On linear and non-linear representations of the generalized Poincaré groups in the class of Lie vector fields

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We study representations of the generalized Poincaré group and its extensions in the class of Lie vector fields acting in a space of $n+m$ independent and one dependent variables. We prove that an arbitrary representation of the group $P(n,m)$ with $\max\{n,m\} \geq 3$ is equivalent to the standard one, while the conformal group $C(n,m)$ has non-trivial nonlinear representations. Besides that, we investigate in detail representations of the Poincaré group $P(2,2)$, extended Poincaré groups $\widetilde{P}(1,2)$, $\widetilde{P}(2,2)$, and conformal groups $C(1,2)$, $C(2,2)$ and obtain their linear and nonlinear representations.

1 Introduction

The central problem to be solved within the framework of the classical Lie approach to investigation of the partial differential equation (PDE)

$$F(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{r}{u}) = 0, \tag{1}$$

where symbol $\underset{k}{u}$ denotes a set of k-th order derivatives of the function $u = u(x)$, is to compute its maximal symmetry group. Sophus Lie developed the universal infinitesimal algorithm which reduced the above problem to solving some linear over-determined system of PDE (see, e.g. [1–3]). The said method enables us to solve the inverse problem of symmetry analysis of differential equations — description of equations invariant under given transformation group. This problem is of great importance of mathematical and theoretical physics. For example, in relativistic field theory motion equations have to obey the Lorentz–Poincaré–Einstein relativity principle. It means that equations considered should be invariant under the Poincaré group $P(1,3)$. That is why, there exists a deep connection between the theory of relativistically-invariant wave equations and representations of the Poincaré group [4–6].

There exists a vast literature on representations of the generalized Poincaré group $P(n,m)$ [6], $n, m \in \mathbb{N}$ but only a few papers are devoted to a study of nonlinear representations. It should be noted that nonlinear representations of the Poincaré and conformal groups often occur as realizations of symmetry groups of nonlinear PDE such as eikonal, Born–Infeld and Monge–Amperé equations (see [3] and references therein). On sets of solutions of some nonlinear heat equations nonlinear representations of the Galilei group are realized [3]. So, nonlinear representations of the transformations groups are intimately connected with nonlinear PDE, and systematic study of these is of great importance.

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In the present paper we obtain the complete description of the Poincaré group $P(n,m)$ (called for bravity the Poincaré group) and of its extensions — the extended Poincaré group $\widetilde{P}(n,m)$ and conformal group $C(n,m)$ acting as Lie transformation groups in the space $\mathbb{R}(n,m)\times\mathbb{R}^1$, where $\mathbb{R}(n,m)$ is the pseudo-Euclidean space with the metric tensor

$$g_{\alpha\beta} = \begin{cases} 1, & \alpha=\beta=\overline{1,n}, \\ -1, & \alpha=\beta=\overline{n+1,n+m}, \\ 0, & \alpha\neq\beta. \end{cases}$$

The paper is organized as follows. In Section 2 we give all necessary notations and definitions. In Section 3 we investigate representations of groups $P(n,m)$, $\widetilde{P}(n,m)$, $C(n,m)$ with $\max\{n,m\}\geq 3$ and prove, in particular, that each representation of the Poincaré group $P(n,m)$ with $\max\{n,m\}\geq 3$ is equivalent to the standard linear representation. In Section 3 we study representations of the above groups with $\max\{n,m\}<3$ and show that groups $\widetilde{P}(1,2)$, $C(1,2)$, $P(2,2)$, $\widetilde{P}(2,2)$, $C(2,2)$ have nontrivial nonlinear representations. It should be noted that nonlinear representations of the groups $P(1,1)$, $\widetilde{P}(1,1)$, $C(1,1)$ were constructed in [9] and of the group $P(1,2)$ — in [10].

2 Notations and definitions

Saying about a representation of the Poincaré group $P(n,m)$ in the class of Lie transformation groups we mean the transformation group

$$x'_\mu = f_\mu(x,u,a), \quad \mu=\overline{1,n+m}, \quad u'=g(x,u,a), \tag{2}$$

where $a=\{a_N,\ N=1,2,\ldots,n+m+C^2_{n+m}\}$ are group parameters preserving the quadratic form $S(x)=g_{\alpha\beta}x_\alpha x_\beta$. Here and below summation over the repeated indices is understood.

It is common knowledge that a problem of description of inequivalent representations of the Lie transformation group (2) can be reduced to a study of inequivalent representations of its Lie algebra [1, 2, 12].

Definition 1. *Set of $n+m+C^2_{n+m}$ differential operators P_μ, $J_{\alpha\beta}=-j_{\beta\alpha}$, $\mu,\alpha,\beta=\overline{1,n+m}$ of the form*

$$Q=\xi_\mu(x,u)\partial_\mu+\eta(x,u)\partial_u \tag{3}$$

satisfying the commutational relations

$$\begin{gathered}[P_\alpha,P_\beta]=0, \quad [P_\alpha,P_{\beta\gamma}]=g_{\alpha\beta}P_\gamma-g_\alpha\gamma P_\beta, \\ [J_{\alpha\beta},J_{\mu\nu}]=g_{\alpha\nu}J_{\beta\mu}+g_{\beta\mu}J_{\alpha\nu}-g_{\alpha\mu}J_{\beta\nu}-g_{\beta\nu}J_{\alpha\mu}\end{gathered} \tag{4}$$

is called a representation of the Poincaré algebra $AP(n,m)$ in the class of Lie vector fields.

In the above formulae

$$\partial_\mu=\frac{\partial}{\partial x_\mu}, \quad \partial_u=\frac{\partial}{\partial u}, \quad [Q_1,Q_2]=Q_1Q_2-Q_2Q_1, \quad \alpha,\beta,\gamma,\mu,\nu=\overline{1,n+m}.$$

Definition 2. *Set of* $1+n+m+C^2_{n+m}$ *differential operators* P_μ, $J_{\alpha\beta}=-J_{\beta\alpha}$, D $(\mu,\alpha,\beta=\overline{1,n+m})$ *of the form (3) satisfying the commutational relations (4) and*

$$[D,J_{\alpha\beta}]=0, \quad [P_\alpha,D]=P_\alpha \quad (\alpha,\beta=\overline{1,n+m}) \tag{5}$$

is called a representation of the extended Poincaré algebra $A\widetilde{P}(n,m)$ *in the class of Lie vector fields.*

Using the Lie theorem [1, 2] one can construct the $(1+n+m+C^2_{n+m})$-parameter Lie transformation group corresponding to the Lie algebra $\{P_\mu,J_{\alpha\beta},D\}$. This transformation group is called a representation of the extended Poincaré group $\widetilde{P}(n,m)$.

Definition 3. *Set of* $1+2(n+m)+C^2_{n+m}$ *differential operators* P_μ, $J_{\alpha\beta}=-J_{\beta\alpha}$, D, K_μ $(\mu,\alpha,\beta=\overline{1,n+m})$ *of the form (3) satisfying the commutational relations (4), (5) and*

$$\begin{aligned}&[K_\alpha,K_\beta]=0, \quad [K_\alpha,J_{\beta\gamma}]=g_{\alpha\beta}K_\gamma-g_{\alpha\gamma}K_\beta,\\ &[P_\alpha,K_\beta]=2(g_{\alpha\beta}D-J_{\alpha\beta}), \quad [D,K_\alpha]=K_\alpha,\end{aligned} \tag{6}$$

is called a representation of the conformal algebra $AC(n,m)$ *in the class of Lie vector fields.*

$(1+2(n+m)+C^2_{n+m})$*-parameter transformation group corresponding to the Lie algebra* $\{P_\mu,J_{\alpha\beta},D,K_\mu\}$ *is called a representation of the conformal group* $C(n,m)$.

Definition 4. *Representation of the Lie transformation group (2) is called linear if functions* f_μ, g *satisfy conditions* $f_\mu=f_\mu(x,a)$ $(\mu=\overline{1,n+m})$, $g=\widetilde{g}(x,a)u$. *If these conditions are not satisfied, representation is called nonlinear.*

Definition 5. *Representation of the Lie algebra in the class of Lie vector fields (3) is called linear if coefficients of its basis elements satisfy the conditions*

$$\xi_\alpha=\xi_\alpha(x), \quad \alpha=\overline{1,n+m}, \quad \eta=\widetilde{\eta}(x)u, \tag{7}$$

otherwise it is called nonlinear.

Using the Lie equations [1, 2] it is easy to establish that if a Lie algebra has a nonlinear representation, its Lie group also has a nonlinear representation and vice versa.

Since commutational relations (4)–(6) are not altered by the change of variables

$$x'_\alpha=F_\alpha(x,u), \quad u'=G(x,u), \tag{8}$$

two representations $\{P_\alpha,J_{\alpha\beta},D,K_\alpha\}$ and $\{P'_\alpha,J'_{\alpha\beta},D',K'_\alpha\}$ are called equivalent provided they are connected by relations (8).

3 Representations of the algebras $AP(n,m)$, $A\widetilde{P}(n,m)$, $AC(n,m)$ with $\max\{n,m\}\geq 3$

Theorem 1. *Arbitrary representation of the Poincaré algebra* $AP(n,m)$ *with* $\max\{n,m\}\geq 3$ *in the class of Lie vector fields is equivalent to the standard representation*

$$P_\alpha=\partial_\alpha, \quad J_{\alpha\beta}=g_{\alpha\gamma}x_\gamma\partial_\beta-g_{\beta\gamma}x_\gamma\partial_\alpha \quad (\alpha,\beta=\overline{1,n+m}). \tag{9}$$

Proof. By force of the fact that operators P_α commute, there exists the change of variables (8) reducing these to the form $P_\alpha = \partial_\alpha$, $\alpha = \overline{1, n+m}$ (a rather simple proof of this assertion can be found in [1, 3]). Substituting operators $P_\alpha = \partial_\alpha$, $J_{\alpha\beta} = \xi_{\alpha\beta\gamma}(x,u)\partial_\gamma + \eta_{\alpha\beta}(x,u)\partial_u$ into relations $[P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta}P_\gamma - g_{\alpha\gamma}P_\beta$ and equating coefficients at the linearly-independent operators ∂_α, ∂_u we get a system of PDE for unknown functions $\xi_{\alpha\beta\gamma}$, $\eta_{\alpha\beta}$

$$\xi_{\alpha\beta\gamma x_\mu} = g_{\mu\alpha}g_{\gamma\beta} - g_{\mu\beta}g_{\gamma\alpha}, \quad \eta_{\alpha\beta x_\mu} = 0, \quad \alpha, \beta, \gamma, \mu = \overline{1, n+m},$$

whence

$$\xi_{\alpha\beta\gamma} = x_\alpha g_{\gamma\beta} - x_\beta g_{\gamma\alpha} + F_{\alpha\beta\gamma}(u), \quad \eta_{\alpha\beta} = G_{\alpha\beta}(u).$$

Here $F_{\alpha\beta\gamma} = -F_{\beta\alpha\gamma}$, $G_{\alpha\beta} = -G_{\beta\alpha}$ are arbitrary smooth functions, $\alpha, \beta, \gamma = \overline{1, n+m}$.

Consider the third commutational relation from (4) under $1 \le \alpha, \beta, \mu, \nu \le n$, $\beta = \mu$. Equating coefficients at the operator ∂_u, we get the system of nonlinear ordinary differential equations for $G_{\mu\nu}(u)$

$$G_{\alpha\nu} = G_{\alpha\beta}\dot{G}_{\beta\nu} - G_{\beta\nu}\dot{G}_{\alpha\beta} \tag{11a}$$

(no summation over β), where a dot means differentiation with respect to u.

Since (11a) holds under arbitrary $\alpha, \beta, \nu = \overline{1, n}$, we can redenote subscripts in order to obtain the following equations

$$G_{\beta\nu} = G_{\beta\alpha}\dot{G}_{\alpha\nu} - G_{\alpha\nu}\dot{G}_{\beta\alpha}, \tag{11b}$$

$$G_{\alpha\beta} = G_{\alpha\nu}\dot{G}_{\nu\beta} - G_{\nu\beta}\dot{G}_{\alpha\nu} \tag{11c}$$

(no summation over α and ν).

Multiplying (11a) by $G_{\alpha\nu}$, (11b) by $G_{\beta\nu}$, (11c) by $G_{\alpha\beta}$ and summing we get

$$G^2_{\alpha\mu} + G^2_{\beta\mu} + G^2_{\alpha\beta} = 0,$$

whence $G_{\alpha\nu} = G_{\beta\gamma} = G_{\alpha\beta} = 0$.

Since α, β, ν are arbitrary indices satisfying the restriction $1 \le \alpha, \beta, \nu \le n$, we conclude that $G_{\alpha\beta} = 0$ for all $\alpha, \beta = 1, 2, \ldots, n$.

Furthermore, from commutational relations for operators $J_{\alpha\beta}$, $\alpha, \beta = \overline{1, n}$ we get the homogeneous system of linear algebraic equations for functions $F_{\alpha\beta\gamma}(u)$, which general solution reads

$$F_{\alpha\beta\gamma} = F_\alpha(u)g_{\beta\gamma} - F_\beta(u)g_{\alpha\gamma}, \quad \alpha, \beta, \gamma = \overline{1, n},$$

where $F_\alpha(u)$ are arbitrary smooth functions.

Consequently, the most general form of operators P_μ, $J_{\alpha\beta}$ with $1 \le \alpha, \beta \le n$ satisfying (4) is equivalent to the following:

$$P_\mu = \partial_\mu, \quad J_{\alpha\beta} = \big(x_\alpha + F_\alpha(u)\big)\partial_\beta - \big(x_\beta + F_\beta(u)\big)\partial_\alpha.$$

Making in the above operators the change of variables

$$x'_\mu = x_\mu + F_\mu(u), \quad \mu = \overline{1, n}, \quad x'_A = x_A, \quad A = \overline{n+1, n+m}, \quad u' = 0$$

and omitting primes we arrive at the formulae (9) with $1 \le \alpha, \beta \le n$.

Consider the commutator of operators $J_{\alpha\beta}$, $J_{\alpha A}$ under $1 \le \alpha, \beta \le n$, $n+1 \le A \le n+m$

$$\begin{aligned}\left[J_{\alpha\beta}, J_{\alpha A}\right] = [x_\alpha \partial_\beta - x_\beta \partial_\alpha, g_{\alpha\gamma} x_\gamma \partial_A - g_{A\gamma} x_\gamma \partial_\alpha + \\ + F_{\alpha A\gamma}(u)\partial_\gamma + G_{\alpha A}(u)\partial_u] = x_A \partial_\beta - x_\beta \partial_A.\end{aligned} \tag{12a}$$

On the other hand, by force of commutational relations (4) an equality

$$[J_{\alpha\beta}, J_{\alpha A}] = J_{\beta A} \tag{12b}$$

holds. Comparing right-hand sides of (12a) and (12b) we come to conclusion that $F_{\alpha A\gamma} = 0$, $G_{\alpha A} = 0$. Consequently, operators $J_{\alpha A} = -J_{A\alpha}$ with $\alpha = \overline{1,n}$, $A = \overline{n+1, n+m}$ have the form (9).

Analogously, computing the commutator of operators $J_{\alpha A}$, J_{AB} under $1 \le \alpha \le n$, $n+1 \le A, B \le n+m$ and taking into account commutational relations (4) we get $F_{AB\gamma} = 0$, $A, B = \overline{n+1, n+m}$, $\gamma = \overline{1,n}$. Consequently, operators J_{AB} are of the form

$$J_{AB} = x_B \partial_A - x_A \partial_B + G_{AB}(u)\partial_u, \quad A, B = \overline{n+1, n+m}.$$

At last, substituting the results obtained into commutational relations

$$[J_{\alpha A}, J_{\alpha B}] = -J_{AB}$$

(no summation over α), where $\alpha = \overline{1,n}$, $A, B = \overline{n+1, n+m}$, we get

$$G_{AB} = 0, \quad A, B = \overline{n+1, n+m}.$$

Thus, we have proved that there exists the change of variables (8) reducing an arbitrary representation of the Poincaré algebra $AP(n,m)$ with $\max\{n,m\} \ge 3$ to the standard representation (9). Theorem is proved.

Note 1. Poincaré algebra $AP(n,m)$ contains as a subalgebra the Euclid algebra $AE(n)$ with basis elements P_α, $J_{\alpha\beta}$, $\alpha, \beta = \overline{1,n}$. When proving the above theorem we have established that arbitrary representations of the algebra $AE(n)$ with $n \ge 3$ in the class of Lie vector fields are equivalent to the standard representation

$$P_\mu = \partial_\mu, \quad J_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha, \quad \mu, \alpha, \beta = \overline{1,n}.$$

Theorem 2. *Arbitrary representation of the extended Poincaré algebra $A\widetilde{P}(n,m)$ with $\max\{n,m\} \ge 3$ in the class of Lie vector fields is equivalent to the following representation:*

$$P_\alpha = \partial_\alpha, \quad J_{\alpha\beta} = g_{\alpha\gamma} x_\gamma \partial_\beta - g_{\beta\gamma} x_\gamma \partial_\alpha, \quad D = x_\alpha \partial_\alpha + \varepsilon u \partial_u, \tag{13}$$

where $\varepsilon = 0, 1$; $\alpha, \beta, \gamma = \overline{1, n+m}$.

Proof. From theorem 1 it follows that a representation of the Poincaré algebra $AP(n,m) = \langle P_\mu, J_{\alpha\beta} \rangle$ can always be reduced to the form (9). To find the explicit form of the dilatation operator $D = \xi_\mu(x,u)\partial_\mu + \eta(x,u)\partial_u$ we use the commutational relations $[P_\alpha, D] = P_\alpha$. Equating coefficients at linearly-independent operators ∂_μ, ∂_u, we get

$$\xi_{\mu x_\alpha} = \delta_{\mu\alpha}, \quad \eta_{x_\alpha} = 0,$$

where $\delta_{\mu\alpha}$ is a Kronecker symbol; $\mu, \alpha = \overline{1, n+m}$.

Integrating the above equations we have

$$\xi_\mu = x_\mu + F_\mu(u), \quad \eta = G(u),$$

where $P_\mu(u)$, $G(u)$ are arbitrary smooth functions.

Using commutational relations $[J_{\mu\nu}, D] = 0$ we arrive at the following equalities:

$$g_{\mu\gamma} F_\gamma \partial_\nu - g_{\nu\gamma} F_\gamma \partial_\mu = 0; \quad \mu, \nu = \overline{1, n+m},$$

whence $F_\gamma = 0$, $\gamma = \overline{1, n+m}$.

Thus, the most general form of the operator D is the following:

$$D = x_\mu \partial_\mu + G(u)\partial_u.$$

Provided $G(u) = 0$, we get the formulae (13) under $\varepsilon = 0$. If $G(u) = 0$, then after making the change of variables

$$x'_\mu = x_\mu, \quad \mu = \overline{1, n+m}, \quad u' = \int \big(G(u)\big)^{-1} du$$

we obtain the formulae (8) under $\varepsilon = 1$. Theorem is proved.

Theorem 3. *Arbitrary representation of the conformal algebra $AC(n,m)$ with $\max\{n, m\} \geq 3$ in the class of Lie vector fields is equivalent to one of the following representations:*

1) *operators P_μ, $J_{\alpha\beta}$, D are given by (13), and operators K_α have the form*

$$K_\alpha = 2g_{\alpha\beta} x_\beta D - (g_{\mu\nu} x_\mu x_\nu)\partial_\alpha; \tag{14}$$

2) *operators P_μ, $J_{\alpha\beta}$, D are given by (13) with $\varepsilon = 1$, and operators K_α have the form*

$$K_\alpha = 2g_{\alpha\beta} x_\beta D - (g_{\mu\nu} x_\mu x_\nu \pm u^2)\partial_\alpha. \tag{15}$$

Proof. From theorem 2 it follows that the basis of the algebra $A\widetilde{P}(n,m)$ up to the change of variables (8) can be chosen in the form (13).

From the commutational relations for operators $P_\alpha = \partial_\alpha$ and $K_\beta = \xi_{\beta\mu}(x,u)\partial_\mu + \eta_\beta(x,u)\partial_u$ we get the following system of PDE:

$$\xi_{\beta\mu x_\alpha} = 2g_{\alpha\beta} x_\mu - 2g_{\alpha\nu} x_\nu \delta_{\beta\mu} + 2g_{\beta\nu} x_\nu \delta_{\mu\alpha}, \quad \eta_{\beta x_\alpha} = 2\varepsilon g_{\beta\alpha} u.$$

Integrating these we have

$$\xi_{\beta\mu} = 2g_{\beta\nu} x_\nu x_\mu - g_{\alpha\nu} x_\alpha x_\nu \delta_{\beta\mu} + F_{\beta\mu}(u), \quad \eta_\beta = 2\varepsilon x_\beta u + G_\beta(u),$$

where $F_{\beta\mu}$, g_β are arbitrary smooth functions, $\alpha, \beta, \mu, \nu = \overline{1, n+m}$.

Next, we make use of commutational relations $[D, K_\alpha] = K_\alpha$. Direct computation shows that the following equalities hold

$$\begin{aligned}
[D, K_\alpha] = {} & [x_\mu \partial_\mu + \varepsilon u \partial_u, 2g_{\alpha\beta} x_\beta (x_\mu \partial_\mu + \varepsilon u \partial_u) - g_{\mu\nu} x_\mu x_\nu \partial_\alpha + \\
& + F_{\alpha\beta}(u)\partial_\beta + G_\alpha(u)\partial_u] = 2g_{\alpha\beta} x_\beta (x_\mu \partial_\mu + \varepsilon u \partial_u) - \\
& - (g_{\mu\nu} x_\mu x_\nu)\partial_\alpha + (\varepsilon u F_{\alpha\beta u} - F_{\alpha\beta})\partial_\beta + \varepsilon(u G_{\alpha u} - G_\alpha)\partial_u.
\end{aligned}$$

Comparison of the right-hand sides of the above equalities yields the system of PDE

$$2F_{\alpha\beta} = \varepsilon u F_{\alpha\beta u}, \quad G_\alpha = \varepsilon(uG_{\alpha u} - G_\alpha), \quad \alpha,\beta = \overline{1, n+m}. \tag{16}$$

In the following, we will consider the cases $\varepsilon = 0$ and $\varepsilon = 1$ separately.

Case 1, $\varepsilon = 0$. Then it follows from (16) that $F_{\alpha\beta} = 0$, $G_\alpha = 0$, $\alpha,\beta = \overline{1, n+m}$, i.e. operators K_μ are given by (14) with $\varepsilon = 0$. It is not difficult to verify that the rest of commutational relations (6) also holds.

Case 2, $\varepsilon = 1$. Integrating the equations (16) we get

$$F_{\alpha\beta} = C_{\alpha\beta} u^2, \quad G_\alpha = C_\alpha u^2,$$

where $C_{\alpha\beta}$, C_α are arbitrary real constants.

Next, from the commutational relations for K_α, $J_{\mu\nu}$ it follows that

$$C_{\alpha\beta} = C\delta_{\alpha\beta}, \quad C_\alpha = 0,$$

where C is an arbitrary constant, $\alpha,\beta = \overline{1, n+m}$.

Thus, operators K_μ have the form

$$K_\mu = 2g_{\mu\nu}x_\nu D - (g_{\alpha\beta}x_\alpha x_\beta)\partial_\mu + Cu^2\partial_\mu. \tag{17}$$

Easy check shows that the operators (17) commute, whence it follows that all commutational relations of the conformal algebra hold.

If in (17) $C = 0$, then we have the case (14) with $\varepsilon = 1$. If $C \neq 0$, then after rescaling the dependent variable $u' = u|c|^{1/2}$ we obtain the operators (15). Theorem is proved.

Note 2. Nonlinear representations of the conformal algebra given by (13) with $\varepsilon = 1$ and (15) are realized on the set of solutions of the eikonal equations [3, 14]

$$g_{\mu\nu}u_{x_\mu}u_{x_\nu} \pm 1 = 0$$

and on the set of solutions of d'Alembert–eikonal system [15]

$$g_{\mu\nu}u_{x_\mu}u_{x_\nu} \pm 1 = 0, \quad g_{\mu\nu}u_{x_\mu x_\nu} \pm (n+m-1)u^{-1} = 0.$$

Thus, the Poincaré group $P(n,m)$ with $\max\{n,m\} \geq 3$ has no truly nonlinear representations. The only hope to obtain nonlinear representations of the Poincaré group is to study the case when $\max\{n,m\} < 3$.

4 Representations of the algebras $AP(n,m)$, $A\tilde{P}(n,m)$, $AC(n,m)$ with $\max\{n,m\} < 3$

Representations of algebras $AP(1,1)$, $A\tilde{P}(1,1)$, $AC(1,1)$ in the class of Lie vector fields were completely described by Rideau and Winternitz [9]. They have established, in particular, that the Poincaré algebra $AP(1,1)$ has no nonequivalent representations distinct from the standard one (9), while algebras $A\tilde{P}(1,1)$, $AC(1,1)$ admit nonlinear representations. In the paper [10] nonlinear representations of the Poincaré algebra $AP(1,2)$

$$\begin{aligned} &P_\mu = \partial_\mu, \quad J_{12} = x_1\partial_2 + x_2\partial_1 + \partial_u, \\ &J_{13} = x_1\partial_3 + x_3\partial_1 + \cos u\partial_u, \quad J_{23} = x_2\partial_3 - x_3\partial_2 - \sin u\partial_u, \end{aligned} \tag{18}$$

were constructed and besides that, it was proved that an arbitrary representation of the algebra $AP(1,2)$ in the class of Lie vector fields is equivalent either to the standard representation or to (18).

In the paper [11] we have constructed nonlinear representations of the algebras $AP(2,2)$ and $AC(2,2)$.

Theorem 4. *Arbitrary representation of the Poincaré algebra $AP(2,2)$ in the class of Lie vector fields is equivalent to the following representation:*

$$\begin{aligned} &P_\mu = \partial_\mu, \quad \mu = \overline{1,4}, \\ &J_{12} = x_1\partial_2 - x_2\partial_1 + \varepsilon\partial_u, \quad J_{13} = x_3\partial_1 + x_1\partial_3 + \varepsilon\cos u\partial_u, \\ &J_{14} = x_4\partial_1 + x_1\partial_4 \mp \varepsilon\sin u\partial_u, \quad J_{23} = x_3\partial_2 + x_2\partial_3 + \varepsilon\sin u\partial_u, \\ &J_{24} = x_4\partial_2 + x_2\partial_4 \pm \varepsilon\cos u\partial_u, \quad J_{34} = x_4\partial_3 - x_3\partial_4 \pm \varepsilon\partial_u, \end{aligned} \tag{19}$$

where $\varepsilon = 0, 1$.

Proof. When, proving the theorem 1, we have established that the operators P_μ, $J_{\alpha\beta}$ can be reduced to the form

$$P_\mu = \partial_\mu, \quad J_{\mu\nu} = g_{\mu\alpha}x_\alpha\partial_\nu - g_{\nu\alpha}x_\alpha\partial_\mu + F_{\mu\nu\alpha}(u)\partial_\alpha + G_{\mu\nu}(u)\partial_u, \tag{19a}$$

where $F_{\mu\nu\alpha} = -F_{\nu\mu\alpha}$, $G_{\mu\nu} = -G_{\nu\mu}$ are arbitrary smooth functions; $\mu, \nu, \alpha = \overline{1,4}$.

Consider the triplet of operators J_{12}, J_{13}, J_{23}. From commutational relations (4) we obtain the following system of nonlinear ordinary differential equations for functions G_{12}, G_{13}, G_{23}:

$$\begin{aligned} &G_{23} = G_{13}\dot G_{12} - G_{12}\dot G_{13}, \quad G_{13} = G_{12}\dot G_{23} - G_{23}\dot G_{12}, \\ &G_{12} = G_{13}\dot G_{23} - G_{23}\dot G_{13}, \end{aligned} \tag{20}$$

(a dot means differentiations with respect to u).

Multiplying the first equation of the system (20) by G_{23}, the second — by G_{13} and the third — by G_{12} and summing we get an equality

$$G_{12}^2 = G_{13}^2 + G_{23}^2. \tag{21}$$

In the following one has to consider cases $G_{12} \neq 0$ and $G_{12} = 0$ separately.

Case 1, $G_{12} \neq 0$. General solution of the algebraic equation (21) reads

$$G_{12} = f(u), \quad G_{13} = f(u)\cos g(u), \quad G_{23} = f(u)\sin g(u), \tag{22}$$

where $f(u)$, $g(u)$ are arbitrary smooth functions.

Substitution of (22) into (20) yields $\dot g f^2 = f$. Since $f(u) = g_{12} \neq 0$, the equality $\dot g = f^{-1}$ holds. Consequently, the general solution of the system (20) is of the form

$$G_{12} = \dot g^{-1}, \quad G_{13} = \dot g^{-1}\cos g, \quad G_{23} = \dot g^{-1}\sin g,$$

where $g = g(u)$ is an arbitrary smooth function.

On making the change of variables

$$x'_\alpha = x_\alpha, \quad \alpha = \overline{1,4}, \quad u' = g(u),$$

which does not alter the structure of operators P_μ, $J_{\mu\nu}$ (19a), we reduce operators J_{12}, J_{23}, J_{13} to the form

$$\begin{aligned} J_{12} &= x_1\partial_2 - x_2\partial_1 + \partial_u + \widetilde{F}_{12\alpha}(u)\partial_\alpha, \\ J_{23} &= x_3\partial_2 + x_2\partial_3 + (\sin u)\partial_u + \widetilde{F}_{23\alpha}(u)\partial_\alpha, \\ J_{13} &= x_3\partial_1 + x_1\partial_3 + (\cos u)\partial_u + \widetilde{F}_{13\alpha}(u)\partial_\alpha, \end{aligned} \tag{23}$$

where $\widetilde{F}_{12\alpha}$, $\widetilde{F}_{23\alpha}$, $\widetilde{F}_{13\alpha}$, $\alpha = \overline{1,4}$ are arbitrary smooth functions.

Substitution of (23) into (4) yields the system of linear ordinary differentional equations, which for general solution reads

$$\begin{aligned} &\widetilde{F}_{121} = \dot{V} + W, \quad \widetilde{F}_{122} = \dot{W} - V, \quad \widetilde{F}_{123} = \dot{Q}, \quad \widetilde{F}_{131} = \dot{V}\cos u - Q, \\ &\widetilde{F}_{132} = \dot{W}\cos u, \quad \widetilde{F}_{133} = \dot{Q}\cos u - V, \quad \widetilde{F}_{231} = \dot{V}\sin u, \quad \widetilde{F}_{232} = \dot{W}\sin u - Q, \\ &\widetilde{F}_{233} = \dot{Q}\sin u - W, \quad \widetilde{F}_{124} = R, \quad \widetilde{F}_{134} = R\cos u - C_1\sin u, \\ &\widetilde{F}_{234} = R\sin u + C_1\cos u. \end{aligned}$$

Here V, W, Q, R are arbitrary smooth functions on u, C_1 is an arbitrary constant.

The change of variables

$$\begin{aligned} &x'_1 = x_1 - V(u), \quad x'_2 = x_2 - W(u), \\ &x'_3 = x_3 - Q(u), \quad x'_4 = x_4 - \textstyle\int R(u)du, \quad u' = u \end{aligned}$$

reduce operators J_{12}, J_{23}, J_{13} to the form

$$\begin{aligned} J_{12} &= x_1\partial_2 - x_2\partial_1 + \partial_u, \\ J_{13} &= x_3\partial_1 + x_1\partial_3 - C_1\sin u\partial_u + \cos u\partial_u, \\ J_{23} &= x_3\partial_2 + x_2\partial_3 + C_1\sin u\partial_u + \sin u\partial_u, \end{aligned} \tag{24}$$

the rest of basis elements of the algebra $AP(2,2)$ having the form (19a).

Computing commutational relations (4) for operators J_{ab}; $\alpha, \beta = \overline{1,4}$ given by formulae (19a) with $\mu = \overline{1,3}$, $\nu = 4$ and (24) we obtain system of equations for unknown functions $F_{\mu_4\alpha}$, G_{μ_4}; $\alpha = \overline{1,4}$; $\mu = \overline{1,3}$. General solution of the system reads

$$\begin{aligned} &G_{14} = \mp\sin u, \quad G_{24} = \pm\cos u, \quad G_{34} = \pm 1, \quad C_1 = 0, \\ &F_{141} = F_{242} = F_{343} = C_2, \quad F_{\alpha 4\beta} = 0, \quad \alpha = \beta, \end{aligned}$$

where C_2 is an arbitrary constant.

Substituting the result obtain into the formulae (19a) and making the change of variables

$$x'_\alpha = x_\alpha, \quad \alpha = \overline{1,3}; \quad x'_4 = x_4 + C_2; \quad u' = u$$

we conclude that operators $J_{\alpha 4}$, $\alpha = \overline{1,3}$ are given by (19) with $\varepsilon = 1$.

Case 2, $\boldsymbol{G_{12} = 0}$. In this case from (21) it follows that $G_{12} = G_{13} = G_{23} = 0$. Computing commutators of operators J_{12}, J_{14} and J_{12}, J_{24} we get $G_{14} = G_{24}$. Next, computing commutator of operators J_{13}, J_{23} we came to conclusion that $G_{34} = 0$.

Substitution of operators $J_{\mu\nu}$ from (19a) with $G_{\mu\eta} = 0$, $\mu, \nu = \overline{1,4}$ into commutational relations (4) yields a homogenerous system of linear algebraic equations for functions $F_{\mu\nu\alpha}$. Its general solution can be represented in the form

$$F_{\mu\nu\alpha} = F_\mu(u) g_{\nu\alpha} - F_\nu(u) g_{\mu\alpha}, \quad \mu, \nu, \alpha = \overline{1,4},$$

where $F_\mu(u)$ are arbitrary smooth functions.

Consequently, operators (19a) take the form

$$P_\mu = \partial_\mu, \quad J_{\alpha\beta} = g_{\alpha\gamma}(x_\gamma + F_\gamma(u))\partial_\beta - g_{\beta\gamma}(x_\gamma + F_\gamma(u))\partial_\alpha.$$

Making in the above operators the change of variables $x'_\mu = x_\mu + F_\mu(u)$, $\mu = \overline{1,4}$, $u' = u$ we arrive at formulae (19) with $\varepsilon = 0$. Theorem is proved.

Theorem 5. *Arbitrary representations of the extended Poincaré algebra $A\widetilde{P}(2,2)$ in the class of Lie vector fields is equalent to one of the following representations:*

1) P_μ, $J_{\alpha\beta}$ *are of the form (19) with* $\varepsilon = 1$, $D = x_\mu \partial_\mu$;

2) P_μ, $J_{\alpha\beta}$ *are of the form (19) with* $\varepsilon = 0$, $D = x_\mu\partial_\mu + \varepsilon_1 u \partial_u$, $\varepsilon_1 = 0, 1$.

Theorem 6. *Arbitrary representation of the conformal algebra $AC(2,2)$ in thew class of Lie vector field is equivalent to one of the following representations:*

1) P_μ, $J_{\alpha\beta}$ *are of the form (19) with* $\varepsilon = 0$,

$$D = x_\alpha \partial_\alpha + \varepsilon_1 u \partial_u, \quad \varepsilon_1 = 0, 1,$$
$$K_\alpha = 2 g_{\alpha\beta} x_\beta D - (g_{\mu\nu} x_\mu x_\nu)\partial_\alpha;$$

2) P_μ, $J_{\alpha\beta}$ *are of the form (19) with* $\varepsilon = 0$,

$$D = x_\alpha \partial_\alpha + u \partial_u,$$
$$K_\alpha = 2 g_{\alpha\beta} x_\beta D - (g_{\mu\nu} x_\mu x_\nu \pm u^2)\partial_\alpha;$$

3) P_α, $J_{\mu\nu}$ *are of the form (19) with* $\varepsilon = 1$,

$$D = x_\alpha \partial_\alpha,$$
$$K_1 = 2x_1 D - (g_{\mu\nu} x_\mu x_\nu)\partial_1 + 2(x_2 + x_3 \cos u \mp x_4 \sin u)\partial_u,$$
$$K_2 = 2x_2 D - (g_{\mu\nu} x_\mu x_\nu)\partial_2 + 2(-x_1 + x_3 \sin u \pm x_4 \cos u)\partial_u,$$
$$K_3 = -2x_3 D - (g_{\mu\nu} x_\mu x_\nu)\partial_3 + 2(\pm x_4 + x_1 \cos u - x_2 \sin u)\partial_u,$$
$$K_4 = -2x_4 D - (g_{\mu\nu} x_\mu x_\nu)\partial_4 + 2(\mp x_4 \pm x_1 \sin u \mp x_2 \cos u)\partial_u,$$

where $\mu, \alpha, \beta, \nu = 1, 2, 3, 4$.

Proofs of the theorems 5 and 6 are similar to the proofs of the theorems 2, 3 that is why they are omitted.

In conclusion of the Section we adduce all nonequivalent representations of the extended Poincaré algebra $A\widetilde{P}(1,2)$ [10]

1) P_μ, $J_{\alpha\beta}$ are of the form (9),

$$D = x_\mu \partial_\mu + \varepsilon u \partial_u, \quad \varepsilon = 0, 1;$$

2) P_μ, $J_{\alpha\beta}$ are of the form (18),

$$D = x_\mu \partial_\mu$$

and the conformal algebra $AC(1,2)$ [10]

1) P_μ, $J_{\alpha\beta}$ are of the form (9),

$$D = x_\mu \partial_\mu + \varepsilon u \partial_u, \quad \varepsilon = 0, 1,$$
$$K_\alpha = 2g_{\alpha\beta} x_\beta D - (g_{\mu\nu} x_\mu x_\nu) \partial_\alpha;$$

2) P_μ, $J_{\alpha\beta}$ are of the form (9),

$$D = x_\mu \partial_\mu + u \partial_u,$$
$$K_\alpha = 2g_{\alpha\beta} x_\beta D - (g_{\mu\nu} x_\mu x_\nu \pm u^2) \partial_\alpha;$$

3) P_μ, $J_{\alpha\beta}$ are of the form (18),

$$D = x_\mu \partial_\mu,$$
$$K_1 = 2x_1 D - (g_{\mu\nu} x_\mu x_\nu) \partial_1 + 2(x_2 + x_3 \cos u) \partial_u,$$
$$K_2 = -2x_2 D - (g_{\mu\nu} x_\mu x_\nu) \partial_2 + 2(-x_1 + x_3 \sin u) \partial_u,$$
$$K_3 = -2x_3 D - (g_{\mu\nu} x_\mu x_\nu) \partial_3 - 2(x_1 \cos u + x_2 \sin u) \partial_u.$$

Here $\mu, \alpha, \beta, \nu = 1, 2, 3$.

5 Conclusion

Thus, we have obtained the complete description of nonequivalent representations of the generalized Poincaré group $P(n,m)$ by operators of the form (3). This fact makes a problem of constructing Poincaré-invariant equations of the form (1) purely algorithmic. To obtain all nonequivalent Poincaré-invariant equations on the order N, one has to construct complete sets of functionally-independent differential invariants of the order N for each nonequivalent representation [1, 2].

For example, each $P(n,m)$-invariant first-order PDE with $\max\{n,m\} \geq 3$ can be reduced by appropriate change of variables (2) to the eikonal equation

$$g_{\mu\nu} u_{x_\mu} u_{x_\nu} = F(u), \tag{25}$$

where $F(u)$ is an arbitrary smooth function.

Equation (26) with an arbitrary $F(u)$ is invariant under the algebra $AP(n,m)$ having the basis elements (9). Provided $F(u) = 0$, $n = m = 2$, it admits also the Poincaré algebra with the basis elements (19) [11].

Another interesting example is provided by $P(1,n)$-invariant PDE ($n \geq 3$). In [16] a complete basis of functionally-independent differential invariants of the order 2 of the algebra $AP(1,n)$ with the basis elements (9) has been constructed. Since each representation of the algebra $AP(1,n)$ with $n \geq 3$ is equivalent to (9), the above mentioned result gives the exhaustive description of Poincaré-invariant equations (1) in the Minkowski space $\mathbb{R}(1,n)$.

It would be of interest to apply the technique developed in [15] to construct PDE of the order higher than 1 which are invariant under the Poincaré algebra $AP(2,2)$ with the basis elements (19).

In the present papers we have studied representations of the Poincaré algebra in spaces with one dependent variable. But no less important is to investigate nonlinear representations of the Poincaré algebra in spaces with more number of dependent variables [17]. Linear representations of such a kind are realized on sets of solutions of the complex d'Alembert, of Maxwell, and of Dirac equations. If nonlinear representations in question would be obtained, one could construct principially new Poincaré-invariant mathematical models for describing real physical processes.

We intend to study the above mentioned problems in our future publications. Besides that, we will construct nonlinear representations of the Galilei group $G(1,n)$, which plays in Galilean relativistic quantum mechanics the same role as the Poincaré group in relativistic field theory.

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