New solutions of the wave equation by reduction to the heat equation

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In this article we make a new connection between the linear wave equation and the linear heat equation. In this way we are able to construct new solutions of the linear wave equation, using symmetries and conditional symmetries of the heat equation.

1. Introduction

The linear wave equation in $(1+n)$-dimensional timespace $\mathbb{R}(1,n)$

$$\Box u = \frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \cdots - \frac{\partial^2 u}{\partial x_n^2} = -m^2 u \tag{1}$$

is fundamental to mathematical physics: it describes spinless mesons when $n=3$, and is the paradigm of a hyperbolic equation. Its symmetry properties are also known [1, 2], and one has the following result concerning the Lie point symmetries of (1):

Proposition 1. *The maximal Lie point symmetry algebra of equation (1) has basis*

$$P_\mu = \partial_\mu, \quad I = u\partial_u, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu \tag{2}$$

when $m \neq 0$ *and*

$$\begin{aligned} &P_\mu = \partial_\mu, \quad I = u\partial_u, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \\ &D = x^\mu\partial_\mu, \quad K_\mu = 2x_\mu D - x^2\partial_\mu - 2x_\mu u\partial_u \end{aligned} \tag{3}$$

when $m = 0$*, where*

$$\begin{aligned} &\partial_u = \frac{\partial}{\partial u}, \quad \partial_\mu = \frac{\partial}{\partial x^\mu}, \quad x_\mu = g_{\mu\nu}x^\nu, \\ &g_{\mu\nu} = \mathrm{diag}\,(1,-1,\ldots,-1), \quad \mu,\nu = 0,1,2,\ldots,n. \end{aligned}$$

The symmetries can be used to build ansatzes for exact solutions of (1), which then reduce the equation to a partial differential equation with fewer independent variables or even to an ordinary differential equation [1, 2]. These ansatzes and reductions are based on a subalgebra analysis of parts of the symmetry algebra. The reduced equations do not always have nice symmetry properties, so that a full analysis of the resulting equations has not been carried out to this date. In this article we study a reduction which, as far as we know, has not been done before, and which links up solutions of the wave equation (1) in $\mathbb{R}(1,n)$ with those of the linear heat equation in $\mathbb{R}(1,n-1)$. We consider equation (1) with real u: the complex case with nonlinearities is studied in [3].

In [1, 2, 4], the reduction of the nonlinear wave equation

$$\Box u = F(u) \tag{1a}$$

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is considered and its reduction (to equations with a smaller number of independent variables) is studied with respect to the following algebras: $AP(1,n) = \langle P_\mu, J_{\mu\nu}\rangle$ when $F(u)$ is arbitrary; $A\tilde{P}(1,n) = \langle P_\mu, J_{\mu\nu}, D\rangle$ when $F(u) = \lambda u^p$ with p an arbitrary constant; $AC(1,3) = \langle P_\mu, J_{\mu\nu}, D, K_\mu\rangle$ when $F(u) = \lambda u^3$.

The linear equation (1), unlike the nonlinear one (1a), admits a new symmetry operator: $I = u\partial_u$ so that (1) is invariant under the algebras $\langle P_\mu, J_{\mu\nu}, I\rangle$ for $m \neq 0$ and $\langle P_\mu, J_{\mu\nu}, I, D, K_\mu\rangle$ for $m = 0$. However, until now, reductions of (1) have been based only on subalgebras of $\langle P_\mu, J_{\mu\nu}\rangle$ and $\langle P_\mu, J_{\mu\nu}, D, K_\mu\rangle$. In this paper we take the subalgebra $\langle P_\mu, I\rangle$ in both cases, it allows us to reduce the hyperbolic equation (1) to the parabolic heat equation and, in this way, we are then able to exploit the exact solutions of the heat equation to construct solutions of the wave equation. This is the central result of our paper. It may at first sight seem rather strange that a Poincaré-invariant equation is reducible (with an appropriate ansatz) to one that is Galilei-invariant. However, it is known (see [5]) that the Galilei algebra can be found within the Poincaré algebra, so that one may even expect the original equation to 'contain' a Galilei-invariant one.

2. Reduction to the heat equation

In this paper we limit ourselves to $(1+3)$-dimensional time-space $\mathbb{R}(1,3)$, but the generalization of our result to higher dimensions is obvious as the reduction remains the same.

We now turn to the construction of the ansatz which reduces (1) to the heat equation. Equation (1) is invariant under the operators P_μ, I and is therefore also invariant under any constant linear combination of them:

$$\tau^\mu\partial_\mu + ku\partial_u,$$

where k, τ^μ are constants. This latter operator then gives us the following invariant-surface condition

$$\tau^\mu u_\mu = ku$$

which gives the Lagrangian system

$$\frac{dx_\mu}{\tau_\mu} = \frac{du}{ku}$$

and it is not difficult to show that this, in turn, is equivalent to the Lagrangian system

$$\frac{d(cx)}{c\tau} = \frac{du}{ku} \tag{4}$$

for any constant four-vector c, with $cx = c^\mu x_\mu$, $c\tau = c^\mu\tau^\mu$. Choose now τ so that $\tau^2 = \tau^\mu\tau_\mu = 0$, namely τ is light-like, and choose four-vectors β, δ, ϵ so that

$$\beta^2 = \delta^2 = -1, \quad \epsilon^2 = -\frac{m^2}{k^2}, \quad \tau\beta = \tau\delta = \beta\delta = \beta\epsilon = \delta\epsilon = 0, \quad \tau\epsilon = 1. \tag{5}$$

On choosing c in (4) to be τ, β, δ, ϵ we obtain the system

$$\frac{d(\tau x)}{0} = \frac{d(\beta x)}{0} = \frac{d(\delta x)}{0} = \frac{d(\epsilon x)}{1} = \frac{du}{ku}. \tag{6}$$

The general integral of (6) is given by

$$u = e^{k(\epsilon x)} v(\tau x, \beta x, \delta x), \tag{7}$$

where v is a smooth function of its arguments (we assume that all our operations are smooth, at least locally). Treating (7) as an ansatz for equation (1), we find, on substituting (7) into (1), writing $t = \tau x$, $y_1 = \beta x$, $y_2 = \delta x$, performing some elementary computations and using (5), that v satisfies the linear heat equation (we have chosen $k = \frac{1}{2}$ for convenience)

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial y_1^2} + \frac{\partial^2 v}{\partial y_2^2}. \tag{8}$$

The Cauchy problem for equation (8) is well posed for $t > 0$, and (8) has solutions which are singular for $t = 0$. This then leads to a similar problem for the wave equation when $\tau x = 0$, which is a characteristic ($\tau^2 = 0$), so that the initial-value problem for (8) at $t = 0$ is related to the initial-value problem of (1) on a characteristic. This latter is known as Goursat's problem, and has been studied in [12], to which we refer the reader for more details.

The linear heat equation in $(1+1)$ spacetime dimensions has been studied extensively: its symmetry properties [2, 6, 7] and its conditional symmetries (also known as non-classical symmetries [6], Q-conditional symmetries in [2]) are known. The symmetry algebra of the linear heat equation in $1+2$ timespace can be found in [7] but for the sake of completeness, we give it in the following proposition.

Proposition 2. *The maximal Lie point symmetry algebra of equation (8) is the extended Galilei algebra $AG_3(1,2)$ with a basis given by the following vector fields*

$$\begin{gathered} T = \partial_t \qquad P_a = -\partial_{y_a}, \quad G_a = t\partial_{y_a} - \frac{1}{2} y_a v \partial_v, \quad M = -\frac{1}{2} v\partial_v, \\ J_{12} = y_1 \partial_{y_2} - y_2 \partial_{y_1}, \quad D = 2t\partial_t + y_1\partial_{y_1} + y_2 \partial_{y_2} - v\partial_v, \\ S = t^2 \partial_t + t y_1 \partial_{y_1} + t y_2 \partial_{y_2} - \left(t + \frac{1}{4}\left(y_1^2 + y_2^2\right)\right) v\partial_v. \end{gathered} \tag{9}$$

Remark 1. We have not included the symmetry $v \to v + v_1$, where v_1 is an arbitrary solution of (8).

If we had considered equation (1) in $\mathbb{R}(1,4)$, then we would have obtained the linear heat equation in $1+3$ dimensions with our reduction. Note also that there is a Lie-algebraic reduction of (1) in $\mathbb{R}(1,4)$ to equation (1) in $\mathbb{R}(1,3)$, which amounts to omitting dependency on one of the spatial variables. In this way, we are able to use the wave equation in $\mathbb{R}(1,4)$ as a bridge in constructing solutions of the wave equation in $\mathbb{R}(1,3)$ from those of the heat equation in $1+3$ dimensions.

The invariance of equation (8) under the group $G_2(1,2)$ which the above algebra generates then allows us to obtain a nine-parameter family of exact solutions whenever one solution is given.

The commutation relations of the algebra (9) are

$$\begin{gathered} [P_a, G_b] = \delta_{ab} M, \quad [P_1, J_{12}] = P_2, \quad [P_2, J_{12}] = -P_1, \\ [P_a, D] = P_a, \quad [P_a, S] = G_a, \quad [P_a, T] = 0, \quad [M, X] = 0 \text{ for all } X \in AG_3(2), \\ [G_a, G_b] = 0, \quad [D, G_a] = G_a, \quad [T, G_a] = P_a, \quad [S, G_a] = 0, \\ [J_{12}, T] = [J_{12}, D] = [J_{12}, S] = 0, \quad [T, D] = 2T, \quad [T, S] = D, \quad [D, S] = 2S. \end{gathered}$$

Clearly, we see that the subalgebra $\langle P_a, G_a, M\rangle$, $a = 1, 2$ is an ideal (maximal and solvable, and therefore the radical of the algebra [8, 9]). Our algebra is seen to be the semi-direct sum $\langle J_{12}, S, T, D\rangle + \langle P_a, G_a, M\rangle$. In turn, we can verify that $\langle S, T, D\rangle$ is a semi-simple Lie algebra which we can take as being a realization of $ASL(2,\mathbb{R})$, the Lie algebra of $SL(2,\mathbb{R})$. To see this, we take $X_1 = \frac{1}{2}D$, $X_2 = \frac{1}{2}(T-S)$, $X_3 = \frac{1}{2}(T+S)$ as a new basis, and obtain the commutation relations of $SL(2,\mathbb{R})$:

$$[X_1, X_2] = -X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2.$$

Thus we obtain

$$\langle J_{12}, S, T, D\rangle = \langle J_{12}\rangle \oplus \langle S, T, D\rangle = \langle J_{12}\rangle \oplus ASL(2,\mathbb{R})$$

which is the Lie algebra of $O(2) \otimes SL(2,\mathbb{R})$.

The elements of the group $G_2(1,2)$ are considered as transformations of a space with local coordinates (t, y_1, y_2, v) and points with these coordinates are mapped to points (t', y'_1, y'_2, v'). The finite transformations defining this action are obtained by solving the corresponding Lie equations. For the subalgebra $\langle J_{12}, S, T, D\rangle = \langle J_{12}, X_1, X_2, X_3\rangle$ we solve the Lie equations as follows:

$$\begin{aligned} J_{12}: \quad & \frac{dt'}{d\rho} = 0, \quad \frac{dy'_1}{d\rho} = -y'_2, \quad \frac{dy'_2}{d\rho} = y'_1, \quad \frac{dv'}{d\rho} = 0, \\ & t'|_{\rho=0} = t, \quad y'_a|_{\rho=0} = y_a, \quad v'|_{\rho=0} = v \end{aligned}$$

which gives the finite transformations

$$t' = t, \quad y'_1 = y_1\cos\rho - y_2\sin\rho, \quad y'_2 = y_1\sin\rho + y'_2\cos\rho, \quad v' = v.$$

Then we have the corresponding equations for X_1, X_2, X_3

$$\begin{aligned} X_1: \quad & t' = e^{\nu_1}t = \frac{e^{\nu_1/2}t + 0}{0\cdot t + e^{-\nu_1/2}}, \quad y'_a = e^{\nu_1/2}y_a = \frac{y_a}{0\cdot t + e^{-\nu_1/2}}, \\ & v' = e^{-\nu_1/2}v, \\ X_2: \quad & t' = \frac{t\cosh\nu_2 + \sinh\nu_2}{t\sinh\nu_2 + \cosh\nu_2}, \quad y'_a = \frac{y_a}{t\sinh\nu_2 + \cosh\nu_2}, \\ & v' = v(t\sinh\nu_2 + \cosh\nu_2)\exp\left(\frac{(y_1^2 + y_2^2)\sinh\nu_2}{4(t\sinh\nu_2 + \cosh\nu_2)}\right), \\ X_3: \quad & t' = \frac{t\cos\nu_3 + \sin\nu_3}{\cos\nu_3 - t\sin\nu_3}, \quad y'_a = \frac{y_a}{\cos\nu_3 - t\sin\nu_3}, \\ & v' = v(\cos\nu_3 - t\sin\nu_3)\exp\left(-\tfrac{(y_1^2+y_2^2)\sin\nu_3}{4(\cos\nu_3 - t\sin\nu_3)}\right). \end{aligned}$$

Thus, we see that the action of the group generated by $\langle J_{12}, S, T, D\rangle$ can be given in the form

$$\begin{aligned} & t' = \frac{\zeta t + \eta}{\kappa t + \sigma}, \quad y'_1 = \frac{y_1\varepsilon\cos\rho - y_2\varepsilon\sin\rho}{\kappa t + \sigma}, \quad y'_2 = \frac{y_1\sin\rho + y_2\cos\rho}{\kappa t + \sigma}, \\ & v' = (\kappa t + \sigma)v\exp\left(\frac{\kappa(y_1^2 + y_2^2)}{4(\kappa t + \sigma)}\right) \end{aligned}$$

with $\zeta\sigma - \eta\kappa = 1$, and $\varepsilon = \pm 1$ corresponds to the possibility of space reflections under which (8) is manifestly invariant (the group $O(2)$ has two components). The parameters ζ, η, κ, σ correspond to the action of $SL(2,\mathbb{R})$.

Solving the Lie equations defined by each of the other infinitesimal generators in (9), we obtain finite transformations such that $(t, y_1, y_2, v) \to (t', y_1', y_2', v')$ as follows:

$$
\begin{aligned}
G_i: \quad & t' = t, \quad y_i' = \mu_i t + y_i, \quad y_j' = y_j \quad \text{for } j \neq i, \\
& v' = v \exp\left(-\frac{1}{2}\left(\frac{\mu_i^2}{2}t + \mu_i y_i\right)\right), \\
P_i: \quad & t' = t, \quad y_i' = y_i - \lambda_i, \quad y_j' = y_j \quad \text{for } j \neq i, \quad v' = v, \\
M: \quad & t' = t, \quad y_i' = y_i, \quad v' = v \exp\left(-\frac{1}{2}\theta\right).
\end{aligned}
$$

3. Subalgebras and ansatzes

Having obtained and discussed the symmetry algebra of equation (8), we now pass to listing the subalgebras of $AG_2(1,2)$ which are inequivalent up to conjugation by $G_2(1,2)$, and giving the corresponding reduced equations. In those cases where it is possible, we integrate these equations. The method of obtaining subalgebras up to conjugation is described in [4, 10]; here we simply present our results. The reductions we have obtained have been verified with MAPLE.

3.1. Reduction to ordinary differential equations by two-dimensional subalgebras. Here we list the subalgebras, with restrictions on any parameters entering into the algebra, and then we give the corresponding ansatz and finally the differential equation which arises, with its solution. In all the cases, we can take the real and imaginary parts of the solutions, as the reduced equations are linear. This is understood when complex arguments appear.

3.1.1.

$$
\langle P_2, T + \alpha M\rangle \ (\alpha = 0, \pm 1): \quad v = e^{-\alpha t/2}\varphi(\omega), \quad \omega = y_1, \quad \ddot{\varphi} + \frac{1}{2}\alpha\varphi = 0.
$$

Integrating this reduced equation, we find the following cases

$$
\begin{aligned}
& \varphi = C_1\omega + C_2 \quad \text{for } \alpha = 0, \\
& \varphi = C_1 \exp\left(\frac{\omega}{\sqrt{2}}\right) + C_2 \exp\left(-\frac{\omega}{\sqrt{2}}\right) \quad \text{for } \alpha = -1, \\
& \varphi = C_1 \cos\left(\frac{\omega}{\sqrt{2}} + C_2\right) \quad \text{for } \alpha = 1.
\end{aligned}
$$

From these we obtain the following exact solutions of (8):

$$
\begin{aligned}
& v = C_1 y_2 + C_2 \quad \text{for } \alpha = 0, \\
& v = e^{t/2}\left(C_1 \exp\left(\frac{y_1}{\sqrt{2}}\right) + C_2 \exp\left(-\frac{y_2}{\sqrt{2}}\right)\right) \quad \text{for } \alpha = -1, \\
& v = e^{-t/2} C_1 \cos\left(\frac{y_1}{\sqrt{2}} + C_2\right) \quad \text{for } \alpha = 1
\end{aligned}
$$

with C_1, C_2 being arbitrary constants.

3.1.2.

$$
\begin{aligned}
& \langle D + (2\alpha+1)M, T\rangle \ (\alpha \in \mathbb{R}): \quad v = y_1^{-(\alpha+3/2)}\varphi(\omega), \quad \omega = \frac{y_2}{y_1}, \\
& (\omega^2+1)\ddot{\varphi} + (5+2\alpha)\omega\dot{\varphi} + \left(\frac{3}{2}+\alpha\right)\left(\frac{5}{2}+\alpha\right)\varphi = 0.
\end{aligned}
$$

For $\alpha = -\frac{5}{2}$ we have

$$\varphi = C_1\omega + C_2.$$

If $\alpha = -\frac{3}{2}$ then

$$\varphi = C_1 \arctan\omega + C_2.$$

For $\alpha \neq -\frac{3}{2}, -\frac{5}{2}$ then

$$\varphi = C_1(1+\omega^2)^{-(\alpha/2+3/4)} \cos\left(\left(\frac{3}{2}+\alpha\right)\arctan\omega + C_2\right).$$

The exact solutions being:

$$v = C_1 y_2 + C_2 y_2, \quad \alpha = -\frac{5}{2},$$
$$v = C_1 \arctan\frac{y_2}{y_1} + C_2, \quad \alpha = -\frac{3}{2},$$
$$v = C_1(y_1^2+y_2^2)^{-(\alpha/2+3/4)} \cos\left(\left(\frac{3}{2}+\alpha\right)\arctan\frac{y_2}{y_1} + C_2\right) \quad \alpha \neq -\frac{3}{2}, -\frac{5}{2}.$$

3.1.3.

$$\langle D + (4\alpha+1)M, P_2\rangle \ (\alpha\in\mathbb{R}): \quad v = t^{-(\alpha+3/4)}\varphi(\omega), \quad \omega = \frac{y_1^2}{t},$$
$$4\omega\ddot\varphi + (2+\omega)\dot\varphi + \left(\frac{3}{4}+\alpha\right)\varphi = 0.$$

If we make the transformation $\omega \to \xi = -\frac{\omega}{4}$ in this ODE, we obtain

$$\xi\varphi'' + \left(\frac{1}{2}-\xi\right)\varphi'' - \left(\alpha + \frac{3}{4}\right)\varphi = 0,$$

where φ' denotes differentiation with respect to ξ. The solutions of this equation are given in terms of the Pochhammer–Barnes confluent hypergeometric function (see for example Vol. 1, ch. 6 of [11])

$$\Phi(a;b;z) = \sum_{n=0}^{\infty}\frac{(a)_n z^n}{(b)_n n!}$$

with $b \neq 0$ and where $(a)_n = a(a+1)(a+2)\cdots(a+n-1), n \geq 1$. We find then [11]

$$\varphi = C_1\Phi\left(\alpha+\frac{3}{4};\frac{1}{2};-\frac{1}{4}\omega\right) + C_2\left(-\frac{1}{4}\omega\right)^{1/2}\Phi\left(\alpha+\frac{5}{4};\frac{3}{2};-\frac{1}{4}\omega\right).$$

Thus we find the exact solution

$$v = t^{-\left(\alpha+\frac{3}{4}\right)}\left[C_1\Phi\left(\alpha+\frac{3}{4};\frac{1}{2}-\frac{y_1^2}{4t}\right) + C_2\left(-\frac{y_1^2}{4t}\right)^{1/2}\Phi\left(\alpha+\frac{5}{4};\frac{3}{2};-\frac{y_1^2}{4t}\right)\right].$$

3.1.4.

$$\langle G_1, P_2\rangle: \quad v = \exp\left(-\frac{y_1^2}{4t}\right)\varphi(\omega), \quad \omega = t, \quad \dot\varphi + \frac{1}{2\omega}\varphi = 0$$

which integrates to give the exact solution

$$v = C|t|^{-1/2} \exp\left(-\frac{y_1^2}{4t}\right).$$

3.1.5.

$$\langle P_2, T + G_1\rangle : \quad v = \exp\left(\frac{t^3}{6} - \frac{y_1 t}{2}\right)\varphi(\omega), \quad \omega = t^2 - 2y_1,$$

$$16\ddot{\varphi} - \omega\varphi = 0.$$

To treat this ODE, first write $\varphi = \sqrt{\omega}\psi(z)$ with $z = \omega^{3/2}/6$. Then ψ satisfies

$$\psi'' + \frac{1}{z}\psi' - \left(1 + \frac{1}{9z^2}\right)\psi = 0$$

which is the equation for the Bessel function $J_{\pm 1/3}(iz)$ (these two are linearly independent solutions) (see Vol. 2, section 7.2.2 of [11]). Consequently, we have

$$v = \left(t^2 - 2y_1\right)^{1/2} \exp\left(\frac{t^3}{6} - \frac{ty_1}{2}\right) \times$$
$$\times \left[C_1 J_{1/3}\left(\frac{i\left(t^2 - 2y_1\right)^{3/2}}{6}\right) + C_2 J_{-1/3}\left(\frac{i\left(t^2 - 2y_1\right)^{3/2}}{6}\right)\right]$$

as an exact solution of the heat equation.

3.1.6.

$$\langle J_{12} + \alpha D - \alpha(4\beta + 2)M, T\rangle \quad (\alpha > 0,\ \beta \in \mathbb{R}),$$

$$v = \left(y_1^2 + y_2^2\right)^{\beta}\varphi(\omega), \quad \omega = \alpha \arctan\left(\frac{y_1}{y_2}\right) + \frac{1}{2}\ln\left(y_1^2 + y_2^2\right),$$

$$(\alpha^2 + 1)\ddot{\varphi} + 4\beta\dot{\varphi} + 4\beta^2\varphi = 0.$$

Integrating this equation, we obtain

$$\varphi = C_1\omega + C_2 \quad \text{for} \quad \beta = 0$$

and

$$\varphi = C_1 \exp\left(-\frac{2\beta\omega}{1 + \alpha^2}\right)\cos\left(-\frac{2\alpha\beta\omega}{1 + \alpha^2} + C_2\right) \quad \text{for} \quad \beta \neq 0.$$

These then give us the exact solutions

$$v = C_1\left[\alpha \arctan\left(\frac{y_1}{y_2}\right) + \frac{1}{2}\ln\left(y_1^2 + y_2^2\right)\right] + C_2 \quad \text{for} \quad \beta = 0,$$

$$v = C_1\left(y_1^2 + y_2^2\right)^{\beta}\exp\left(-\frac{2\beta\omega}{1 + \alpha^2}\right)\cos\left(\frac{2\alpha\beta\omega}{1 + \alpha^2} + C_2\right) \quad \text{for} \quad \beta \neq 0,$$

where

$$\omega = \alpha \arctan\left(\frac{y_1}{y_2}\right) + \frac{1}{2}\ln\left(y_1^2 + y_2^2\right).$$

3.1.7.

$$\langle J_{12} + 2\alpha M, D - (4\beta + 2)M\rangle \quad (\alpha \ge 0,\ \beta \in \mathbb{R}),$$
$$v = t^\beta \exp\left(\alpha \arctan \frac{y_1}{y_2}\right)\varphi(\omega), \quad \omega = \frac{y_1^2 + y_2^2}{t},$$
$$\omega^2\ddot{\varphi} + \left(\omega + \frac{\omega^2}{4}\right)\dot{\varphi} + \left(\frac{\alpha^2}{4} - \frac{\beta\omega}{4}\right)\varphi = 0.$$

This equation gives

$$\ddot{\varphi} + \left(\frac{1}{4} + \frac{1}{\omega}\right)\dot{\varphi} + \left(\frac{\alpha^2}{4\omega^2} - \frac{\beta}{4\omega}\right)\varphi = 0.$$

Its solutions can be given in terms of Whittaker functions $W(k; m; z)$ (see Vol. 1, ch. 6, pp. 248–251 of [11]) and one obtains

$$\varphi = \frac{e^{-\omega/8}}{\sqrt{\omega}} W\left(-(\beta + 1/2); \frac{i\alpha}{2}; \frac{\omega}{4}\right)$$

and hence

$$v = \frac{t^{\beta+1/2}}{\sqrt{y_1^2 + y_2^2}} \exp\left(-\frac{y_1^2 + y_2^2}{8t}\right)\exp\left(\alpha \arctan \frac{y_1}{y_2}\right) W\left(-(\beta + 1/2); \frac{i\alpha}{2}; \frac{y_1^2 + y_2^2}{4t}\right).$$

3.1.8.

$$\langle J_{12} + 2\alpha M, T + \beta M\rangle \quad (\alpha \ge 0, \beta = 0, \pm 1),$$
$$v = \exp\left(\alpha \arctan \frac{y_1}{y_2} - \frac{\beta t}{2}\right)\varphi(\omega), \quad \omega = y_1^2 + y_2^2,$$
$$\omega^2\ddot{\varphi} + \omega\dot{\varphi} + \left(\frac{\alpha^2}{4} + \frac{\beta\omega}{8}\right)\varphi = 0.$$

We have the following cases:

$$\varphi = C_1 + C_2 \log \omega \quad \text{for} \quad \alpha = \beta = 0,$$
$$\varphi = C_1 \cos\left(-\frac{\alpha}{2}\log \omega + C_2\right) \quad \text{for} \quad \alpha \ne 0,\ \beta = 0,$$
$$\varphi = J_{i\alpha}\left(\sqrt{\frac{\beta\omega}{2}}\right) \quad \text{for} \quad \alpha \ge 0,\ \beta \ne 0.$$

Consequently, we have the following solutions of (8)

$$v = C_1 + C_2 \log\left(y_1^2 + y_2^2\right) \quad \text{for} \quad \alpha = \beta = 0,$$
$$v = \exp\left(\alpha \arctan \frac{y_1}{y_2}\right) C_1 \cos\left(-\frac{\alpha}{2}\log\left(y_1^2 + y_2^2\right) + C_2\right) \quad \text{for} \quad \alpha \ne 0,\ \beta = 0,$$
$$v = \exp\left(\alpha \arctan \frac{y_1}{y_2} - \frac{\beta t}{2}\right) J_{i\alpha}\left(\sqrt{\frac{\beta\left(y_1^2 + y_2^2\right)}{2}}\right) \quad \text{for} \quad \alpha \ge 0,\ \beta \ne 0.$$

3.1.9.

$$\langle J_{12}+S+T+2\alpha M, G_1+P_2\rangle \quad (\alpha\in\mathbb{R}),$$
$$v=\left(t^2+1\right)^{-1/2}\exp\left[\left(\frac{1-t^2}{4t}\right)\left(\frac{y_1+ty_2}{t^2+1}\right)^2-\frac{y_1^2}{4t}-\alpha\arctan t\right]\varphi(\omega),$$
$$\omega=\frac{y_1+ty_2}{t^2+1},\quad \ddot\varphi+(\alpha+\omega^2)\varphi=0.$$

This equation is known as the Weber equation. Its solutions are the real and imaginary parts of the functions

$$D_{-\sqrt{\alpha}}(\pm(1+i)\omega),$$

where $D_\nu(z)$ are the Weber–Hermite (parabolic cylinder) functions (Vol. 2, ch. 8, section 8.2 of [11]). This gives the following exact solutions of (9):

$$v=\left(t^2+1\right)^{-1/2}\exp\left[\left(\frac{1-t^2}{4t}\right)\left(\frac{y_1+ty_2}{t^2+1}\right)^2-\frac{y_1^2}{4t}-\alpha\arctan t\right]\times$$
$$\times D_{\sqrt{\alpha}}\left(\pm(1+i)\frac{y_1+ty_2}{t^2+1}\right)$$

and the real and imaginary parts of this function give us exact solutions of the heat equation (9).

3.1.10.

$$\langle J_{12}+2\alpha M, S+T+2\beta M\rangle \quad (\alpha\geq 0,\ \beta\in\mathbb{R}),$$
$$v=\left(t^2+1\right)^{-1/2}\exp\left[-\beta\arctan t+\alpha\arctan\frac{y_1}{y_2}-\frac{t\left(y_1^2+y_2^2\right)}{4\left(t^2+1\right)}\right]\varphi(\omega),$$
$$\omega=\frac{y_1^2+y_2^2}{t^2+1},\quad \ddot\varphi+\frac{1}{\omega}\dot\varphi+\left(\frac{1}{16}+\frac{\beta}{4\omega}+\frac{\alpha^2}{4\omega^2}\right)\varphi=0.$$

The solutions of this equation can be given in terms of Whittaker functions [11], and we obtain the following exact solutions of the heat equation as a result:

$$v=\left(y_1^2+y_2^2\right)^{-1/2}\exp\left[-\beta\arctan t+\alpha\arctan\frac{y_1}{y_2}-\frac{t(y_1^2+y_2^2)}{4(t^2+1)}\right]\times$$
$$\times W\left(\frac{i\beta}{8};\frac{i\alpha}{2};\frac{i\left(y_1^2+y_2^2\right)}{2\left(t^2+1\right)}\right).$$

In the above cases we have been able to describe exact solutions of (8) in terms of elementary functions or confluent hypergeometric functions. Using the notation introduced in equations (8) and (7), we are thus able to construct strikingly new exact solutions of the linear wave equation (1).

3.2. Reduction to partial differential equations by one-dimensional subalgebras. Here we list the subalgebras, the relevant parameters, ansatzes and reduced equations, without constructing their exact solutions. We use φ_1 to denote the partial derivative with respect to ω_1, and φ_{22} means the second derivative with respect to ω_2, and so on.

3.2.1.

$$\langle P_2\rangle: \quad v=\varphi(\omega_1,\omega_2), \quad \omega_1=t, \quad \omega_2=y_1, \quad \varphi_1=\varphi_{22}.$$

This is the heat equation in $1+1$ spacetime dimensions. The symmetries and conditional symmetries of the heat equation are well known. A discussion of these can be found in [6] and in appendix 7 of [2].

3.2.2.

$$\langle G_1+P_2\rangle: \quad v=\exp\left(-\frac{y_1^2}{4t}\right)\varphi(\omega_1,\omega_2), \quad \omega_1=t, \quad \omega_2=y_1+ty_2,$$
$$(1+\omega_1^2)\varphi_{22}-\varphi_1-\frac{\omega_2}{\omega_1}\varphi_2-\frac{1}{2\omega_1}\varphi=0.$$

3.2.3.

$$\langle T+\alpha M\rangle \quad (\alpha=0,\pm1): \quad v=\exp\left(-\frac{\alpha t}{2}\right)\varphi(\omega_1,\omega_2), \quad \omega_1=y_1, \quad \omega_2=y_2,$$
$$\varphi_{11}+\varphi_{22}+\frac{1}{2}\alpha\varphi=0.$$

This equation is the Laplace equation for $\alpha=0$. Solutions can be obtained by using separation of variables.

3.2.4.

$$\langle T+G_1\rangle: \quad v=\exp\left(\frac{t^3}{6}-\frac{y_1t}{2}\right)\varphi(\omega_1,\omega_2), \quad \omega_1=t^2-2y_1, \quad \omega_2=y_2,$$
$$4\varphi_{11}+\varphi_{22}-\frac{1}{4}\omega_1\varphi=0.$$

3.2.5.

$$\langle J_{12}+2\alpha M\rangle \quad (\alpha\geq 0),$$
$$v=\exp\left(\alpha\arctan\frac{y_1}{y_2}\right)\varphi(\omega_1,\omega_2), \quad \omega_1=y_1^2+y_2^2, \quad \omega_2=t,$$
$$4\omega_1^2\varphi_{11}+4\omega_1\varphi_1+\omega_1\varphi_2+\alpha^2\varphi=0.$$

3.2.6.

$$\langle J_{12}+T+2\alpha M\rangle \quad (\alpha\in\mathbb{R}),$$
$$v=\exp(-\alpha t)\varphi(\omega_1,\omega_2), \quad \omega_1=y_1^2+y_2^2, \quad \omega_2=t+\arctan\frac{y_1}{y_2},$$
$$4\omega_1^2\varphi_{11}+\varphi_{22}+4\omega_1\varphi_1-\omega_1\varphi_2+\alpha\omega_1\varphi=0.$$

3.2.7.

$$\langle J_{12}+\frac{\alpha}{2}D+\alpha(2\beta-1)M\rangle \quad (\alpha\geq 0, \ \beta\geq 1/2),$$
$$v=t^\beta\varphi(\omega_1,\omega_2), \quad \omega_1=\log t+\alpha\arctan\frac{y_1}{y_2}, \quad \omega_2=\frac{y_1^2+y_2^2}{t},$$
$$\alpha^2\varphi_{11}+4\omega_2^2\varphi_{22}-\omega_2\varphi_1+(4\omega_2+\omega_2^2)\varphi_2+\beta\omega_2\varphi=0.$$

3.2.8.

$$\langle D+(4\alpha-2)M\rangle \quad (\alpha \geq 1/2): \quad v = t^{-\alpha}\varphi(\omega_1,\omega_2), \quad \omega_1 = \frac{y_1^2}{t}, \quad \omega_2 = \frac{y_2^2}{t},$$

$$4\omega_1\varphi_{11} + 4\varphi_2\varphi_{22} + (2+\omega_1)\varphi_1 + (2+\omega_2)\varphi_2 + \alpha\varphi = 0.$$

3.2.9.

$$\langle S+T+\alpha J_{12}+2\beta M\rangle \quad (\alpha > 0,\ \beta \in \mathbb{R}),$$

$$v = \left(t^2+1\right)^{-1/2}\exp\left[-\beta\arctan t - \frac{t(y_1^2+y_2^2)}{4\,(t^2+1)}\right]\varphi(\omega_1,\omega_2)$$

$$\omega_1 = \frac{y_1^2+y_2^2}{t^2+1}, \quad \omega_2 = \arctan\frac{y_1}{y_2} + \alpha\arctan t,$$

$$4\omega_1\varphi_{11} + \frac{1}{\omega_1}\varphi_{22} + 4\varphi_1 - \alpha\varphi_2 + \left(\beta + \frac{\omega_1}{4}\right)\varphi = 0.$$

3.2.10.

$$\langle S+T+2\alpha M\rangle \quad (\alpha \in \mathbb{R}),$$

$$v = \left(t^2+1\right)^{-1/2}\exp\left[-\alpha\arctan t - \frac{t(y_1^2+y_2^2)}{4(t^2+1)}\right]\varphi(\omega_1,\omega_2),$$

$$\omega_1 = \frac{y_1^2}{t^2+1}, \quad \omega_2 = \frac{y_2^2}{t^2+1},$$

$$4\omega_1\varphi_{11} + 4\omega_2\varphi_{22} + 2\varphi_1 + 2\varphi_2 + \left(\alpha + \frac{\omega_1+\omega_2}{4}\right)\varphi = 0.$$

3.2.11.

$$\langle S+T+J_{12}+\alpha(G_1+P_2)\rangle \quad (\alpha > 0),$$

$$v = (t^2+1)^{-1/2}\exp\left[\frac{(1-t^2)(y_1+ty_2)^2}{4t(t^2+1)^2} - \frac{y_1^2}{4t}\right]\varphi(\omega_1,\omega_2),$$

$$\omega_1 = \frac{y_1+ty_2}{t^2+1}, \quad \omega_2 = \frac{ty_1-y_2}{t^2+1} = \alpha\arctan t,$$

$$\varphi_{11} + \varphi_{22} - (2\omega_1-\alpha)\varphi_2 + \omega_1^2\varphi = 0.$$

4. Some conditional symmetries of the 2 + 1 heat equation

In this section we give the conditional symmetries of equation (8). The defining equations are nonlinear coupled partial differential equations, which we do not solve, except in one case, leaving the others for consideration in a later publication. We have the following result.

Proposition 3. *Equation (8) is conditionally invariant under*

$$X = \xi^0\frac{\partial}{\partial t} + \xi^1\frac{\partial}{\partial y_1} + \xi^2\frac{\partial}{\partial y_2} + \eta\frac{\partial}{\partial v}$$

when the coefficients satisfy the following conditions:

$$(i) \quad \xi^0 = 1: \quad \xi^1_{y_1} = \xi^2_{y_2}, \quad \xi^1_{y_2} = -\xi^2_{y_1}, \quad \eta = Av + B,$$

where ξ^1, ξ^2, A, B are functions of t, y_1, y_2 and satisfy the system

$$\xi^1_t + 2\xi^1\xi^1_{y_1} + 2A_{y_1} = 0, \quad \xi^2_t + 2\xi^2\xi^2_{y_2} + 2A_{y_2} = 0,$$

$$A_t = A_{y_1y_1} + A_{y_2y_2} - 2A\xi^2_{y_2}, \quad B_t = B_{y_1y_1} + B_{y_2y_2} - 2B\xi^2_{y_2}.$$

(ii) $\xi^0 = 0$, $\xi^1 = 1:$ $\xi^2_{y_2} = \xi^2\xi^2_{y_1}$, $\eta = Av + B$,

where ξ^2, A, B are functions of t, y_1, y_2 and satisfy the system

$$
\begin{aligned}
&\xi^2_t - \xi^2_{y_1y_1} - \xi^2_{y_2y_2} + 2\xi^2_{y_1}\xi^2_{y_2} - 2\xi^2 A_{y_1} - 2A\xi^2_{y_1} = 0,\\
&A_t = A_{y_1y_1} + A_{y_2y_2} + 2AA_{y_1} - 2A_{y_2}\xi^2_{y_1},\\
&B_t = B_{y_1y_1} + B_{y_2y_2} + 2BA_{y_1} - 2By_2\xi^2_{y_1}.
\end{aligned}
$$

(iii) $\xi^0 = \xi^1 = 0$, $\xi^2 = 1:$ $\eta = Av + B$,

where A is a function of t, y_2 only, and B is a function of t, y_1, y_2 and satisfy the equations

$$A_t = A_{y_2y_2} + 2AA_{y_2}, \quad B_t = B_{y_1y_1} + B_{y_2y_2} + 2BA_{y_2}.$$

As is clear in the above three cases, the systems of equations involved are highly nonlinear, and cannot be solved in general. However, the equation for the function A in case (iii) is recognized to be the Burgers equation. This equation can be linearized by the Hopf–Cole transformation $A = w_{y_2}/w$, where w is a solution of the heat equation $w_t = w_{y_2y_2}$ (see for example [2]). The solutions obtained in this way can then be used to build ansatzes first for the $2+1$ heat equation (8) and then, in turn, the linear wave equation (1), using the ansatz (7).

Ansatzes can also be obtained from the symmetry algebra of the Burgers equation. Indeed, the symmetry algebra of the equation

$$A_t = A_{y_2y_2} + 2AA_{y_2} \tag{10}$$

is generated by the operators

$$
\begin{aligned}
&\partial_t, \quad \partial_{y_2}, \quad 2t\partial_{y_2} - \partial_A, \quad 2t\partial_t + y_2\partial_{y_2} - A\partial_A,\\
&t^2\partial_t + ty_2\partial_{y_2} - \left(tA + \frac{y_2}{2}\right)\partial_A.
\end{aligned} \tag{11}
$$

The operator (11) gives the ansatz

$$A = -\frac{y_2}{2t} + \frac{1}{t}\psi\left(\frac{y_2}{t}\right) \tag{12}$$

which gives, on substituting into (10), the equation

$$\ddot\psi + 2\psi\dot\psi = 0$$

for ψ, where the dot denotes differentiation with respect to the variable $\omega = y_2/t$. This equation readily integrates to

$$\dot\psi + \psi^2 = c,$$

where c is a constant. This gives us three cases:

$$c = 0: \quad \psi = t/(kt + y_2), \tag{13}$$

where k is a constant;

$$c = a^2, \quad a > 0: \quad \psi = a\left(t\exp\left(\frac{2ay_2}{t}\right) - 1\right)\Big/\left(t\exp\left(\frac{2ay_2}{t}\right) + 1\right) \tag{14}$$

with $l \neq 0$ a constant;

$$c = -a^2, \quad a > 0: \quad \psi = -a\tan\left(a^2 + \frac{ay_2}{t}\right). \tag{15}$$

Substituting these into (12), one obtains exact solutions of (10). We use these exact solutions for A together with theorem 3 (iii) (with $B = 0$) as follows. The equation (8) is conditionally invariant under

$$\partial_{y_2} + Av\partial_v \tag{16}$$

and this gives us an ansatz for v to be substituted into (8), and this, in turn, gives us an exact solution of (8) which, when we combine it with (7), gives an exact solution of (1). We list the results of these stages for each of the equations (13)–(15).

The ansatz for v from (13) is

$$v = (kt + y_2)\exp\left(-y_2^2/4t\right)\Phi(t, y_1),$$

where $\Phi(t, y_1)$ satisfies

$$\Phi_t = \Phi_{y_1y_1} - \frac{3}{2}\Phi$$

and consequently we find that v is given by

$$v = (kt + y_2)\exp\left(-y_2^2/4t - 3t/2\right)\Phi(t, y_1),$$

where $\Psi(t, y_1)$ satisfies the $(1+1)$-dimensional heat equation.

The ansatz for v from (14) is

$$v = e^{a^2/t}\left[l\exp\left(-(y_2 - 2a)^2/4t\right) + \exp\left(-(y_2 + 2a)^2/4t\right)\right]\Phi(t, y_1),$$

where $\Phi(t, y_1)$ satisfies

$$\Phi_t + \left(\frac{1}{2t} - \frac{a^2}{t^2}\right)\Phi = \Phi_{y_1y_1}$$

and using this we eventually find that v is given by

$$v = \frac{1}{\sqrt{t}}\left[le^{ay_2/t} + e^{-ay_2/t}\right]\exp\left(-\left(4a^2 + y_2^2\right)/4t\right)\Psi(t, y_1), \tag{17}$$

where $\Psi(t, y_1)$ satisfies the $(1+1)$-dimensional heat equation.

The ansatz for v from (15) is

$$v = \cos\left(a^2 + \frac{ay_2}{t}\right)\exp\left(-y_2^2/4t\right)\Phi(t, y_1),$$

where $\Phi(t, y_1)$ satisfies

$$\Phi_t + \left(\frac{1}{2t} - \frac{a^2}{t^2}\right)\Phi = \Phi_{y_1y_1},$$

so that we obtain

$$v = \frac{1}{\sqrt{t}}\cos\left(a^2 + \frac{ay_2}{t}\right)\exp\left(-\left(4a^2 + y_2^2\right)/4t\right)\Psi(t, y_1), \tag{18}$$

where $\Psi(t, y_1)$ satisfies the $(1+1)$-dimensional heat equation.

We can now combine equations (17)–(19) with equation (7) to obtain new solutions of (1):

$$u = [k(\tau x) + (\delta x)] \exp\left(\frac{(\epsilon x)}{2} - \frac{(\delta x)^2}{4(\tau x)} - \frac{3(\tau x)}{2}\right) \Psi((\tau x), (\beta x)),$$

$$u = \frac{1}{\sqrt{(\tau x)}} \left[l e^{a(\delta x)/(\tau x)} + e^{-a(\delta x)/(\tau x)}\right] \exp\left(\frac{(\epsilon x)}{2} - \frac{(4a^2 + (\delta x)^2)}{4(\tau x)}\right) \Psi((\tau x), (\beta x)),$$

$$u = \frac{1}{\sqrt{(\tau x)}} \cos\left(a^2 + \frac{a(\delta x)}{(\tau x)}\right) \exp\left(\frac{(\epsilon x)}{2} - \frac{(4a^2 + (\delta x)^2)}{4(\tau x)}\right) \Psi((\tau x), (\beta x)),$$

where $\Psi(t, x)$ is any solution of the $(1+1)$-dimensional heat equation.

One can, in principle, perform the same procedure for the other conditional symmetry operators defined in theorem 3; however, it is first necessary to obtain some exact solutions of the systems. These latter are quite nonlinear and require further treatment, and we leave this to a future publication.

5. Conclusion

We have been able to give a new reduction of the linear wave equation in $1+3$ timespace dimensions to a linear heat equation in $1+2$ timespace dimensions, that is, a reduction of a hyperbolic equation to a parabolic one. The further reductions of this heat equation by two-dimensional subalgebras (inequivalent under the action of $G_2(1,2)$) to ordinary differential equations leads to exact solutions in terms of special functions. These are of interest in their own right. Conditional symmetries can also be used to obtain new exact solutions. Using these solutions of the heat equation, one can construct new solutions of the linear wave equation. In concluding, we remark that the complex nonlinear wave equation

$$\Box\Psi + F(|\Psi|, \partial_\mu|\Psi|\partial^\mu|\Psi|)\Psi = 0,$$

where F is an arbitrary smooth function of its arguments and Ψ is a complex function, can be reduced by the same ansatz as (7) (but with k imaginary) to a nonlinear Schrödinger equation with the same nonlinearity. Some of these equations admit soliton solutions. We report on these results in [3].

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