, \quad Q_{x_2} = 0, \quad x_0 Q_{x_3} + x_3 Q_{x_0} - SQ = 0, \tag{3.11}$$

where $Q = \|Q_{\mu\nu}(x)\|^3_{\mu,\nu=0}$ is a $(4\times 4)$-matrix.

From the first two equations of system (3.11) it follows that $Q = Q(x_0, x_3)$. Since $S$ is a constant matrix, a solution of the third equation can be looked for in the form (see, for example, [4])

$$Q = \exp\{f(x_0, x_3)S\}.$$

Substituting this expression into (3.11) we get

$$(x_0 f_{x_3} + x_3 f_{x_0} - 1)\exp\{fS\} = 0$$

or, equivalently,

$$x_0 f_{x_3} + x_3 f_{x_0} = 1,$$

whence $f = \ln(x_0 + x_3)$.

Consequently, a particular solution of equations (3.11) reads

$$Q = \exp\{\ln(x_0 + x_3)S\}.$$

Using an evident identity $S = S^3$ we get the following equalities:

$$\begin{aligned}Q &= \sum_{n=0}^{\infty}(n!)^{-1}(\ln(x_0 + x_3))^n S^n = \\ &= I + S[\ln(x_0 + x_3) + (3!)^{-1}(\ln(x_0 + x_3))^3 + \cdots] + \\ &\quad + S^2[(2!)^{-1}(\ln(x_0 + x_3))^2 + (4!)^{-1}(\ln(x_0 + x_3))^4 + \cdots] = \\ &= I + S\sinh(\ln(x_0 + x_3)) + S^2(\cosh(\ln(x_0 + x_3)) - 1),\end{aligned}$$

where $I$ is a unit $(4 \times 4)$-matrix.

Substitution of the obtained expressions for functions $w(x)$, $Q_{\mu\nu}(x)$ into (3.8) yields the ansatz for the Yang–Mills field $A^a_\mu(x)$ invariant under the algebra $\langle P_1, P_2, J_{03}\rangle$

$$\begin{aligned}&A^a_0 = B^a_0(x_0^2 - x_3^2)\cosh\ln(x_0 + x_3) + B^a_3(x_0^2 - x_3^2)\sinh\ln(x_0 + x_3),\\ &A^a_1 = B^a_1(x_0^2 - x_3^2), \quad A^a_2 = B^a_2(x_0^2 - x_3^2),\\ &A^a_3 = B^a_3(x_0^2 - x_3^2)\cosh\ln(x_0 + x_3) + B^a_0(x_0^2 - x_3^2)\sinh\ln(x_0 + x_3).\end{aligned} \tag{3.12}$$

Substituting (3.12) into YME we get a system of ODE for functions $B^a_\mu$. If we will succeed in constructing its general or particular solutions, then substituting it into formulae (3.12) we get an exact solution of YME. But such a solution will have an unpleasant feature: independent variables $x_\mu$ will be included into it in asymmetrical way. At the same time, in the initial equation (1.1) all independent variables are on equal rights. To remove this defect one has to apply solution generation procedure by transformations from the Lorentz group. As a result, we will obtain an ansatz for the Yang–Mills field in the manifestly-covariant form with symmetrical dependence on $x_\mu$.

In the same way, we construct the rest of ansatzes invariant under three-dimensional subalgebras of the Poincaré algebra. They are represented in the unified form

$$\begin{aligned}A^a_\mu(x) = \{&(a_\mu a_\nu - d_\mu d_\nu)\cosh\theta_0 + (d_\mu a_\nu - d_\nu a_\mu)\sinh\theta_0 + \\ &+ 2(a_\mu + d_\mu)[(\theta_1\cos\theta_3 + \theta_2\sin\theta_3)b_\nu + (\theta_2\cos\theta_3 - \theta_1\sin\theta_3)c_\nu + \\ &+ (\theta_1^2 + \theta_2^2)e^{-\theta_0}(a_\nu + d_\nu)] + (b_\mu c_\nu - b_\nu c_\mu)\sin\theta_3 - \\ &- (c_\mu c_\nu + b_\mu b_\nu)\cos\theta_3 - 2e^{-\theta_0}(\theta_1 b_\mu + \theta_2 c_\mu)(a_\nu + d_\nu)\}B^{a\nu}(w).\end{aligned} \tag{3.13}$$

Here $\theta_\mu$, $\mu = \overline{0,3}$, $w$ are some functions whose explicit form is determined by the choice of a subalgebra of the Poincaré algebra $AP(1,3)$.

Below, we adduce a complete list of 3-dimensional $P(1,3)$-inequivalent subalgebras of the Poincaré algebra following [7]

$$\begin{array}{ll}
L_1 = \langle P_0, P_1, P_2\rangle; & L_2 = \langle P_1, P_2, P_3\rangle;\\
L_3 = \langle P_0 + P_3, P_1, P_2\rangle; & L_4 = \langle J_{03} + \alpha J_{12}, P_1, P_2\rangle;\\
L_5 = \langle J_{03}, P_0 + P_3, P_1\rangle; & L_6 = \langle J_{03} + P_1, P_0, P_3\rangle;\\
L_7 = \langle J_{03} + P_1, P_0 + P_3, P_2\rangle; & L_8 = \langle J_{12} + \alpha J_{03}, P_0, P_3\rangle;\\
L_9 = \langle J_{12} + P_0, P_1, P_2\rangle; & L_{10} = \langle J_{12} + P_3, P_1, P_2\rangle;\\
L_{11} = \langle J_{12} + P_0 - P_3, P_1, P_2\rangle; & L_{12} = \langle G_1, P_0 + P_3, P_2 + \alpha P_1\rangle;\\
L_{13} = \langle G_1 + P_2, P_0 + P_3, P_1\rangle; & L_{14} = \langle G_1 + P_0 - P_3, P_0 + P_3, P_2\rangle;\\
L_{15} = \langle G_1 + P_0 - P_3, P_0 + P_3, P_1 + \alpha P_2\rangle; & L_{16} = \langle J_{12}, J_{03}, P_0 + P_3\rangle;\\
L_{17} = \langle G_1 + P_2, G_2 - P_1 + \alpha P_2, P_0 + P_3\rangle; & L_{18} = \langle J_{03}, G_1, P_2\rangle;\\
L_{19} = \langle G_1, J_{03}, P_0 + P_3\rangle; & L_{20} = \langle G_1, J_{03} + P_2, P_0 + P_3\rangle;\\
L_{21} = \langle G_1, J_{03} + P_1 + \alpha P_2, P_0 + P_3\rangle; & L_{22} = \langle G_1, G_2, J_{03} + \alpha J_{12}\rangle;\\
L_{23} = \langle G_1, P_0 + P_3, P_1\rangle; & L_{24} = \langle J_{12}, P_1, P_2\rangle;\\
L_{25} = \langle J_{03}, P_0, P_3\rangle; & L_{26} = \langle J_{12}, J_{13}, J_{23}\rangle;\\
L_{27} = \langle J_{01}, J_{02}, J_{12}\rangle. &
\end{array} \tag{3.14}$$

Here $G_i = J_{0i} - J_{i3}$ $(i = 1, 2)$, $\alpha \in \mathbb{R}$.

Ansatzes for the Yang–Mills field $A^a_\mu(x)$ are of the form (3.13), functions $\theta_\mu(x)$, $\mu = \overline{0,3}$, $w(x)$ being determined by one of the following formulae:

$$\begin{array}{ll}
L_1: & \theta_\mu = 0, \quad w = dx; \quad L_2: \quad \theta_\mu = 0, \quad w = ax; \quad L_3: \quad \theta_\mu = 0, w = kx;\\
L_4: & \theta_0 = -\ln|kx|, \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = \alpha\ln|kx|, \quad w = (ax)^2 - (dx)^2;\\
L_5: & \theta_0 = -\ln|kx|, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad w = cx;\\
L_6: & \theta_0 = -bx, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad w = cx;\\
L_7: & \theta_0 = -bx, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad w = bx - \ln|kx|;\\
L_8: & \theta_0 = \alpha\arctan(bx(cx)^{-1}), \quad \theta_1 = \theta_2 = 0,\\
& \theta_3 = -\arctan(bx(cx)^{-1}), \quad w = (bx)^2 + (cx)^2;\\
L_9: & \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = -ax, \quad w = dx;\\
L_{10}: & \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = dx, \quad w = ax;\\
L_{11}: & \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = -\dfrac{1}{2}kx, \quad w = ax - dx;\\
L_{12}: & \theta_0 = 0, \quad \theta_1 = \dfrac{1}{2}(bx - \alpha cx)(kx)^{-1}, \quad \theta_2 = \theta_3 = 0, \quad w = kx;\\
L_{13}: & \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = \frac{1}{2}cx, \quad w = kx;\\
L_{14}: & \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\dfrac{1}{4}kx, \quad w = 4bx + (kx)^2;\\
L_{15}: & \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\dfrac{1}{4}kx, \quad w = 4(\alpha bx - cx) + \alpha(kx)^2;\\
L_{16}: & \theta_0 = -\ln|kx|, \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = -\arctan(bx(cx)^{-1}),\\
& w = (bx)^2 + (cx)^2;\\
L_{17}: & \theta_0 = \theta_3 = 0, \quad \theta_1 = \dfrac{1}{2}(cx + (\alpha + kx)bx)(1 + kx(\alpha + kx))^{-1},
\end{array} \tag{3.15}$$

$$\theta_2 = -\frac{1}{2}(bx - cxkx)(1 + kx(\alpha + kx))^{-1}, \quad w = kx;$$

$$L_{18}: \quad \theta_0 = -\ln|kx|, \quad \theta_1 = \frac{1}{2}bx(kx)^{-1}, \quad \theta_2 = \theta_3 = 0,$$
$$w = (ax)^2 - (bx)^2 - (dx)^2;$$

$$L_{19}: \quad \theta_0 = -\ln|kx|, \quad \theta_1 = \frac{1}{2}bx(kx)^{-1}, \quad \theta_2 = \theta_3 = 0, w = cx;$$

$$L_{20}: \quad \theta_0 = -\ln|kx|, \quad \theta_1 = \frac{1}{2}bx(kx)^{-1}, \quad \theta_2 = \theta_3 = 0, \quad w = \ln|kx| - cx;$$

$$L_{21}: \quad \theta_0 = -\ln|kx|, \quad \theta_1 = \frac{1}{2}(bx - \ln|kx|)(kx)^{-1}, \quad \theta_2 = \theta_3 = 0,$$
$$w = \alpha\ln|kx| - cx;$$

$$L_{22}: \quad \theta_0 = -\ln|kx|, \quad \theta_1 = \frac{1}{2}bx(kx)^{-1}, \quad \theta_2 = \frac{1}{2}cx(kx)^{-1},$$
$$\theta_3 = \alpha\ln|kx|, \quad w = (ax)^2 - (bx)^2 - (cx)^2 - (dx)^2.$$

Here $ax = a_\mu x^\mu$, $bx = b_\mu x^\mu$, $cx = c_\mu x^\mu$, $dx = d_\mu x^\mu$, $\mu = \overline{0,3}$, $kx = ax + dx$.

**Note.** Basis elements of subalgebras $L_{23}$, $L_{24}$, $L_{25}$, $L_{26}$, $L_{27}$ do not satisfy (3.3). That is why, ansatzes invariant under these subalgebras are partially-invariant solutions and are not considered here.

# 4 Reduction of the Yang–Mills equations

In order to reduce YME to ODE it is necessary to substitute ansatz (3.13) into (1.1) and convolute the expression obtained with $Q^\mu_\alpha(x)$. As a result, we get a system of twelve nonlinear ODE for functions $B^a_\nu(w)$ of the form

$$\begin{aligned} &k_{\mu\gamma}\ddot{\vec{B}}^\gamma + l_{\mu\gamma}\dot{\vec{B}}^\gamma + m_{\mu\gamma}\vec{B}^\gamma + eg_{\mu\nu\gamma}\dot{\vec{B}}^\nu \times \vec{B}^\gamma + eh_{\mu\nu\gamma}\vec{B}^\nu \times \vec{B}^\gamma + \\ &\quad + e^2\vec{B}_\gamma \times (\vec{B}^\gamma \times \vec{B}_\mu) = \vec{0}. \end{aligned} \tag{4.1}$$

Coefficients of the reduced ODE are given by the following formulae:

$$\begin{aligned} &k_{\mu\gamma} = g_{\mu\gamma}F_1 - G_\mu G_\gamma, \quad l_{\mu\gamma} = g_{\mu\gamma}F_2 + 2S_{\mu\gamma} - G_\mu H_\gamma - G_\mu \dot{G}_\gamma, \\ &m_{\mu\gamma} = R_{\mu\gamma} - G_\mu \dot{H}_\gamma, \quad g_{\mu\nu\gamma} = g_{\mu\gamma}G_\nu + g_{\nu\gamma}G_\mu - 2g_{\mu\nu}G_\gamma, \\ &h_{\mu\nu\gamma} = (1/2)(g_{\mu\gamma}H_\nu - g_{\mu\nu}H_\gamma) - T_{\mu\nu\gamma}, \end{aligned} \tag{4.2}$$

where $g_{\mu\nu}$ is a metric tensor of the Minkowski space $\mathbb{R}(1,3)$ and $F_1, F_2, G_\mu, \ldots, T_{\mu\nu\gamma}$ are functions on $w$ determined by the relations

$$\begin{aligned} &F_1 = w_{x_\mu}w_{x^\mu}, \quad F_2 = \Box w, \quad G_\mu = Q_{\alpha\mu}w_{x_\alpha}, \quad H_\mu = Q_{\alpha\mu x_\alpha}, \\ &S_{\mu\nu} = Q^\alpha_\mu Q_{\alpha\nu x_\beta}w_{x^\beta}, \quad R_{\mu\nu} = Q^\alpha_\mu \Box Q_{\alpha\nu}, \\ &T_{\mu\nu\gamma} = Q^\alpha_\mu Q_{\alpha\nu x_\beta}Q_{\beta\gamma} + Q^\alpha_\nu Q_{\alpha\gamma x_\beta}Q_{\beta\mu} + Q^\alpha_\gamma Q_{\alpha\mu x_\beta}Q_{\beta\nu}. \end{aligned} \tag{4.3}$$

Substituting functions $Q_{\mu\nu}(x)$ from (3.13), where $\theta_\mu(x)$, $w(x)$ are determined by one of the formulae (3.15) into (4.2), (4.3) we obtain coefficients of the corresponding systems of ODE (4.1)

$$L_1: \quad k_{\mu\gamma} = -g_{\mu\gamma} - d_\mu d_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0,$$
$$g_{\mu\nu\gamma} = g_{\mu\gamma}d_\nu + g_{\nu\gamma}d_\mu - 2g_{\mu\nu}d_\gamma, \quad h_{\mu\nu\gamma} = 0;$$

$$
\begin{aligned}
L_2: \quad & k_{\mu\gamma} = g_{\mu\gamma} - a_\mu a_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0,\\
& g_{\mu\nu\gamma} = g_{\mu\gamma} a_\nu + g_{\nu\gamma} a_\mu - 2g_{\mu\nu} a_\gamma, \quad h_{\mu\nu\gamma} = 0;\\
L_3: \quad & k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0, \quad g_{\mu\nu\gamma} = g_{\mu\gamma} k_\nu + g_{\nu\gamma} k_\mu - 2g_{\mu\nu} k_\gamma,\\
& h_{\mu\nu\gamma} = 0;\\
L_4: \quad & k_{\mu\gamma} = 4g_{\mu\gamma} w - a_\mu a_\gamma (w+1)^2 - d_\mu d_\gamma (w-1)^2 - (a_\mu d_\gamma + a_\gamma d_\mu)(w^2-1),\\
& l_{\mu\gamma} = 4(g_{\mu\gamma} + \alpha(b_\mu c_\gamma - c_\mu b_\gamma)) - 2k_\mu (a_\gamma - d_\gamma + k_\gamma w), \quad m_{\mu\gamma} = 0,\\
& g_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}(a_\nu - d_\nu + k_\nu w) + g_{\nu\gamma}(a_\mu - d_\mu + k_\mu w) -\\
& \qquad - 2g_{\mu\nu}(a_\gamma - d_\gamma + k_\gamma w)),\\
& h_{\mu\nu\gamma} = \frac{\epsilon}{2}[g_{\mu\gamma} k_\nu - g_{\mu\nu} k_\gamma] + \alpha\epsilon[(b_\mu c_\nu - c_\mu b_\nu) k_\gamma + (b_\nu c_\gamma - c_\nu b_\gamma) k_\mu +\\
& \qquad + (b_\gamma c_\mu - c_\gamma b_\mu) k_\nu];\\
L_5: \quad & k_{\mu\gamma} = -g_{\mu\gamma} - c_\mu c_\gamma, \quad l_{\mu\gamma} = -\epsilon c_\mu k_\gamma, \quad m_{\mu\gamma} = 0,\\
& g_{\mu\nu\gamma} = g_{\mu\gamma} c_\nu + g_{\nu\gamma} c_\mu - 2g_{\mu\nu} c_\gamma, \quad h_{\mu\nu\gamma} = \frac{\epsilon}{2}(g_{\mu\gamma} k_\nu - g_{\mu\nu} k_\gamma);\\
L_6: \quad & k_{\mu\gamma} = -g_{\mu\gamma} - c_\mu c_\gamma, \quad l_{\mu\gamma} = 0,\\
& m_{\mu\gamma} = -(a_\mu a_\gamma - d_\mu d_\gamma), \quad g_{\mu\nu\gamma} = g_{\mu\gamma} c_\nu + g_{\nu\gamma} c_\mu - 2g_{\mu\nu} c_\gamma,\\
& h_{\mu\nu\gamma} = -[(a_\mu d_\nu - a_\nu d_\mu) b_\gamma + (a_\nu d_\gamma - a_\gamma d_\nu) b_\mu + (a_\gamma d_\mu - a_\mu d_\gamma) b_\nu];\\
L_7: \quad & k_{\mu\gamma} = -g_{\mu\gamma} - (b_\mu - \epsilon k_\mu)(b_\gamma - \epsilon k_\gamma), \quad l_{\mu\gamma} = -2(a_\mu d_\gamma - a_\gamma d_\mu),\\
& m_{\mu\gamma} = -(a_\mu a_\gamma - d_\mu d_\gamma),\\
& g_{\mu\nu\gamma} = g_{\mu\gamma}(b_\nu - \epsilon k_\nu) + g_{\nu\gamma}(b_\mu - \epsilon k_\mu) - 2g_{\mu\nu}(b_\gamma - \epsilon k_\gamma),\\
& h_{\mu\nu\gamma} = -[(a_\mu d_\nu - a_\nu d_\mu) b_\gamma + (a_\nu d_\gamma - a_\gamma d_\nu) b_\mu + (a_\gamma d_\mu - a_\mu d_\gamma) b_\nu]; \qquad (4.4)\\
L_8: \quad & k_{\mu\gamma} = -4w(g_{\mu\gamma} + c_\mu c_\gamma), \quad l_{\mu\gamma} = -4(g_{\mu\gamma} + c_\mu c_\gamma),\\
& m_{\mu\gamma} = -\frac{1}{w}(\alpha^2(a_\mu a_\gamma - d_\mu d_\gamma) + b_\mu b_\gamma),\\
& g_{\mu\nu\gamma} = 2\sqrt{w}(g_{\mu\gamma} c_\nu + g_{\nu\gamma} c_\mu - 2g_{\mu\nu} c_\gamma),\\
& h_{\mu\nu\gamma} = \frac{1}{2\sqrt{w}}(g_{\mu\gamma} c_\nu - g_{\mu\nu} c_\gamma) + \frac{\alpha}{\sqrt{w}}((a_\mu d_\nu - a_\nu d_\mu) b_\gamma +\\
& \qquad + (a_\nu d_\gamma - d_\nu a_\gamma) b_\mu + (a_\gamma d_\mu - a_\mu d_\gamma) b_\nu);\\
L_9: \quad & k_{\mu\gamma} = -g_{\mu\gamma} - d_\mu d_\gamma, \quad l_{\mu\gamma} = 0,\\
& m_{\mu\gamma} = b_\mu b_\gamma + c_\mu c_\gamma, \quad g_{\mu\nu\gamma} = g_{\mu\gamma} d_\nu + g_{\nu\gamma} d_\mu - 2g_{\mu\nu} d_\gamma,\\
& h_{\mu\nu\gamma} = a_\gamma(b_\mu c_\nu - c_\mu b_\nu) + a_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + a_\nu(b_\gamma c_\mu - c_\gamma b_\mu);\\
L_{10}: \quad & k_{\mu\gamma} = g_{\mu\gamma} - a_\mu a_\gamma, \quad l_{\mu\gamma} = 0,\\
& m_{\mu\gamma} = -(b_\mu b_\gamma + c_\mu c_\gamma), \quad g_{\mu\nu\gamma} = g_{\mu\gamma} a_\nu + g_{\nu\gamma} a_\mu - 2g_{\mu\nu} a_\gamma,\\
& h_{\mu\nu\gamma} = -[d_\gamma(b_\mu c_\nu - c_\mu b_\nu) + d_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + d_\nu(b_\gamma c_\mu - c_\gamma b_\mu)];\\
L_{11}: \quad & k_{\mu\gamma} = -(a_\mu - d_\mu)(a_\gamma - d_\gamma), \quad l_{\mu\gamma} = -2(b_\mu c_\gamma - c_\mu b_\gamma), \quad m_{\mu\gamma} = 0,\\
& g_{\mu\nu\gamma} = g_{\mu\gamma}(a_\nu - d_\nu) + g_{\nu\gamma}(a_\mu - d_\mu) - 2g_{\mu\nu}(a_\gamma - d_\gamma),\\
& h_{\mu\nu\gamma} = \tfrac{1}{2}[k_\gamma(b_\mu c_\nu - c_\mu b_\nu) + k_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + k_\nu(b_\gamma c_\mu - c_\gamma b_\mu)];\\
L_{12}: \quad & k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = -\frac{1}{w} k_\mu k_\gamma, \quad m_{\mu\gamma} = -\frac{\alpha^2}{w^2} k_\mu k_\gamma,\\
& g_{\mu\nu\gamma} = g_{\mu\gamma} k_\nu + g_{\nu\gamma} k_\mu - 2g_{\mu\nu} k_\gamma,\\
& h_{\mu\nu\gamma} = \frac{1}{2w}(g_{\mu\gamma} k_\nu - g_{\mu\nu} k_\gamma) +
\end{aligned}
$$

$$+\frac{\alpha}{w}((k_\mu b_\nu - k_\nu b_\mu)c_\gamma + (k_\nu b_\gamma - k_\gamma b_\nu)c_\mu + (k_\gamma b_\mu - k_\mu b_\gamma)c_\nu);$$

$L_{13}:$ $k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = 0, \quad m_{\mu\gamma} = -k_\mu k_\gamma,$
$g_{\mu\nu\gamma} = g_{\mu\gamma}k_\nu + g_{\nu\gamma}k_\mu - 2g_{\mu\nu}k_\gamma, \quad h_{\mu\nu\gamma} = -((k_\mu b_\nu - k_\nu b_\mu)c_\gamma +$
$+ (k_\nu b_\gamma - k_\gamma b_\nu)c_\mu + (k_\gamma b_\mu - k_\mu b_\gamma)c_\nu);$

$L_{14}:$ $k_{\mu\gamma} = -16(g_{\mu\gamma} + b_\mu b_\gamma), \quad l_{\mu\gamma} = m_{\mu\gamma} = h_{\mu\nu\gamma} = 0,$
$g_{\mu\nu\gamma} = 4(g_{\mu\gamma}b_\nu + g_{\nu\gamma}b_\mu - 2g_{\mu\nu}b_\gamma);$

$L_{15}:$ $k_{\mu\gamma} = -16[(1+\alpha^2)g_{\mu\gamma} + (c_\mu - \alpha b_\mu)(c_\gamma - \alpha b_\gamma)],$
$l_{\mu\gamma} = m_{\mu\gamma} = h_{\mu\nu\gamma} = 0,$
$g_{\mu\nu\gamma} = -4[g_{\mu\gamma}(c_\nu - \alpha b_\nu) + g_{\nu\gamma}(c_\mu - \alpha b_\mu) - 2g_{\mu\nu}(c_\gamma - \alpha b_\gamma)];$

$L_{16}:$ $k_{\mu\gamma} = -4w(g_{\mu\gamma} + c_\mu c_\gamma), \quad l_{\mu\gamma} = -4(g_{\mu\gamma} + c_\mu c_\gamma) - 2\epsilon k_\gamma c_\mu \sqrt{w},$
$m_{\mu\gamma} = -\frac{1}{w}b_\mu b_\gamma, \quad g_{\mu\nu\gamma} = 2\sqrt{w}(g_{\mu\gamma}c_\nu + g_{\nu\gamma}c_\mu - 2g_{\mu\nu}c_\gamma),$
$h_{\mu\nu\gamma} = \frac{1}{2}[\epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma) + \frac{1}{\sqrt{w}}(g_{\mu\gamma}c_\nu - g_{\mu\nu}c_\gamma)];$

$L_{17}:$ $k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = -\frac{2w+\alpha}{w(w+\alpha)+1}k_\mu k_\gamma,$
$m_{\mu\gamma} = -4k_\mu k_\gamma(1 + w(\alpha + w))^{-2}, \quad g_{\mu\nu\gamma} = g_{\mu\gamma}k_\nu + g_{\nu\gamma}k_\mu - 2g_{\mu\nu}k_\gamma,$
$h_{\mu\nu\gamma} = \frac{1}{2}(\alpha + 2w)(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma)(1 + w(\alpha + w))^{-1} -$
$2(1 + w(w+\alpha))^{-1}((k_\mu b_\nu - k_\nu b_\mu)c_\gamma + (k_\nu b_\gamma - k_\gamma b_\nu)c_\mu +$
$+ (k_\gamma b_\mu - k_\mu b_\gamma)c_\nu);$

$L_{18}:$ $k_{\mu\gamma} = 4wg_{\mu\gamma} - (k_\mu w + a_\mu - d_\mu)(k_\gamma w + a_\gamma - d_\gamma),$
$l_{\mu\gamma} = 6g_{\mu\gamma} + 4(a_\mu d_\gamma - a_\gamma d_\mu) - 3k_\gamma(k_\mu w + a_\mu - d_\mu), \quad m_{\mu\gamma} = -k_\mu k_\gamma,$
$g_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}(k_\nu w + a_\nu - d_\nu) + g_{\nu\gamma}(k_\mu w + a_\mu - d_\mu) -$
$- 2g_{\mu\nu}(k_\gamma w + a_\gamma - d_\gamma)),$
$h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma);$

$L_{19}:$ $k_{\mu\gamma} = -g_{\mu\gamma} - c_\mu c_\gamma, \quad l_{\mu\gamma} = 2\epsilon k_\gamma c_\mu, \quad m_{\mu\gamma} = -k_\mu k_\gamma,$
$g_{\mu\nu\gamma} = g_{\mu\gamma}c_\nu + g_{\nu\gamma}c_\mu - 2g_{\mu\nu}c_\gamma, \quad h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma);$

$L_{20}:$ $k_{\mu\gamma} = -g_{\mu\gamma} - (c_\mu - \epsilon k_\mu)(c_\gamma - \epsilon k_\gamma), \quad l_{\mu\gamma} = 2\epsilon k_\gamma c_\mu - 2k_\mu k_\gamma,$
$m_{\mu\gamma} = -k_\mu k_\gamma,$
$g_{\mu\nu\gamma} = g_{\mu\gamma}(\epsilon k_\nu - c_\nu) + g_{\nu\gamma}(\epsilon k_\mu - c_\mu) - 2g_{\mu\nu}(\epsilon k_\gamma - c_\gamma),$
$h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma);$

$L_{21}:$ $k_{\mu\gamma} = -g_{\mu\gamma} - (c_\mu - \alpha\epsilon k_\mu)(c_\gamma - \alpha\epsilon k_\gamma), \quad l_{\mu\gamma} = 2(\epsilon k_\gamma c_\mu - \alpha k_\mu k_\gamma),$
$m_{\mu\gamma} = -k_\mu k_\gamma,$
$g_{\mu\nu\gamma} = -g_{\mu\gamma}(c_\nu - \alpha\epsilon k_\nu) - g_{\nu\gamma}(c_\mu - \alpha\epsilon k_\mu) + 2g_{\mu\nu}(c_\gamma - \alpha\epsilon k_\gamma),$
$h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma);$

$L_{22}:$ $k_{\mu\gamma} = 4wg_{\mu\gamma} - (a_\mu - d_\mu + k_\mu w)(a_\gamma - d_\gamma + k_\gamma w),$
$l_{\mu\gamma} = 4[2g_{\mu\gamma} + \alpha(b_\mu c_\gamma - c_\mu b_\gamma) - a_\mu a_\gamma + d_\mu d_\gamma - wk_\mu k_\gamma],$
$m_{\mu\gamma} = -2k_\mu k_\gamma,$
$g_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}(a_\nu - d_\nu + k_\nu w) + g_{\nu\gamma}(a_\mu - d_\mu + k_\mu w) -$
$- 2g_{\mu\nu}(a_\gamma - d_\gamma + k_\gamma w),$

$$h_{\mu\nu\gamma} = \frac{3\epsilon}{2}(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma) - \epsilon\alpha[k_\gamma(b_\mu c_\nu - c_\mu b_\nu) + + k_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + k_\nu(b_\gamma c_\mu - c_\gamma b_\mu)];$$

where $k_\mu = a_\mu + d_\mu$, $\epsilon = 1$ for $ax + dx > 0$ and $\epsilon = -1$ for $ax + dx < 0$.

## 5 Exact solutions of the Yang–Mills equations

When applying the symmetry reduction procedure to the nonlinear Dirac equation, we succeeded in constructing general solutions of a large part of reduced systems of ODE. In the case involved we are not so lucky. Nevertheless, we obtain some particular solutions of equations (4.2), (4.4).

The principal idea of our approach to integration of systems of ODE (4.2), (4.4) is rather simple and quite natural. It is a reduction of these systems by the number of components with the aid of ad hoc substitutions. Using this trick we construct particular solutions of equations 1, 2, 5, 8, 14, 15, 16, 18, 19, 20, 21, 22 ($\alpha = 0$). Below we adduce substitutions for $\vec{B}_\mu(w)$ and corresponding equations.

1. $\vec{B}_\mu = a_\mu\vec{e}_1 f(w) + b_\mu\vec{e}_2 g(w) + c_\mu\vec{e}_3 h(w)$,
$\ddot{f} - e^2(g^2 + h^2)f = 0, \quad \ddot{g} + e^2(f^2 - h^2)g = 0, \quad \ddot{h} + e^2(f^2 - g^2)h = 0.$

2. $\vec{B}_\mu = b_\mu\vec{e}_1 f(w) + c_\mu\vec{e}_2 g(w) + d_\mu\vec{e}_3 h(w)$,
$\ddot{f} + e^2(g^2 + h^2)f = 0, \quad \ddot{g} + e^2(f^2 + h^2)g = 0, \quad \ddot{h} + e^2(f^2 + g^2)h = 0.$

5. $\vec{B}_\mu = k_\mu\vec{e}_1 f(w) + b_\mu\vec{e}_2 g(w), \quad \ddot{f} - e^2g^2f = 0, \ddot{g} = 0.$

8.1. $(\alpha = 0)$ $\vec{B}_\mu = k_\mu\vec{e}_1 f(w) + b_\mu\vec{e}_2 g(w)$,
$4w\ddot{f} + 4\dot{f} - e^2g^2f = 0, \quad 4w\ddot{g} + 4\dot{g} - w^{-1}g = 0.$

8.2. $\vec{B}_\mu = a_\mu\vec{e}_1 f(w) + d_\mu\vec{e}_2 g(w) + b_\mu\vec{e}_3 h(w)$,

$$\begin{aligned} &4w\ddot{f} + 4\dot{f} - \frac{\alpha^2}{w}f - \frac{2\alpha e}{\sqrt{w}}gh + e^2(h^2 + g^2)f = 0, \\ &4w\ddot{g} + 4\dot{g} + \frac{\alpha^2}{w}g + \frac{2\alpha e}{\sqrt{w}}fh + e^2(f^2 - h^2)g = 0, \\ &4w\ddot{h} + 4\dot{h} - w^{-1}h + \frac{2\alpha e}{\sqrt{w}}fg + e^2(f^2 - g^2)h = 0. \end{aligned} \tag{5.1}$$

14.1. $\vec{B}_\mu = a_\mu\vec{e}_1 f(w) + d_\mu\vec{e}_2 g(w) + c_\mu\vec{e}_3 h(w)$,
$16\ddot{f} - e^2(h^2 + g^2)f = 0, \quad 16\ddot{g} + e^2(f^2 - h^2)g = 0,$
$16\ddot{h} + e^2(f^2 - g^2)h = 0.$

14.2. $\vec{B}_\mu = k_\mu\vec{e}_1 f(w) + c_\mu\vec{e}_2 g(w), \quad 16\ddot{f} - e^2g^2f = 0, \ddot{g} = 0.$

15.1. $\vec{B}_\mu = a_\mu\vec{e}_1 f(w) + d_\mu\vec{e}_2 g(w) + (1 + \alpha^2)^{-\frac{1}{2}}(\alpha c_\mu + b_\mu)\vec{e}_3 h(w)$,
$16(1 + \alpha^2)\ddot{f} - e^2(h^2 + g^2)f = 0, \quad 16(1 + \alpha^2)\ddot{g} + e^2(f^2 - h^2)g = 0,$
$16(1 + \alpha^2)\ddot{h} + e^2(f^2 - g^2)h = 0.$

15.2. $\vec{B}_\mu = k_\mu\vec{e}_1 f(w) + (1 + \alpha^2)^{-\frac{1}{2}}(\alpha c_\mu + b_\mu)\vec{e}_2 g(w)$,
$16(1 + \alpha^2)\ddot{f} - e^2fg^2 = 0, \quad \ddot{g} = 0.$

16. $\vec{B}_\mu = k_\mu \vec{e}_1 f(w) + b_\mu \vec{e}_2 g(w)$,
$4w\ddot{f} + 4\dot{f} - e^2 g^2 f = 0, \quad 4w\ddot{g} + 4\dot{g} - w^{-1} g = 0.$
18. $\vec{B}_\mu = b_\mu \vec{e}_1 f(w) + c_\mu \vec{e}_2 g(w)$,
$4w\ddot{f} + 6\dot{f} + e^2 g^2 f = 0, \quad 4w\ddot{g} + 6\dot{g} + e^2 f^2 g = 0.$
19. $\vec{B}_\mu = k_\mu \vec{e}_1 f(w) + b_\mu \vec{e}_2 g(w)$,
$\ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0.$
20. $\vec{B}_\mu = k_\mu \vec{e}_1 f(w) + b_\mu \vec{e}_2 g(w)$,
$\ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0.$
21. $\vec{B}_\mu = k_\mu \vec{e}_1 f(w) + b_\mu \vec{e}_2 g(w)$,
$\ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0.$
22. $(\alpha = 0)$ $\vec{B}_\mu = b_\mu \vec{e}_1 f(w) + c_\mu \vec{e}_2 g(w)$,
$4w\ddot{f} + 8\dot{f} + e^2 g^2 f = 0, \quad 4w\ddot{g} + 8\dot{g} + e^2 f^2 g = 0.$

In the above formulae we use designations $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$, $\vec{e}_3 = (0,0,1)$.

Thus, combining symmetry reduction by the number of independent variables and reduction by the number of dependent variables we reduce YME to rather simple ODE. It is worth reminding that effectiveness of the widely used ansatz for the Yang–Mills field suggested by t'Hooft et al [2] is closely connected with the fact that it reduces the system of twelve PDE to one nonlinear wave equation.

Next, we will briefly consider a procedure of integration of equations (5.1).

Substitution $f = 0$, $g = h = u(w)$ reduces the system of ODE 1 from (5.1) to the equation

$$\ddot{u} = e^2 u^3, \tag{5.2}$$

which is integrated in elliptic functions [8]. Besides that, ODE (5.2) has a solution which is expressed in terms of elementary functions $u = \sqrt{2}(ew - C)^{-1}$, $C \in \mathbb{R}^1$.

ODE 2 with $f = g = h = u(w)$ reduces to the form $\ddot{u} + 2e^2 u^3 = 0$.

This equation is also integrated in elliptic functions [8].

Integrating the second equation of system of ODE 5 we get $g = C_1 w + C_2$, $C_i \in \mathbb{R}^1$. If $C_1 \neq 0$, then the constant $C_2$ can be neglected, and we may put $C_2 = 0$. Provided $C_1 \neq 0$, the first equation from system 5 reads

$$\ddot{f} - e^2 C_1^2 w^2 f = 0. \tag{5.3}$$

A general solution of ODE (5.3) is given by formula $f = w^{1/2} Z_{\frac{1}{4}}(\frac{ie}{2} C_1 w^2)$.

Hereafter, we use the designation $Z_\nu(w) = C_3 J_\nu(w) + C_4 Y_\nu(w)$, where $J_\nu$, $Y_\nu$ are Bessel functions, $C_3$, $C_4$ are arbitrary constants.

In the case $C_1 = 0$, $C_2 \neq 0$ a general solution of the first equation from system 5 reads $f = C_3 \cosh C_2 ew + C_4 \sinh C_2 ew$, where $C_3$, $C_4$ are arbitrary constants.

At last, provided $C_1 = C_2 = 0$, a general solution of the first equation from system 5 has the form $f = C_3 w + C_4$, $C_3, C_4 \in \mathbb{R}^1$.

A general solution of the second ODE from system 8.1 is of the form $g = C_1 \sqrt{w} + C_2 (\sqrt{w})^{-1}$, where $C_1$, $C_2$ are arbitrary constants.

Substituting the expression obtained into the first equation we get

$$4w^2\ddot{f} + 4w\dot{f} - e^2(C_1w + C_2)^2 f = 0. \tag{5.4}$$

Under $C_1, C_2 \neq 0$ a solution of ODE (5.4) is not known. In the remaining cases its general solution reads

$$\begin{aligned}
&a) \quad C_1 \neq 0, \quad C_2 = 0 \qquad f = Z_0\left[\frac{ie}{2}C_1 w\right],\\
&b) \quad C_1 = 0, \quad C_2 \neq 0 \qquad f = C_3 w^{\frac{eC_2}{2}} + C_4 w^{-\frac{eC_2}{2}},\\
&c) \quad C_1 = 0, \quad C_2 = 0 \qquad f = C_3 \ln w + C_4.
\end{aligned}$$

Here $C_3$, $C_4$ are arbitrary constants.

We do not succeed in obtaining particular solutions of system 8.2. Equations 14.1 coincide with equations 1, if one changes $e$ by $\frac{e}{4}$. Similarly, equations 14.2 coincide with equations 5, if one changes $e$ by $\frac{e}{4}$. Next, equations 15.1 coincide with equations 1 and equations 15.2 – with equations 5, if one replaces $e$ by $\frac{e}{4}(1+\alpha^2)^{-\frac{1}{2}}$.

System of ODE 16 coincides with system 8.1 and systems 19, 20, 21 – with system 5. We did not succeed in integrating equations 18.

At last, system 22 ($\alpha = 0$) with the substitution $f = g = u(w)$ reduces to the form

$$w\ddot{u} + 2\dot{u} + \frac{e^2}{4}u^3 = 0. \tag{5.5}$$

ODE (5.5) is Emden–Fowler equation and the function $u = e^{-1}w^{-\frac{1}{2}}$, is its particular solution.

Substituting the results obtained into corresponding formulae from (5.1) and then into the ansatz (3.13), we get exact solutions of the nonlinear YME (1.1). Let us note that solutions of systems of ODE 5, 8.1, 14.2, 15.2, 16, 19, 20, 21 satisfying the condition $g = 0$ give rise to Abelian solutions of YME. We do not adduce them and present only non-Abelian solutions of YME.

1. $\vec{A}_\mu = (\vec{e}_2 b_\mu + \vec{e}_3 c_\mu)\sqrt{2}(edx - \lambda)^{-1}$;
2. $\vec{A}_\mu = (\vec{e}_2 b_\mu + \vec{e}_3 c_\mu)\left[\lambda\,\mathrm{sn}\left(\frac{\sqrt{2}}{2}e\lambda dx\right)\mathrm{dn}\left(\frac{\sqrt{2}}{2}e\lambda dx\right)\right]\left[\mathrm{cn}\left(\frac{\sqrt{2}}{2}e\lambda dx\right)\right]^{-1}$;
3. $\vec{A}_\mu = (\vec{e}_2 b_\mu + \vec{e}_3 c_\mu)\lambda[\mathrm{cn}\,(e\lambda dx)]^{-1}$;
4. $\vec{A}_\mu = (\vec{e}_1 b_\mu + \vec{e}_2 c_\mu + \vec{e}_3 d_\mu)\lambda\,\mathrm{cn}\,(e\lambda ax)$;
5. $\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}\sqrt{cx}Z_{\frac{1}{4}}\left[\frac{i}{2}e\lambda(cx)^2\right] + \vec{e}_2 b_\mu \lambda cx$;
6. $\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}[\lambda_1\cosh(e\lambda cx) + \lambda_2\sinh(e\lambda cx)] + \vec{e}_2 b_\mu\lambda$;
7. $\vec{A}_\mu = \vec{e}_1 k_\mu Z_0\left[\frac{i}{2}e\lambda((bx)^2 + (cx)^2)\right] + \vec{e}_2(b_\mu cx - c_\mu bx)\lambda$;
8. $\vec{A}_\mu = \vec{e}_1 k_\mu[\lambda_1((bx)^2 + (cx)^2)^{\frac{e\lambda}{2}} + \lambda_2((bx)^2 + (cx)^2)^{-\frac{e\lambda}{2}}] + \vec{e}_2(b_\mu cx - c_\mu bx)\lambda((bx)^2 + (cx)^2)^{-1}$;

$$
\begin{aligned}
9.\quad \vec{A}_\mu &= \left[\vec{e}_2\left(\frac{1}{8}(d_\mu - k_\mu(kx)^2) + \frac{1}{2}b_\mu kx\right) + \vec{e}_3 c_\mu\right]\lambda\,\mathrm{sn}\left(\frac{e\sqrt{2}}{8}\lambda(4bx + (kx)^2)\right)\times \\
&\quad \times \mathrm{dn}\left(\frac{e\sqrt{2}}{8}\lambda(4bx + (kx)^2)\right)\left(\mathrm{cn}\left(\frac{e\sqrt{2}}{8}\lambda(4bx + (kx)^2)\right)\right)^{-1}; \\
10.\quad \vec{A}_\mu &= \left[\vec{e}_2\left(\frac{1}{8}(d_\mu - k_\mu(kx)^2) + \frac{1}{2}b_\mu kx\right) + \vec{e}_3 c_\mu\right]\times \\
&\quad \times \lambda\left[\mathrm{cn}\left(\frac{e\sqrt{2}\lambda}{8}(4bx + (kx)^2)\right)\right]^{-1}; \\
11.\quad \vec{A}_\mu &= \left[\vec{e}_2\left(\frac{1}{8}(d_\mu - k_\mu(kx)^2) + \frac{1}{2}b_\mu kx\right) + \vec{e}_3 c_\mu\right]\times \\
&\quad \times 4\sqrt{2}(e(4bx + (kx)^2) - \lambda)^{-1}; \\
12.\quad \vec{A}_\mu &= \vec{e}_1 k_\mu\sqrt{4bx + (kx)^2}\,Z_{\frac{1}{4}}\left(\frac{ie\lambda}{8}(4bx + (kx)^2)^2\right) + \vec{e}_2 c_\mu\lambda(4bx + (kx)^2); \\
13.\quad \vec{A}_\mu &= \vec{e}_1 k_\mu\left(\lambda_1\cosh\left(\frac{e\lambda}{4}(4bx + (kx)^2)\right) + \right. \\
&\quad \left. + \lambda_2\sinh\left(\frac{e\lambda}{4}(4bx + (kx)^2)\right)\right) + \vec{e}_2 c_\mu\lambda; \\
14.\quad \vec{A}_\mu &= \left\{\vec{e}_2\left(d_\mu - \frac{1}{8}k_\mu(kx)^2 - \frac{1}{2}b_\mu kx\right) + \right. \\
&\quad \left. + \vec{e}_3\left(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx\right)(1+\alpha^2)^{-\frac{1}{2}}\right\}\times \\
&\quad \times \lambda\,\mathrm{sn}\left[\frac{e\lambda\sqrt{2}}{8}(4(\alpha bx - cx) + \alpha(kx)^2)(1+\alpha^2)^{-\frac{1}{2}}\right]\times \\
&\quad \times \mathrm{dn}\left[\frac{e\lambda\sqrt{2}}{8}(4(\alpha bx - cx) + \alpha(kx)^2)(1+\alpha^2)^{-\frac{1}{2}}\right]\times \\
&\quad \times\left\{\mathrm{cn}\left[\frac{e\lambda\sqrt{2}}{8}(4(\alpha bx - cx) + \alpha(kx)^2)(1+\alpha^2)^{-\frac{1}{2}}\right]\right\}^{-1}; \\
15.\quad \vec{A}_\mu &= \left\{\vec{e}_2\left(d_\mu - \frac{1}{8}k_\mu(kx)^2 - \frac{1}{2}b_\mu kx\right) + \right. \\
&\quad \left. + \vec{e}_3\left(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx\right)(1+\alpha^2)^{-\frac{1}{2}}\right\}\times \\
&\quad \times\left\{\mathrm{cn}\left[\frac{e\lambda}{4}(4(\alpha bx - cx) + \alpha(kx)^2)(1+\alpha^2)^{-\frac{1}{2}}\right]\right\}^{-1}; \\
16.\quad \vec{A}_\mu &= \left\{\vec{e}_2\left(d_\mu - \frac{1}{8}k_\mu(kx)^2 - \frac{1}{2}b_\mu kx\right) + \right. \\
&\quad \left. + \vec{e}_3\left(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx\right)(1+\alpha^2)^{-\frac{1}{2}}\right\}\times \\
&\quad \times 4\sqrt{2}(1+\alpha^2)^{\frac{1}{2}}[e(4(\alpha bx - cx) + \alpha(kx)^2)]^{-1}; \\
17.\quad \vec{A}_\mu &= \vec{e}_1 k_\mu\left\{\sqrt{4(\alpha bx - cx) + \alpha(kx)^2}\times\right.
\end{aligned}
\tag{5.6}
$$

$$\times Z_{\frac{1}{4}}\left(\frac{ie\lambda}{8}(4(\alpha bx - cx) + \alpha(kx)^2)^2(1+\alpha^2)^{-\frac{1}{2}}\right)\Big\}+$$
$$+\vec{e}_2\left(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx\right)\lambda(4(\alpha bx - cx) + \alpha(kx)^2)(1+\alpha^2)^{-\frac{1}{2}};$$

18. $$\vec{A}_\mu = \vec{e}_1 k_\mu\Big\{\lambda_1\cosh\left[\frac{e\lambda}{4}(1+\alpha^2)^{-\frac{1}{2}}(4(\alpha bx - cx) + \alpha(kx)^2)\right]+$$
$$+\lambda_2\sinh\left[\frac{e\lambda}{4}(1+\alpha^2)^{-\frac{1}{2}}(4(\alpha bx - cx) + \alpha(kx)^2\right]\Big\}+$$
$$+\vec{e}_2\left(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx\right)\lambda(1+\alpha^2)^{-\frac{1}{2}};$$

19. $$\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1} Z_0\left[\frac{ie\lambda}{2}((bc)^2 + (cx)^2)\right] + \vec{e}_2(b_\mu cx - c_\mu bx)\lambda;$$

20. $$\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}[\lambda_1((bx)^2+(cx)^2)^{\frac{e\lambda}{2}} + \lambda_2((bx)^2+(cx)^2)^{-\frac{e\lambda}{2}}]+$$
$$+\vec{e}_2(b_\mu cx - c_\mu bx)\lambda((bx)^2+(cx)^2)^{-1};$$

21. $$\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}\sqrt{cx}Z_{\frac{1}{4}}\left(\frac{ie\lambda}{2}(cx)^2\right) + \vec{e}_2(b_\mu - k_\mu bx(kx)^{-1})\lambda cx;$$

22. $$\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}[\lambda_1\cosh(\lambda ecx) + \lambda_2\sinh(\lambda ecx)] + \vec{e}_2(b_\mu - k_\mu bx(kx)^{-1})\lambda;$$

23. $$\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}\sqrt{\ln|kx| - cx}\,Z_{\frac{1}{4}}\left(\frac{ie\lambda}{2}(\ln|kx| - cx)^2\right)+$$
$$+\vec{e}_2(b_\mu - k_\mu bx(kx)^{-1})\lambda(\ln|kx| - cx);$$

24. $$\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}[\lambda_1\cosh(\lambda e(\ln|kx| - cx)) + \lambda_2\sinh(\lambda e(\ln|kx| - cx))]+$$
$$+\vec{e}_2(b_\mu - k_\mu bx(kx)^{-1})\lambda;$$

25. $$\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}\sqrt{\alpha\ln|kx| - cx}\,Z_{\frac{1}{4}}\left(\frac{ie\lambda}{2}(\alpha\ln|kx| - cx)^2\right)+$$
$$+\vec{e}_2(b_\mu - k_\mu(bx - \ln|kx|)(kx)^{-1})\lambda(\alpha\ln|kx| - cx);$$

26. $$\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}[\lambda_1\cosh(\lambda e(\alpha\ln|kx| - cx))+$$
$$+\lambda_2\sinh(\lambda e(\alpha\ln|kx| - cx))] + \vec{e}_2(b_\mu - k_\mu(bx - \ln|kx|)(kx)^{-1})\lambda;$$

27. $$\vec{A}_\mu = \{\vec{e}_1(b_\mu - k_\mu bx(kx)^{-1}) + \vec{e}_2(c_\mu - k_\mu cx(kx)^{-1})\}e^{-1}(x_\mu x^\mu)^{-\frac{1}{2}};$$

28. $$\vec{A}_\mu = \{\vec{e}_1(b_\mu - k_\mu bx(kx)^{-1}) + \vec{e}_2(c_\mu - k_\mu cx(kx)^{-1})\}f(x_\mu x^\mu),$$
$$w\ddot{f} + 2\dot{f} + (e^2 f^3/4) = 0, \quad w = x_\mu x^\mu = (ax)^2 - (bx)^2 - (cx)^2 - (dx)^2.$$

In the above formulae $Z_\alpha(w)$ is the Bessel function; sn, dn, cn are Jacobi elliptic functions having the modulus $\frac{\sqrt{2}}{2}$; $\lambda, \lambda_1, \lambda_2 = \text{const}$.

In the present paper we do not analyze in detail the obtained solution. We only note that the solutions numbered by 27 is nothing more but the meron solution of YME [2]. In the Euclidean space meron and instanton solutions were obtained by Alfaro, Fubini, Furlan [9] and Belavin, Polyakov, Schwartz, Tyupkin [10] with the use of the ansatz suggested by 't Hooft [11], Corrigan and Fairlie [12] and Wilczek [13].

Another important point is that we can obtain new exact solutions of YME by applying to solutions (5.6) the solution generation technique. We do not adduce corresponding formulae because of their cumbersomity.

# 6 Some generalizations

It was noticed in [14] that group-invariant solutions of nonlinear PDE could provide us with rather general information about the structure of solutions of the equation under study. Using this fact, we constructed in [4, 14] a number of new exact solutions of the nonlinear Dirac equation which could not be obtained by symmetry reduction procedure. We will demonstrate that the same idea will be effective for constructing new solutions of YME.

Solutions of YME numbered by 7, 8, 19, 20 can be presented in the following unified form:

$$\vec{A}_\mu = k_\mu \vec{B}(kx, cx) + b_\mu \vec{C}(kx, cx), \tag{6.1}$$

where $kx = k_\mu x^\mu$, $cx = c_\mu x^\mu$, $k_\mu = a_\mu + d_\mu$.

Substituting the ansatz (6.1) into YME and splitting the equality obtained with respect to linearly-independent four-vectors with components $k_\mu$, $b_\mu$, $c_\mu$, we get

$$\begin{aligned} &1. \quad \vec{C}_{w_1 w_1} = \vec{0}, \\ &2. \quad \vec{C} \times \vec{C}_{w_1} = \vec{0}, \\ &3. \quad \vec{B}_{w_1 w_1} + e\vec{C}_{w_0} \times \vec{C} + e^2 \vec{C} \times (\vec{C} \times \vec{B}) = \vec{0}. \end{aligned} \tag{6.2}$$

Here we use designations $w_0 = kx$, $w_1 = cx$.

A general solution of the first two equations from (6.2) is given by one of the formulae

$$\begin{aligned} &\text{I.} \quad \vec{C} = \vec{f}(w_0), \\ &\text{II.} \quad \vec{C} = (w_1 + v_0(w_0))\vec{f}(w_0), \end{aligned}$$

where $v_0$, $\vec{f}$ are arbitrary smooth functions.

Consider the case $\vec{C} = \vec{f}(w_0)$. Substituting this expression into the third equation from (6.2) we have

$$\vec{B}_{w_1 w_1} + e\vec{f}_{w_0} \times \vec{f} + e^2 \vec{f}(\vec{f}\vec{B}) - e^2 \vec{f}^2 \vec{B} = \vec{0}. \tag{6.3}$$

Since equations (6.3) do not contain derivatives of $\vec{B}$ with respect to $w_0$, they can be considered as a system of ODE with respect to the variable $w_1$. Multiplying (6.3) by $\vec{f}$ we arrive at the relation $(\vec{B}\vec{f})_{w_1 w_1} = 0$, whence

$$\vec{B}\vec{f} = v_1(w_0) w_1 + v_2(w_0). \tag{6.4}$$

In (6.4) $v_1$, $v_2$ are arbitrary smooth enough functions.

With account of (6.4) system (6.3) reads

$$\vec{B}_{w_1 w_1} - e^2 \vec{f}^2 \vec{B} = e\vec{f} \times \vec{f}_{w_0} - e^2 (v_1 w_1 + v_2)\vec{f}.$$

The above linear system of ODE is easily integrated. Its general solution is given by the formula

$$\begin{aligned} \vec{B} = {}& \vec{g}(w_0) \cosh e|\vec{f}| w_1 + \vec{h}(w_0) \sinh e|\vec{f}| w_1 + \\ & + e^{-1} |\vec{f}|^{-2} \vec{f}_{w_0} \times \vec{f} + |\vec{f}|^{-2} (v_1 w_1 + v_2) \vec{f}, \end{aligned} \tag{6.5}$$

where $\vec{g}$, $\vec{h}$ are arbitrary smooth functions.

Substituting (6.5) into (6.4) we get the following restrictions on the choice of the functions $\vec{g}$, $\vec{h}$:

$$\vec{f}\vec{g} = 0, \quad \vec{f}\vec{h} = 0. \tag{6.6}$$

Thus, provided $\vec{C}_{w_1} = 0$, a general solution of the system of ODE (6.3) is given by the formulae (6.5), (6.6). Substituting (6.5) into the initial ansatz (6.1) we obtain the following family of exact solutions of YME:

$$\vec{A}_\mu = k_\mu\{\vec{g}(kx)\cosh e|\vec{f}|cx + \vec{h}(kx)\sinh e|\vec{f}|cx + \\ + e^{-1}|\vec{f}|^{-2}\dot{\vec{f}} \times \vec{f} + (v_1(kx)cx + v_2(kx))\vec{f}\} + b_\mu\vec{f}$$

where $\vec{f}(kx)$, $\vec{g}(kx)$, $\vec{h}(kx)$, $v_1(kx)$, $v_2(kx)$ are arbitrary smooth functions satisfying (6.6), $\dot{\vec{f}} = \frac{d\vec{f}}{d\omega_0}$.

The case $\vec{C} = (w_1 + v_0(w_0))\vec{f}(w_0)$ is treated in analogous way. As a result, we obtain the following family of exact solutions of YME:

$$\vec{A}_\mu = k_\mu\left\{(cx + v_0(kx))^{\frac{1}{2}}\left[\vec{g}(kx)J_{\frac{1}{4}}\left(\frac{ie}{2}|\vec{f}|(\vec{c}\vec{x} + v_0(kx))^2\right) + \right.\right. \\ \left. + \vec{h}(kx)Y_{\frac{1}{4}}\left(\frac{ie}{2}|\vec{f}|(cx + v_0(kx))^2\right)\right] + \\ \left. + (v_1(kx)cx + v_2(kx))\vec{f} + e^{-1}|\vec{f}|^{-2}\dot{\vec{f}} \times \vec{f}\right\} + b_\mu(cx + v_0(kx))\vec{f},$$

where $\vec{f}(kx)$, $\vec{g}(kx)$, $\vec{h}(kx)$, $v_0(kx)$, $v_1(kx)$, $v_2(kx)$ are arbitrary smooth functions satisfying (6.6), $J_{\frac{1}{4}}(w)$, $Y_{\frac{1}{4}}(w)$ are the Bessel functions.

Another effective ansatz for the Yang–Mills field is obtained if one replaces in (6.1) $cx$ by $bx$

$$\vec{A}_\mu = k_\mu\vec{B}(kx, bx) + b_\mu\vec{C}(kx, bx). \tag{6.7}$$

Substitution of (6.7) into YME yields the following system of PDE for $\vec{B}$, $\vec{C}$:

$$\vec{B}_{w_1w_1} - \vec{C}_{w_0w_1} - e(\vec{B} \times \vec{C}_{w_1} + 2\vec{B}_{w_1} \times \vec{C} + \vec{C} \times \vec{C}_{w_0}) + e^2\vec{C} \times (\vec{C} \times \vec{B}) = \vec{0}. \tag{6.8}$$

We succeeded in integrating system (6.8), provided $\vec{C} = \vec{f}(w_0)$. Substituting the result obtained into (6.7), we come to the following family of exact solutions of YME:

$$\vec{A}_\mu = k_\mu\{(\vec{g} + |\vec{f}|^{-1}\vec{g} \times \vec{f}bx)\cos(e|\vec{f}|bx) + (\vec{h} + |\vec{f}|^{-1}\vec{h} \times \vec{f}bx)\sin(e|\vec{f}|bx) + \\ + e^{-1}|\vec{f}|^{-2}\dot{\vec{f}} \times \vec{f} + (v_1(kx)bx + v_2(kx))\vec{f}\} + b_\mu\vec{f},$$

where $\vec{f}(kx)$, $\vec{g}(kx)$, $\vec{h}(kx)$, $v_1(kx)$, $v_2(kx)$ are arbitrary smooth functions.

Besides that, we obtained the following class of exact solutions of YME:

$$\vec{A}_\mu = k_\mu\vec{e}_1v_0(kx)u^2(bx) + b_\mu\vec{e}_2u(bx),$$

where $\vec{e}_1 = (1, 0, 0)$, $\vec{e}_2 = (0, 1, 0)$; $v_0(kx)$ is an arbitrary smooth function; $u(bx)$ is a solution of the nonlinear ODE $\ddot{u} = e^2u^5$, which is integrated in elliptic functions.

In conclusion of this Section we will obtain a generalization of the plane-wave Coleman solution [15] ]

$$\vec{A}_\mu = k_\mu(\vec{f}(kx)bx + \vec{g}(kx)cx). \tag{6.9}$$

It is not difficult to verify that (6.9) satisfy YME with arbitrary $\vec{f}$, $\vec{g}$.

Evidently, solution (6.9) is a particular case of the ansatz

$$\vec{A}_\mu = k_\mu \vec{B}(kx, bx, cx). \tag{6.10}$$

Substituting (6.10) into YME we get

$$\vec{B}_{w_1 w_1} + \vec{B}_{w_2 w_2} = \vec{0}, \tag{6.11}$$

where $w_1 = bx$, $w_2 = cx$.

Integrating the Laplace equations (6.11) and substituting the result obtained into (6.10) we have

$$\vec{A}_\mu = k_\mu(\vec{U}(kx, bx + icx) + \vec{U}(kx, bx - icx)).$$

Here $\vec{U}(kx, z)$ is an arbitrary analytical with respect to $z$ function. Choosing $\vec{U} = \frac{1}{2}(\vec{f}(kx) - i\vec{g}(kx))z$ we get Coleman solution (6.9).

## 7 Conclusion

Thus, starting from the invariance of YME under the Poincaré group we have obtained wide families of its exact solutions including arbitrary functions. In our future papers we intend to describe exact solutions of YME invariant under the extended Poincaré group and conformal group.

Besides that, we will study exact solutions which correspond to the conditional and non-local symmetries of the Yang–Mills equations (1.1)

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