Symmetry properties, reduction and exact solutions of biwave equations

W.I. FUSHCHYCH, R.Z. ZHDANOV, O.V. ROMAN

We have studied symmetry properties of the biwave equations $\Box^2 u = F(u)$ and the systems of wave equations which are equivalent to them. Reduction of the nonlinear biwave equations with the use of subalgebras of the extended Poincaré algebra $\widetilde{AP}(1,1)$ and the conformal algebra $C(1,1)$ was carried out. Some exact solutions of these equations were obtained.

It was suggested in [1] to describe different physical processes with the help of nonlinear partial equations of high order, namely

$$\Box^l u = F\Big(u, \frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu}\Big). \tag{1}$$

Here and further $\Box = \partial^2/\partial x_0 - \partial^2/\partial x_1 - \cdots - \partial^2/\partial x_n$ is d'Alembertian in $(n+1)$-dimensional pseudo-Euclidean space $\mathbb{R}(1,n)$ with metric tensor $g_{\mu\nu} = \mathrm{diag}(1,-1,\ldots,-1)$, $\mu,\nu = \overline{0,n}$; $\Box^l = \Box(\Box^{l-1})$, $l \in \mathbb{N}$; $x_\mu = x^\nu g_{\mu\nu}$; $F(\cdot,\cdot)$ is an arbitrary smooth function; $u = u(x)$ is a real function; the summation over the repeated indices from 0 to n is understood.

Equations (1) were considered from different points of view in [2, 3, 4], where the pseudodifferential equations of type (1) were also studied (in this case l is fractional or negative).

Assuming $l = 1$ and $F = F(u)$ in (1) we obtain the standard wave equation

$$\Box u = F(u) \tag{2}$$

which describes a scalar spinless uncharged particle in quantum field theory. Symmetry properties of equation (2) were studied in [4, 5, 6] and wide classes of its exact solutions with certain concrete values of the function $F(u)$ were obtained in [4, 5, 7, 8, 9].

In this paper we restrict ourselves by considering the biwave equation

$$\Box^2 u = F(u) \tag{3}$$

which is one of the simplest equations of type (1) of high order ($l = 2$, $F = F(u)$).

1 Symmetry classification of the biwave equation

In order to carry out a symmetry classification of equation (3) we shall establish at first the maximal transformation group admitted by equation (3) provided $F(u)$ is an arbitrary function. After that we shall determine all the functions $F(u)$ when equation (3) admits more extended symmetry.

Results of symmetry classification of equation (3) are cited in the following statements.

Preprint LiTH-MAT-R-95-20, Department of Mathematics, Linköping University, Sweden, 16 p.

Lemma 1 *The maximal invariance group of equation (3) with an arbitrary function $F(u)$ is the Poincaré group $P(1,n)$ generated by the operators*

$$P_\mu = \frac{\partial}{\partial x_\mu}, \quad J_{\mu\nu} = x^\mu \frac{\partial}{\partial x_\nu} - x^\nu \frac{\partial}{\partial x_\mu}, \quad \mu,\nu = \overline{0,n}. \tag{4}$$

Theorem 1 *All the equations of type (3) admitting more extended invariance algebra than the Poincaré algebra $AP(1,n)$ are equivalent one of the following:*

$$1.\ \Box^2 u = \lambda_1 u^k, \quad \lambda_1 \neq 0,\ k \neq 0,1; \tag{5}$$

$$2.\ \Box^2 u = \lambda_2 e^u, \quad \lambda_2 \neq 0; \tag{6}$$

$$3.\ \Box^2 u = \lambda_3 u, \quad \lambda_3 \neq 0; \tag{7}$$

$$4.\ \Box^2 u = 0. \tag{8}$$

Here λ_1, λ_2, λ_3 are arbitrary constants.

Theorem 2 *The symmetry of the equations (5)–(8) is described in the following way:*

1. (a) The maximal invariance group of equation (5) when $k \neq (n+5)/(n-3)$, $k \neq 0,1$ is the extended Poincaré group $\widetilde{P}(1,n)$ generated by the operators (4) and

$$D = x_\mu \frac{\partial}{\partial x_\mu} + \frac{4}{1-k} u \frac{\partial}{\partial u}.$$

(b) The maximal invariance group of equation (5) when $k = (n+5)/(n-3)$, $n \neq 3$ is the conformal group $C(1,n)$ generated by the operators (4) and

$$\begin{gathered} D^{(1)} = x_\mu \frac{\partial}{\partial x_\mu} + \frac{3-n}{2} u \frac{\partial}{\partial u}, \\ K_\mu^{(1)} = 2x^\mu D^{(1)} - (x_\nu x^\nu) \frac{\partial}{\partial x_\mu}. \end{gathered} \tag{9}$$

2. (a) The maximal invariance group of equation (6) when $n \neq 3$ is the extended Poincaré group $\widetilde{P}(1,n)$ generated by the operators (4) and

$$D^{(2)} = x_\mu \frac{\partial}{\partial x_\mu} - 4\frac{\partial}{\partial u}. \tag{10}$$

(b) The maximal invariance group of equation (6) when $n = 3$ is the conformal group $C(1,n)$ generated by the operators (4) and

$$K_\mu^{(2)} = 2x^\mu D^{(2)} - (x_\nu x^\nu) \frac{\partial}{\partial x_\mu}. \tag{11}$$

3. The maximal invariance group of equation (7) is generated by the operators (4) and

$$Q = h(x) \frac{\partial}{\partial u}, \quad I = u \frac{\partial}{\partial u},$$

where $h(x)$ is an arbitrary solution of equation (7).

4. The maximal invariance group of equation (8) is generated by the operators (4), (9) and

$$Q = q(x)\frac{\partial}{\partial u}, \quad I = u\frac{\partial}{\partial u},$$

where $q(x)$ is an arbitrary solution of equation (8).

The proof of Lemma 1 and Theorems 1, 2 is carried out by means of the infinitesimal algorithm of S. Lie [4, 10]. Since it requires very cumbersome computations we only give a general scheme of the proof.

In the Lie approach the infinitesimal operator of equation (3) invariance group is of the form

$$X = \xi^\mu(x,u)\frac{\partial}{\partial x_\mu} + \eta(x,u)\frac{\partial}{\partial u}. \tag{12}$$

The invariance criterion of equation (3) under group generated by the operators (12) is

$$\underset{4}{X}(\Box^2 u - F(u))\Big|_{\Box^2 u = F(u)} = 0, \tag{13}$$

where $\underset{4}{X}$ is the 4-th prolongation of the operator X.

Splitting equation (13) with respect to the independent variables, we come to the system of partial differential equations for functions $\xi^\mu(x,u)$ and $\eta(x,u)$:

$$\begin{aligned}
&\xi^\mu_u = 0, \quad \eta_{uu} = 0, \quad \mu = \overline{0,n}, \\
&\xi^i_0 = \xi^0_i, \quad \xi^i_j = -\xi^j_i, \quad i \neq j, \ i,j = \overline{1,n}, \\
&\xi^0_0 = \xi^1_1 = \cdots = \xi^n_n, \\
&2\eta_{\nu u} = (3-n)\xi^\nu_{00}, \quad \nu = \overline{0,n},
\end{aligned} \tag{14}$$

$$\Box^2\eta - \eta F'(u) + F(u)(\eta_u - 4\xi^0_0) = 0. \tag{15}$$

Besides, when $n = 1$, there are additional equations:

$$\eta_{00u} = 0, \quad \eta_{01u} = 0, \tag{16}$$

that do not follow from equations (14) and (15).

In the above formulae we use the notations $\xi^\mu_\nu = \partial\xi^\mu/\partial x_\nu$, $\eta_\mu = \partial\eta/\partial x_\mu$ and so on.

System (14) is a system of Killing equations. The general solution of equations (14), (16) is of the form:

$$\begin{aligned}
&\xi^\nu = 2x^\nu x_\mu c^\mu - x_\mu x^\mu c^\nu + b_{\nu\mu}x^\mu + dx_\nu + a_\nu, \\
&\eta = ((3-n)c^\mu x_\mu + p)u + \varkappa(x),
\end{aligned} \tag{17}$$

where c^μ, $b_{\nu\mu} = -b_{\mu\nu}$, d, a_ν, p are arbitrary constants, $\varkappa(x)$ is an arbitrary smooth function.

Substituting (17) into the classifying equation (15) and splitting it with respect to u we obtain statements of Lemma 1 and Theorems 1, 2 according to the form of $F(u)$.

It follows from the statements proved that the equation of type (1) is invariant under the extended Poincaré group $\widetilde{P}(1,n)$ iff it is equivalent one of equations (5), (6) or (8). Let us note that the analogous result was obtained for the wave equations (2) in [5].

The following statement also is the consequence of the Theorems but since it is important we adduce it as a Theorem.

Theorem 3 *Equation (3) is invariant under the conformal group $C(1,n)$ iff it is equivalent to the following:*

$$1.\ \Box^2 u = \lambda_1 u^{(n+5)/(n-3)}, \quad n \neq 3; \tag{18}$$

$$2.\ \Box^2 u = \lambda_2 e^u, \quad n = 3. \tag{19}$$

Let us note that conformal invariance of equation (18) was first ascertained in [11] and that of equation (19) was done in [4] by means of Baker–Campbell–Hausdorff formulae.

In conclusion of the Section let us emphasize an important property of the linear biwave equation (8), when $n = 3$, which is the consequence of Theorems 2 and 3.

Corollary *There exist two nonequivalent representations of the Lie algebra of the conformal group $C(1,n)$ on the set of solutions of equation (8) [1, 3, 4]:*

$$\begin{aligned}
1.\quad & P^{(1)}_\mu = P_\mu, \quad J^{(1)}_{\mu\nu} = J_{\mu\nu}, \\
& D^{(1)} = x_\mu \frac{\partial}{\partial x_\mu}, \quad K^{(1)}_\mu = 2x^\mu D^{(1)} - (x_\nu x^\nu)\frac{\partial}{\partial x_\mu}; \\
2.\quad & P^{(2)}_\mu = P_\mu, \quad J^{(2)}_{\mu\nu} = J_{\mu\nu}, \\
& D^{(2)} = x_\mu \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial u}, \quad K^{(2)}_\mu = 2x^\mu D^{(2)} - (x_\nu x^\nu)\frac{\partial}{\partial x_\mu},
\end{aligned}$$

where the operators P_μ, $J_{\mu\nu}$ are determined in (4).

2 Symmetry classification of system of wave equations

Introducing a new variable $v = \Box u$ in (3) we get the system of partial differential equations

$$\begin{aligned}
&\Box u = v, \\
&\Box v = F(u),
\end{aligned} \tag{20}$$

which is equivalent to the biwave equation (3).

Symmetry properties of the system (20) are investigated by analogy with the previous Section. So we only formulate statements analogous to the preceding ones without proving them.

Lemma 2 *The maximal invariance group of the system (20) with an arbitrary function $F(u)$ is the Poincaré group $P(1,n)$ generated by the operators (4).*

Theorem 4 *All the systems of type (20) admitting more extended invariance algebra than the Poincaré algebra $AP(1,n)$ are equivalent one of the following:*

$$1.\quad \begin{aligned} &\Box u = v,\\ &\Box v = \lambda_1 u^k, \quad \lambda_1 \neq 0,\ k \neq 0, 1; \end{aligned} \tag{21}$$

$$2.\quad \begin{aligned} &\Box u = v,\\ &\Box v = \lambda_2 u, \quad \lambda_2 \neq 0; \end{aligned} \tag{22}$$

$$3.\quad \begin{aligned} &\Box u = v,\\ &\Box v = 0. \end{aligned} \tag{23}$$

Theorem 5 *The symmetries of the systems (21)–(23) is described in the following way:*

1. The maximal invariance group of the system (21) is the extended Poincaré group $\widetilde{P}(1,n)$ generated by the operators (4) and

$$D = x_\mu \frac{\partial}{\partial x_\mu} + \frac{4}{1-k} u \frac{\partial}{\partial u} + \frac{2(1+k)}{1-k} v \frac{\partial}{\partial v}.$$

2. The maximal invariance group of the system (22) is generated by the operators (4) and

$$Q_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad Q_2 = v\frac{\partial}{\partial u} + \lambda_2 u \frac{\partial}{\partial v}, \quad Q_3 = h_1(x)\frac{\partial}{\partial u} + h_2(x)\frac{\partial}{\partial v},$$

where $(h_1(x), h_2(x))$ is an arbitrary solution of the system (22).

3. The maximal invariance group of the system (23) is generated by the operators (4) and

$$D = x_\mu \frac{\partial}{\partial x_\mu} + 2u\frac{\partial}{\partial u}, \quad Q_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},$$
$$Q_2 = v\frac{\partial}{\partial u}, \quad Q_3 = q_1(x)\frac{\partial}{\partial u} + q_2(x)\frac{\partial}{\partial v},$$

where $(q_1(x), q_2(x))$ is an arbitrary solution of the system (23).

It follows from the foregoing statements that unlike the biwave equations, the extended Poincaré group $\widetilde{P}(1,n)$ is the invariance group of the system (20) only in two cases, namely, when (20) is equivalent to (21) or (23). Moreover, the system (20) is not invariant under the conformal group for any functions $F(u)$. Therefore, in the class of Lie operators, the invariance algebras of the biwave equations and the corresponding systems of wave equations are essentially different.

3 Reduction and exact solutions of the equation $\Box^2 u = \lambda e^u$

As follows from Theorem 2 the maximal invariance group of the equation (6), when $n = 1$ is the extended Poincaré group $\widetilde{P}(1,1)$ with generators

$$P_0 = \frac{\partial}{\partial x_0}, \quad P_1 = \frac{\partial}{\partial x_1}, \quad J_{01} = x^0 \frac{\partial}{\partial x_1} - x^1 \frac{\partial}{\partial x_0}, \tag{24}$$

$$D^{(2)} = x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} - 4\frac{\partial}{\partial u}. \qquad (25)$$

It is known that if an equation admits the symmetry operator

$$X = \xi^\mu(x)\frac{\partial}{\partial x_\mu} + \eta(x)\frac{\partial}{\partial u} \qquad (26)$$

then its solutions can be found in the form [4]:

$$u(x) = \varphi(\omega) + g(x). \qquad (27)$$

For the substitution (27) to be an ansatz for the equation with the symmetry operator (26), the functions $\omega(x)$ and $g(x)$ are to satisfy the following conditions:

$$\xi^\mu(x)\frac{\partial \omega}{\partial x_\mu} = 0, \quad \xi^\mu(x)\frac{\partial g(x)}{\partial x_\mu} = \eta(x).$$

To obtain all the $\widetilde{P}(1,1)$-nonequivalent ansatzes (27) we have to describe all the nonequivalent one-dimensional subalgebras of the Lie algebra $\widetilde{AP}(1,1)$ spanned by the operators (24) and (25) (see [4, 9]). In the paper we make use of classification given in [9] and omitting rather cumbersome computations we write $\widetilde{P}(1,1)$-nonequivalent ansatzes in Table 1.

Table 1.

N	Algebra	Invariant variables ω	Ansatz
1°	$D - J_{01}$	$x_0 + x_1$	$u = \varphi(\omega) - 2\ln(x_0 - x_1)$
2°	$D + \alpha J_{01},\ \alpha \neq -1$	$(1+\alpha)\ln(x_1 - x_0) - (1-\alpha)\ln(x_0 + x_1)$	$u = \varphi(\omega) - \dfrac{4}{\alpha+1}\ln(x_0 + x_1)$
3°	$D - J_{01} + P_0$	$\ln(x_0 - x_1 + 1/2) - 2(x_0 + x_1)$	$u = \varphi(\omega) - 2\ln(x_0 - x_1 + 1/2)$
4°	J_{01}	$x_0^2 - x_1^2$	$u = \varphi(\omega)$
5°	P_0	x_1	$u = \varphi(\omega)$
6°	$P_0 + P_1$	$x_0 - x_1$	$u = \varphi(\omega)$

Remark. Inequivalent subalgebras listed in Table 1 are built by taking account of the obvious fact that equation (6) is invariant under the transformations of the form:

$$\begin{array}{l} x_0' \to x_0, \\ x_1' \to -x_1; \end{array} \quad \text{and} \quad \begin{array}{l} x_0' \to x_1, \\ x_1' \to x_0. \end{array} \qquad (28)$$

Substituting ansatzes obtained in (6) we get the following equations for the function $\varphi(\omega)$:

1° $\quad 0 = \lambda e^\varphi,$

2° $\quad \varphi^{(4)}(\alpha^2-1)^2 + 2\varphi^{(3)}\alpha(1-\alpha^2) - \varphi^{(2)}(1-\alpha^2) = \dfrac{\lambda}{16}\exp\left(\varphi + \dfrac{2\omega}{\alpha+1}\right),$

3° $\quad \varphi^{(4)} - \varphi^{(3)} = \dfrac{\lambda}{64}e^\varphi,$

4° $\quad \varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + 2\varphi^{(2)} = \dfrac{\lambda}{16}e^\varphi,$

5° $\quad \varphi^{(4)} = \lambda e^\varphi,$

6° $\quad 0 = \lambda e^\varphi.$

Equation 5° has the partial solution

$$\varphi = \ln\left(\frac{24}{\lambda}(\omega + c)^{-4}\right), \quad \lambda > 0,$$

that leads us to the following exact solutions of equation (6):

$$u = \ln\left(\frac{24}{\lambda}(x_0 + c_1)^{-4}\right), \quad \lambda > 0,$$
$$u = \ln\left(\frac{24}{\lambda}(x_1 + c_2)^{-4}\right), \quad \lambda > 0. \tag{29}$$

Here c, c_1, c_2 are arbitrary constants. This solutions are invariant under the operators P_0 and P_1 accordingly.

To finish the Section let us note that the solutions (29) can be obtained by making use of the ansatz in Liouville form [4]:

$$u = \ln\left\{\frac{24}{\lambda}\frac{\Big(\dot\varphi_1(\omega_1)\dot\varphi_2(\omega_2)\Big)^2}{\big(\varphi_1(\omega_1) + \varphi_2(\omega_2)\big)^4}\right\}, \quad \omega_1 = x_0 + x_1, \quad \omega_2 = x_0 - x_1,$$

which reduces equation (6) to one of the following systems:

1. $\ddot\varphi_1 = 0, \quad \ddot\varphi_2 = 0;$
2. $\ddot\varphi_1 = \dfrac{2\dot\varphi_1^2}{\varphi_1}, \quad \ddot\varphi_2 = \dfrac{2\dot\varphi_2^2}{\varphi_2}.$

Here $\dot\varphi$ and $\ddot\varphi$ mean the first derivative and the second one of the corresponding argument.

Finding the general solution of the systems we get the following exact solutions of equation (6):

$$u = \ln\left(\frac{24}{\lambda}\frac{(a^2 - b^2)^2}{(ax_0 + bx_1 + c)^4}\right), \tag{30}$$

where a, b, c are arbitrary constants.

Solution (30) can be obtained from (29) by the transformations of the extended Poincaré group with the generators (24) and (25).

4 Reduction and exact solutions of the equation $\Box^2 u = \lambda u^k$

It follows from Theorem 2 that when $n = 1$ the equation (5) is invariant under the extended Poincaré group $\widetilde{P}(1,1)$ with the generators (24) and

$$D = x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} + \frac{4}{1-k}u\frac{\partial}{\partial u}. \tag{31}$$

If an equation admits the symmetry operator

$$X = \xi^\mu(x)\frac{\partial}{\partial x_\mu} + \eta(x)u\frac{\partial}{\partial u} \tag{32}$$

then its solutions can be found in the form [4]:

$$u(x) = f(x)\varphi(\omega) \tag{33}$$

provided functions $\omega(x)$ and $f(x)$ satisfy the following system:

$$\xi^\mu(x)\frac{\partial\omega}{\partial x_\mu} = 0, \quad \xi^\mu(x)\frac{\partial f(x)}{\partial x_\mu} = \eta(x)f(x). \tag{34}$$

With an allowance for invariance of equation (5) under the changes of variables (28) we write $\widetilde{P}(1,1)$-nonequivalent ansatzes of the form (33) in Table 2.

Table 2.

N	Algebra	Invariant variables ω	Ansatz
1°	$D - J_{01}$	$x_0 + x_1$	$u = (x_0 - x_1)^{\frac{2}{1-k}}\varphi(\omega)$
2°	$D + \alpha J_{01},\ \alpha \neq -1$	$(x_0 - x_1)(x_0 + x_1)^{\frac{\alpha-1}{\alpha+1}}$	$u = (x_0 + x_1)^{\frac{4}{(1-k)(\alpha+1)}}\varphi(\omega)$
3°	$D + J_{01} + P_0$	$(x_0 + x_1 + \frac{1}{2})\times$ $\times\exp\left(2(x_1 - x_0)\right)$	$u = \exp\left(\frac{4}{k-1}(x_1 - x_0)\right)\varphi(\omega)$
4°	J_{01}	$x_0^2 - x_1^2$	$u = \varphi(\omega)$
5°	P_0	x_1	$u = \varphi(\omega)$
6°	$P_0 + P_1$	$x_0 + x_1$	$u = \varphi(\omega)$

Let us note that analogous ansatzes were obtained in [4] for the nonlinear wave equation

$$\Box u = \lambda u^k. \tag{35}$$

Substituting the ansatzes obtained to equation (5) we get the following equations for the function $\varphi(\omega)$:

$$1^\circ \quad \frac{1+k}{(1-k)^2}\varphi^{(2)} = \frac{\lambda}{32}\varphi^k,$$

$$2^\circ \quad (\alpha-1)^2\varphi^{(4)}\omega^2 + 2(\alpha-1)(\alpha+1)^2\left(\frac{3k+1}{1-k} + 2\alpha\right)\omega\varphi^{(3)} +$$
$$+ 2\left(\alpha^2 - 4\alpha + 3 + \frac{6\alpha - 10}{1-k} + \frac{8}{(1-k)^2}\right)\varphi^{(2)} = \frac{\lambda}{16}(\alpha+1)^2\varphi^k,$$

$$3^\circ \quad \varphi^{(4)}\omega^2 + \frac{5k-1}{k-1}\varphi^{(3)}\omega + \frac{4k^2}{(1-k)^2}\varphi^{(2)} = \frac{\lambda}{64}\varphi^k,$$

$$4^\circ \quad \varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + 2\varphi^{(2)} = \frac{\lambda}{16}\varphi^k,$$

$$5^\circ \quad \varphi^{(4)} = \lambda\varphi^k,$$

$$6^\circ \quad \lambda\varphi^k = 0.$$

Equations 1°, 2°, 4° have the partial solutions of the form:

$$\varphi = \left(\frac{64}{\lambda}\frac{(k+1)^2}{(k-1)^4}\right)^{\frac{1}{k-1}}\omega^{-\frac{2}{k-1}}, \quad k \neq -1,$$

and equation 5° has the partial solution of the form

$$\varphi = \left(\frac{8}{\lambda}\frac{(k+1)(k+3)(3k+1)}{(k-1)^4}\right)^{\frac{1}{k-1}} \omega^{-\frac{4}{k-1}}, \quad k \neq -1, -3, -\frac{1}{3}$$

which lead us to the following solutions of equation (5):

$$u = \left(\frac{64}{\lambda}\frac{(k+1)^2}{(k-1)^4}\right)^{\frac{1}{k-1}} \Big((x_0+x_1+c_1)(x_0-x_1+c_2)\Big)^{-\frac{2}{k-1}}, \quad k \neq -1,$$
$$u = \left(\frac{8}{\lambda}\frac{(k+1)(k+3)(3k+1)}{(k-1)^4}\right)^{\frac{1}{k-1}} (x_0+c_3)^{\frac{4}{1-k}}, \quad k \neq -1, -3, -\frac{1}{3},$$
$$u = \left(\frac{8}{\lambda}\frac{(k+1)(k+3)(3k+1)}{(k-1)^4}\right)^{\frac{1}{k-1}} (x_1+c_4)^{\frac{4}{1-k}}, \quad k \neq -1, -3, -\frac{1}{3},$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

Note that equation (35) has analogous solutions (see [4]).

5 Reduction and exact solutions of the equation $\Box^2 u = \lambda u^{-3}$

It follows from Theorems 2 and 3 that when $n = 1$ the equation

$$\Box^2 u = \lambda u^{-3} \tag{36}$$

is invariant under the conformal group $C(1,1)$ with the generators (24) and

$$\begin{aligned} D^{(1)} &= x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} + u\frac{\partial}{\partial u}, \\ K_\mu^{(1)} &= 2x^\mu D^{(1)} - (x_\nu x^\nu)\frac{\partial}{\partial x_\mu}, \quad \mu,\nu = 0,1. \end{aligned} \tag{37}$$

By analogy with the previous Section solutions of equation (36) can be found in the form (33) where functions $\omega(x)$ and $f(x)$ are the solutions of the system (34) provided the operator (32) belongs to the invariance algebra of equation (36).

To obtain all the $C(1,1)$–nonequivalent ansatzes we use the one-dimensional nonequivalent subalgebras of the conformal algebra $AC(1,1)$ adduced in [9].

Omitting rather cumbersome computations and taking account of equation (36) being invariant under the changes of variables (28) we write nonequivalent ansatzes in Table 3.

We omit subalgebras not containing conformal the operator (37) since they were considered in the previous Section.

Substituting ansatzes obtained in (36) we get the following equations for the function $\varphi(\omega)$:

1° $\quad \varphi^{(4)} + 2\varphi^{(2)} + \varphi = \dfrac{\lambda}{16}\varphi^{-3};$

2° $\quad (\alpha^2-1)^2\varphi^{(4)} + 2(\alpha^2+1)\varphi^{(2)} + \varphi = \dfrac{\lambda}{16}\varphi^{-3};$

Table 3.

N	Algebra	Invariant variables ω	Ansatz
1°	$P_0+K_0^{(1)}$	$\mathrm{arctg}(x_1-x_0)+$ $+\mathrm{arctg}(x_1+x_0)$	$u=\Big((x_0-x_1)^2+1\Big)^{1/2}\times$ $\times\Big((x_0+x_1)^2+1\Big)^{1/2}\varphi(\omega)$
2°	$P_0+K_0^{(1)}+\alpha(K_1^{(1)}-P_1)$ $0<\alpha<1$	$(\alpha-1)\mathrm{arctg}(x_0-x_1)+$ $+(\alpha+1)\mathrm{arctg}(x_1+x_0)$	$u=\Big((x_0-x_1)^2+1\Big)^{1/2}\times$ $\times\Big((x_0+x_1)^2+1\Big)^{1/2}\varphi(\omega)$
3°	$P_0+K_0^{(1)}+K_1^{(1)}-P_1$	x_0+x_1	$u=\Big((x_0-x_1)^2+1\Big)^{1/2}\times$ $\times\varphi(\omega)$
4°	$2P_1+K_0^{(1)}+K_1^{(1)}$	x_0+x_1+ $+\dfrac12\ln\dfrac{1+x_0-x_1}{1-x_0+x_1}$	$u=\Big((x_0-x_1)^2+1\Big)^{1/2}\times$ $\times\varphi(\omega)$
5°	$2P_1-K_0^{(1)}-K_1^{(1)}$	x_0+x_1+ $+\mathrm{arctg}(x_0-x_1)$	$u=\Big((x_0-x_1)^2+1\Big)^{1/2}\times$ $\times\varphi(\omega)$
6°	$P_0+K_0^{(1)}+K_1^{(1)}-P_1-$ $-\beta(J_{01}+D^{(1)}),\ \beta>0$	$\ln(x_0+x_1)-$ $-\beta\mathrm{arctg}(x_1-x_0)$	$u=\Big((x_0-x_1)^2+1\Big)^{1/2}\times$ $\times\Big(x_0+x_1\Big)^{1/2}\varphi(\omega)$

3° $\quad \varphi^{(2)}=\dfrac{\lambda}{16}\varphi^{-3};$

4° $\quad \varphi^{(4)}-\varphi^{(2)}=\dfrac{\lambda}{16}\varphi^{-3};$

5° $\quad \varphi^{(4)}+\varphi^{(2)}=\dfrac{\lambda}{16}\varphi^{-3};$

6° $\quad 4\beta^2\varphi^{(4)}+(4-\beta^2)\varphi^{(2)}-\varphi=\frac{\lambda}{4}\varphi^{-3}.$

The general solution of equation 3° is of the form

$$\varphi=\pm\sqrt{\frac{(c_1\omega+c_2)^2}{c_1}+\frac{\lambda}{16c_1}},\quad c_1\neq 0;$$
$$\varphi=\pm\sqrt{\frac12\sqrt{-\lambda}\omega+c},$$

where c, c_1, c_2 are arbitrary constants.

Hence we obtain the following exact solutions of equation (36):

$$1.\quad u=\pm\frac{1}{\sqrt2}\left(\frac{\lambda}{a_1}\right)^{1/4}\left|(x_0+x_1+a_2)^2-a_1\right|^{1/2}\left|x_0-x_1+a_3\right|^{1/2},$$
$$2.\quad u=\pm\frac{1}{\sqrt2}\left(\frac{\lambda}{b_1}\right)^{1/4}\left|(x_0-x_1+b_2)^2-b_1\right|^{1/2}\left|x_0+x_1+b_3\right|^{1/2},$$
$$3.\quad u=\pm\frac{1}{2}\left(\frac{\lambda}{c_1c_2}\right)^{1/4}\left|(x_0-x_1+c_3)^2+c_1\right|^{1/2}\left|(x_0+x_1+c_4)^2+c_2\right|^{1/2},$$

where a_i, b_i, c_j, $i=\overline{1,3}$, $j=\overline{1,4}$ are arbitrary constants.

Besides, the expression

$$u = \pm\lambda^{1/4}\big|(x_0 - x_1 + c_1)(x_0 + x_1 + c_2)\big|^{1/2}$$

(c_1, c_2 are arbitrary constants) was proved in Section 4 to be the exact solution of equation (36).

In conclusion let us note that we can obtain the same solutions using the following ansatz

$$u = \varphi_1(\omega_1)\varphi_2(\omega_2), \quad \omega_1 = x_0 + x_1, \quad \omega_2 = x_0 - x_1,$$

which reduces equation (36) to the system of ordinary differential equations for the unknown functions $\varphi_1(\omega_1)$ and $\varphi_2(\omega_2)$, namely

$$\begin{aligned} \ddot{\varphi}_1 &= \frac{c}{4}\varphi_1^{-3}, \\ \ddot{\varphi}_2 &= \frac{\lambda}{4c}\varphi_2^{-3}, \end{aligned} \tag{38}$$

where c is an arbitrary constant.

Acknowledgement. The main part of this work for the authors was made by the financial support by Soros Grant, Grant of the Ukrainian Foundation for Fundamental Research and the Swedish Institute.

1. Fushchych W.I., Symmetry in problems of mathematical physics, in Algebraic-Theoretical Studies in Mathematical Physics, Kiev, Inst. of Math., 1981, 6.
2. Bollini C.G., Giambia J.J., Arbitrary powers of d'Alembertians and the Huygens' principle, *J. Math. Phys.*, 1993, **34**, № 2, 610–621.
3. Fushchych W.I., Selehman M.A., Integro-differential equations invariant under group of Galilei, Poincare, Schrödinger and conformal group, *Proceedings of the Academy of Sciences of Ukrainy*, 1983, № 5, 21–24.
4. Fushchych W., Shtelen W., Serov N., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer Academic Publishers, 1993, 436 p.
5. Fushchych W.I., Serov N.I., The symmetry and some exact solutions of the nonlinear many-dimensional Liouville, d'Alembert and eikonal equations, *J. Phys. A: Math. Gen.*, 1983, № 16, 3645–3658.
6. Ibragimov N.H., Lie groups in some questions of mathematical physics, Novosibirsk, Novosibirsk University, 1972, 200 p.
7. Winternitz P., Grundland A.M., Tuszynski J.A., Exact solutions of the multidimensional classical Φ^6 field equations obtained by symmetry reduction, *J. Math. Phys.*, 1987, **28**, № 9, 2194–2212.
8. Fedorchuk V.M., On reduction and some exact solutions of nonlinear wave equation, in Symmetry and Solutions of Nonlinear Equations of Mathematical Physics, Kiev, Inst. of Math., 1987, 73–76.
9. Fushchych W.I., Barannik L.F., Barannik A.F., Subgroup analysis of Galilei and Poincaré groups and reduction of nonlinear equations, Kiev, Naukova Dumka, 1991, 304 p.
10. Ovsiannikov L.V., Group analysis of differential equations, Moscow, Nauka, 1978, 400p.
11. Serov N.I., Conformal symmetry of nonlinear wave equations, in Algebraic-Theoretical Studies in Mathematical Physics, Kiev, Inst. of Math., 1981, 59.