Symmetry and some exact solutions of non-linear polywave equations

W.I. FUSHCHYCH, O.V. ROMAN, R.Z. ZHDANOV

We have studied the maximal symmetry group admitted by the non-linear polywave equation $\Box^l u = F(u)$. In particular, we establish that equation in question admits the conformal group $C(1,n)$ if and only if $F(u) = \lambda e^u$, $n+1 = 2l$ or $F(u) = \lambda u^{(n+1+2l)/(n+1-2l)}$, $n+1 \neq 2l$. Symmetry reduction for the biwave equation $\Box^2 u = \lambda u^{-3}$ is carried out and some exact solutions are obtained.

Recently a number of works (see, e.g., [1, 2, 3]) have appeared pointing out the possibility to choose linear and non-linear polywave equations

$$\Box^l u = F(u) \tag{1}$$

as possible mathematical models describing an uncharged scalar particle in quantum field theory.

Here $\Box^l = \Box(\Box^{l-1})$, $l \in \mathbb{N}$; $\Box = \partial^2_{x_0} - \partial^2_{x_1} - \cdots - \partial^2_{x_n}$ is d'Alembertian in $(n+1)$-dimensional pseudo-Euclidean space $\mathbb{R}(1,n)$ with metric tensor $g_{\mu\nu} = \mathrm{diag}(1,-1,\ldots,-1)$, $\mu,\nu = \overline{0,n}$; $F(u)$ is an arbitrary smooth function and $u = u(x)$ is a real function (the case $l=1$, $n=1$ has been studied earlier [4], that is why we put $l+n>2$). In the following, a summation over the repeated indices from 0 to n is understood, rising and lowering of the vector indices is performed by means of the tensor $g_{\mu\nu}$, i.e. $x^\mu = g_{\mu\nu}x_\nu$.

But the fact that the non-linear partial differential equation (PDE) in question is of high order makes the prospects of studying such a model rather obscure. Using group properties of equation (1) seems to be the only way to get some non-trivial information about the said equation and its solutions. It occurs that PDE (1) admits wide symmetry group which, in fact, is the same as the one of the standard wave equation

$$\Box u = F(u). \tag{2}$$

The main tool used is the infinitesimal Lie method (see, e.g., [5]). But an application of it to study of symmetry properties of equation (1) is by itself a non-trivial problem in the case $l>1$. It should be emphasized that because of arbitrariness of the order (l) and of the number of independent variables (n) one can not apply symbolic manipulation programs [6, 7]. We have succeeded in constructing the maximal symmetry group admitted by equation (1) using the remarkable combinatorial properties of the prolongation formulae.

Theorem 1. *The maximal invariance group of PDE (1) with arbitrary smooth function $F(u)$ is the Poincaré group $P(1,n)$ generated by the operators*

$$P_\mu = \partial_{x_\mu}, \quad J_{\mu\nu} = x^\mu\partial_{x_\nu} - x^\nu\partial_{x_\mu}, \quad \mu,\nu = \overline{0,n}. \tag{3}$$

Europhys. Lett., 1995, **31**, № 2, P. 75–79.

It is established below that the equation of the type (1) admitting the group, which is more extensive than the Poincaré group, is equivalent up to the change of variables to one of the following equations:

$$1.\ \Box^l u = \lambda_1 u^k, \qquad \lambda_1 \neq 0,\ k \neq 0, 1; \tag{4}$$

$$2.\ \Box^l u = \lambda_2 e^u, \qquad \lambda_2 \neq 0; \tag{5}$$

$$3.\Box^l u = \lambda_3 u, \qquad \lambda_3 \neq 0; \tag{6}$$

$$4.\ \Box^l u = 0. \tag{7}$$

Here λ_1, λ_2, λ_3, k are arbitrary constants.

Maximal invariance groups of the equations (4)–(7) are described by the following statements.

Theorem 2. *Equation (4) has the following symmetry:*

Case 1. $k \neq (n+1+2l)/(n+1-2l)$, $k \neq 0, 1$. *The maximal invariance group of (4) is the extended Poincaré group* $\widetilde{P}(1,n)$ *generated by the operators (3) and*

$$D = x_\mu \partial_{x_\mu} + \frac{2l}{1-k} u\partial_u.$$

Case 2. $k = (n+1+2l)/(n+1-2l)$, $n+1 \neq 2l$. *The maximal invariance group of (4) is the conformal group* $C(1,n)$ *generated by the operators (3) and operators*

$$\begin{aligned} &D^{(1)} = x_\mu \partial_{x_\mu} + \frac{(2l-n-1)}{2} u\partial_u, \\ &K^{(1)}_\mu = 2x^\mu D^{(1)} - (x_\nu x^\nu)\partial_{x_\mu}, \quad \mu,\nu = \overline{0,n}. \end{aligned} \tag{8}$$

Theorem 3. *Equation (5) has the following symmetry:*

Case 1. $n \neq 2l-1$. *The maximal invariance group of (5) is the extended Poincaré group* $\widetilde{P}(1,n)$ *generated by the operators (3) and*

$$D^{(2)} = x_\mu \partial_{x_\mu} - 2l\partial_u. \tag{9}$$

Case 2. $n = 2l-1$. *The maximal invariance group of (5) is the conformal group* $C(1,n)$ *generated by the operators (3) and operators*

$$K^{(2)}_\mu = 2x^\mu D^{(2)} - (x_\nu x^\nu)\partial_{x_\mu}, \quad \mu,\nu = \overline{0,n}. \tag{10}$$

Theorem 4. *The maximal invariance group of the equation (6) is generated by the operators (3) and*

$$Q_\infty = f(x)\partial_u, \quad I = u\partial_u,$$

where $f(x)$ *is an arbitrary solution of PDE (6).*

Theorem 5. *The maximal invariance group of the equation (7) is generated by the operators (3), (8) and*

$$Q_\infty = q(x)\partial_u, \quad I = u\partial_u,$$

where $q(x)$ *is an arbitrary solution of PDE (7).*

The proof of the Theorems 1–5 carried out by means of the infinitesimal algorithm of S. Lie [5] requires very cumbersome computations. That is why, we omit it.

An important consequence of the Theorems 1–5 is the following statement.

Theorem 6. *The non-linear PDE (1) is invariant under the conformal group $C(1,n)$ iff it is equivalent to the following*

$$1. \Box^l u = \lambda_1 u^{\frac{n+1+2l}{n+1-2l}}, \quad n+1 \neq 2l; \tag{11}$$

$$2.\ \Box^l u = \lambda_2 e^u, \quad n+1 = 2l. \tag{12}$$

Remark 1. Conformal invariance of the equation (11) was first ascertained in [8] and that of equation (12) was done in [3] by means of Baker–Campbell–Hausdorff formulae.

Assuming $l = 1$ in (11) we obtain the well-known result [3]; that non-linear wave equation (2) admits the conformal group if it is equivalent to the PDE

$$\Box u = \lambda u^{\frac{n+3}{n-1}} \quad \text{when} \quad n \neq 1.$$

Remark 2. When $l = 2$ it follows from the Theorem 6 that in the four-dimensional space $\mathbb{R}(1,3)$ there is only one $C(1,3)$-invariant equation

$$\Box^2 u = \lambda e^u.$$

One of the important applications of the Lie groups in mathematical physics is the finding exact solutions of non-linear PDE. To this end one has to construct so called invariant solutions [2, 3, 5] which reduce PDE under study to equations with less number of independent variables (in particular, to ordinary differential equations). Integrating these one gets exact solutions of the initial PDE. A procedure described is called symmetry (or group-theoretical) reduction of differential equations. Here we perform symmetry reduction of the conformally-invariant biwave equation in the two-dimensional space $\mathbb{R}(1,1)$:

$$\Box^2 u = \lambda u^{-3}. \tag{13}$$

Making use of inequivalent one-dimensional subalgebras of the conformal algebra $AC(1,1)$ [9] one can obtain the following $C(1,1)$-inequivalent Ansätze which reduce the equation (13) to ordinary differential equations. For each case the reduced equations are given:

1. $u = \varphi(\omega), \quad \omega = x_0$ or $\omega = x_1$,
   $\varphi^{(4)} = \lambda\varphi^{-3}$;
2. $u = \varphi(\omega), \quad \omega = x_0^2 - x_1^2$,
   $\varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + 2\varphi^{(2)} = \dfrac{\lambda}{16}\varphi^{-3}$;
3. $u = (x_0 + x_1)^{1/2}\varphi(\omega), \quad \omega = x_0 - x_1$,
   $\varphi^{(2)} = -\dfrac{\lambda}{4}\varphi^{-3}$;
4. $u = (x_0 + x_1)^{1/(\alpha+1)}\varphi(\omega), \quad \omega = (x_0 - x_1)(x_0 + x_1)^{(\alpha-1)/(\alpha+1)}$;
   $\varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + \dfrac{(\alpha-2)(2\alpha-1)}{(\alpha-1)^2}\varphi^{(2)} = \dfrac{\lambda}{16}\dfrac{(\alpha+1)^2}{(\alpha-1)^2}\varphi^{-3}, \ \alpha > 1$;

5. $u = \exp(x_0 - x_1)\varphi(\omega), \quad \omega = \left(x_0 + x_1 + \dfrac{1}{2}\right)\exp\bigl(-2(x_0 - x_1)\bigr),$
$\varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + \dfrac{9}{4}\varphi^{(2)} = \dfrac{\lambda}{64}\varphi^{-3};$

6. $u = \bigl((x_0 - x_1)^2 + 1\bigr)^{1/2}\varphi(\omega), \quad \omega = x_0 + x_1,$
$\varphi^{(2)} = \dfrac{\lambda}{16}\varphi^{-3};$

7. $u = \bigl((x_0 - x_1)^2 + 1\bigr)^{1/2}\varphi(\omega), \quad \omega = x_0 + x_1 + \arctan(x_0 - x_1),$
$\varphi^{(4)} + \varphi^{(2)} = \dfrac{\lambda}{16}\varphi^{-3};$

8. $u = \bigl((x_0 - x_1)^2 + 1\bigr)^{1/2}\varphi(\omega), \quad \omega = x_0 + x_1 + \dfrac{1}{2}\ln\dfrac{1 + x_0 - x_1}{1 - x_0 + x_1},$
$\varphi^{(4)} - \varphi^{(2)} = \dfrac{\lambda}{16}\varphi^{-3};$

9. $u = \bigl((x_0 - x_1)^2 + 1\bigr)^{1/2}\bigl(x_0 + x_1\bigr)^{1/2}\varphi(\omega),$
$\omega = \ln(x_0 + x_1) - \beta\arctan(x_1 - x_0),$
$4\beta^2\varphi^{(4)} + (4 - \beta^2)\varphi^{(2)} - \varphi = \dfrac{\lambda}{4}\varphi^{-3}, \quad \beta > 0;$

10. $u = \bigl((x_0 - x_1)^2 + 1\bigr)^{1/2}\bigl((x_0 + x_1)^2 + 1\bigr)^{1/2}\varphi(\omega),$
$\omega = (\gamma - 1)\arctan(x_0 - x_1) + (\gamma + 1)\arctan(x_0 + x_1),$
$(\gamma^2 - 1)^2\varphi^{(4)} + 2(\gamma^2 + 1)\varphi^{(2)} + \varphi = \dfrac{\lambda}{16}\varphi^{-3}, \ 0 \le \gamma < 1.$

Integration of the reduced equations gives rise to exact solutions of the non-linear biwave equation (13). Here we present some exact solutions of this equation obtained with the use of Ansätze 3 and 6:

$$
\begin{aligned}
&u = \pm\lambda^{1/4}\bigl(x_0^2 - x_1^2\bigr)^{1/2},\\
&u = \pm\frac{1}{\sqrt{2}}\left(\frac{\lambda}{c_1}\right)^{1/4}\bigl|(x_0 - x_1)^2 - c_1\bigr|^{1/2}(x_0 + x_1)^{1/2},\\
&u = \pm\frac{1}{2}\left(\frac{\lambda}{c_2}\right)^{1/4}\bigl((x_0 - x_1)^2 + 1\bigr)^{1/2}\bigl|(x_0 + x_1)^2 + c_2\bigr|^{1/2},
\end{aligned}
\tag{14}
$$

where c_1, c_2 are arbitrary constants.

Since the conformal group $C(1,1)$ is a maximal symmetry group of equation (13), formulae 1–10 give "maximal" information about its solutions which can be obtained within the framework of the Lie approach. It means that any solution invariant under a subgroup of the symmetry group of PDE (13) can be reduced by a transformation from the group $C(1,1)$ to one of the Ansätze 1–10.

Acknowledgments. One of the authors (RZZ) is supported by the Alexander von Humboldt Foundation.

1. Bollini C.G., Giambia J.J., Arbitrary powers of d'Alembertians and the Huygens' principle, *J. Math. Phys.*, 1993, **34**, 610.
2. Fushchych W.I., Symmetry in problems of mathematical physics, in Algebraic-Theoretical Studies in Mathematical Physics, Kiev, Institute of Mathematics, 1981, 6.
3. Fushchych W.I., Shtelen W.M., Serov N.I., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer, 1993.
4. Fushchych W.I., Serov N.I., The symmetry and some exact solutions of the nonlinear many-dimensional d'Alembert, Liouville and eikonal equations, *J. Phys. A*, 1983, **16**, 3645.
5. Olver P., Applications of Lie groups to differential equations, New York, Springer, 1986.
6. Hereman W., Review of symbolic software for the computation of Lie symmetries of differential equations, *Euromath. Bull.*, 1994, **1**, 45.
7. Mansfield E.L., Clarkson P.A., Applications of the differential algebra package `diffgrob2` to reductions of PDE, in Proceedings of the Fourteenth IMACS World Congress on Computation and Applied Mathematics, Editor W.F. Ames (Georgia Inst. Tech.), 1994, **1**, 336.
8. Serov N.I., Conformal symmetry of nonlinear wave equations, in Algebraic-Theoretical Studies in Mathematical Physics, Kiev, Institute of Mathematics, 1981, 59.
9. Fushchych W.I., Barannik L.F., Barannik A.F., Subgroup analysis of the groups of Galilei and Poincaré and reduction of nonlinear equations, Kiev, Naukova Dumka, 1991.