# Conditional symmetry and anti-reduction of nonlinear heat equation

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Введено поняття $Q$-умовної інваріантності диференціальних рівнянь відносно векторних полів Лі–Беклунда. Це дозволило провести редукцію ряду нелінійних рівнянь теплопровідності до систем звичайних диференціальних рівнянь (нагадаємо, що при редукції з допомогою векторних полів Лі в результаті одержуємо одне редуковане рівняння).

The key idea making it possible to solve a linear heat equation

$$u_t = u_{xx}$$

by the method of separation of variables is reduction of it to two ordinary differential equations (ODE) with the help of the special ansatz (see, e.g. [1, 2])

$$u(t,x) = R(t,x)\varphi_1(\omega_1(t,x)\varphi_2(\omega_2(t,x)).$$

Unfortunately, the method of separation of variables can not be applied to nonlinear second-order partial differential equations. However, some progress is possible if we apply the anti-reduction procedure. The main idea is the same with the one of the method of separation of variables. Namely, we look for a solution of a nonlinear differential equation using the special ansatz reducing it to several equations which have a less number of independent variables [3]. With application to the nonlinear heat equation

$$u_t = [a(u)u_x]_x + F(u) \tag{1}$$

it means that its solution is searched for (1) in the form

$$G(t,x,u,\varphi_0(\omega),\ldots,\varphi_N(\omega)) = 0, \tag{2}$$

where $\omega = \omega(t,x)$ is a new independent variable, $\varphi_0(\omega), \varphi_1(\omega), \ldots, \varphi_N(\omega)$ are smooth functions satisfying some system of ODE.

The principal difficulty of the anti-reduction procedure is the proper choice of the ansatz (2). In the present paper we construct a number of ansatzes reducing nonlinear heat equations of the form (1) to ODE. These ansatzes are obtained by using $Q$-conditional invariance of the equation under study with respect to some Lie–Bäcklund vector field (the definition of $Q$-conditional invariance with respect to the Lie vector field was suggested in [4]).

**Definition.** *We say that Eq. (1) is Q-conditionally invariant under the Lie–Bäcklund vector field*

$$Q = \eta\frac{\partial}{\partial u} + (D_t\eta)\frac{\partial}{\partial u_t} + (D_x\eta)\frac{\partial}{\partial u_x} + \cdots \tag{3}$$

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*if there exist such a finite-order differential operator*

$$X = R_0 + R_1 D_t + R_2 D_x + R_3 D_t^2 + \cdots$$

*and the function $R$ that the equality*

$$Q(u_t - au_{xx} - \dot{a}u_x^2 - F) = X(u_t - au_{xx} - \dot{a}u_x^2 - F) + R\eta \tag{4}$$

*holds.*

In the above formulae (3), (4) $D_t$, $D_x$ are total differentiation operators and $\eta$, $R$, $R_0$, $R_1$, ..., are functions on $t$, $x$, $u$, $u_x$, $u_{xx}$, ....

Roughly speaking, Eq. (1) is $Q$-conditionally invariant with respect to the vector field (3) if the system

$$\begin{cases} Eq.\ (1), \\ \eta(t, x, u, u_x, u_{xx}, \ldots) = 0 \end{cases}$$

is invariant under the vector field (3) in a usual sense. That is why, to study $Q$-conditional invariance of Eq. (1) one can apply the standard infinitesimal algorithm [5]. But the system of determining equations for $\eta$ is nonlinear (let us remind that in the classical Lie approach determining equations are always linear).

We look for conditional symmetry operator (3) with

$$\eta = D_{xg}^2(u), \quad g(u) \in C^2(\mathbb{R}^1, \mathbb{R}^1). \tag{5}$$

**Lemma.** *Eq. (1) is $Q$-conditionally invariant with respect to the Lie–Bäcklund vector field (3), (5) if the functions $a(u)$, $F(u)$, $g(u)$ are given by one of the following formulae:*

$$1)\ a(u) = \ddot{\theta}(u)\theta(u), \quad F(u) = (\lambda_1 + \lambda_2\dot{\theta}(u))(\ddot{\theta}(u))^{-1}, \quad g(u) = \dot{\theta}(u); \tag{6a}$$

$$2)\ a(u) = u\dot{\theta}(u), \quad F(u) = (\lambda_1 + \lambda_2\theta(u))(\dot{\theta}(u))^{-1}, \quad g(u) = \dot{\theta}(u); \tag{6b}$$

$$3)\ a(u) = \dot{\theta}(u), \quad F(u) = (\lambda_1 + \lambda_2\theta(u))(\dot{\theta}(u))^{-1}, \quad g(u) = \dot{\theta}(u). \tag{6c}$$

*In (6) $\lambda_1$, $\lambda_2$ are arbitrary constants, $\theta \in C^3(\mathbb{R}^1, \mathbb{R}^1)$ is an arbitrary function.*

The proof of the lemma is rather tedious, therefore it is omitted. We restrict ourselves by proving that Eq. (1) with $a(u)$, $F(u)$ from (6a)

$$u_t = [\theta(u)\ddot{\theta}(u)u_x]_x + (\lambda_1 + \lambda_2\dot{\theta}(u))(\ddot{\theta}(u))^{-1} \tag{7}$$

is $Q$-conditionally invariant with respect to the Lie–Bäcklund vector field (3) under $\eta = \ddot{\theta}(u)u_{xx} + \dddot{\theta}(u)u_x^2$.

Consider an over-determined system

$$u_t = (\theta\ddot{\theta}u_x)_x + (\lambda_1 + \lambda_2\dot{\theta})\ddot{\theta}^{-1},$$
$$\eta \equiv \ddot{\theta}u_{xx} + \dddot{\theta}u^2x = 0.$$

Introducing a new independent variable $v = \dot{\theta}(u)$, we get

$$v_t = \theta(u)\ddot{\theta}(u)v_{xx} + vv_x^2 + \lambda_1 + \lambda_2 v,$$
$$v_{xx} = 0$$

or, equivalently,

$$\begin{aligned} &v_t = vv_x^2 + \lambda_1 + \lambda_2 v, \\ &v_{xx} = 0. \end{aligned} \tag{8}$$

The Lie–Bäcklund vector field $Q = (\ddot{\theta} u_{xx} + \dddot{\theta} u_x^2)\frac{\partial}{\partial u} + \cdots$ takes the form

$$\tilde{Q} = v_{xx}\frac{\partial}{\partial v} + v_{txx}\frac{\partial}{\partial v_t} + v_{xxx}\frac{\partial}{\partial v_x} + \cdots \tag{9}$$

Acting by the operator (9) on the first equation from (8) we get

$$\tilde{Q}(v_t - vv_x^2 - \lambda_1 - \lambda_2 v) = D_x^2(v_t - vv_x^2 - \lambda_1 - \lambda_2 v) + (4v_x^2 + 2vv_{xx})v_{xx}.$$

Hence, it follows that system (9) is $Q$-conditionally invariant under the Lie–Bäcklund vector field (9).

To construct solution invariant under the Lie–Bäcklund vector field (3), (5) one has to solve an equation $\eta \equiv D_x^2 g(u) = 0$. General solution of the above equation reads

$$g(u) = \varphi_0(t) + x\varphi_1(t), \tag{10}$$

where $\varphi_0$, $\varphi_1$ are arbitrary smooth functions. Replacing in (10) $g(u)$ by $\theta(u)$ we get ansatz for Eq. (7) invariant with respect to the Lie–Bäcklund vector field (3) with

$$\dot{\theta}(u) = \varphi_0(t) + x\varphi_1(t). \tag{11}$$

Substitution of (11) into Eq.(7) yields the system of two ODE for $\varphi_0(t)$, $\varphi_1(t)$

$$\dot{\varphi}_0 = (\lambda_2 + \varphi_1^2)\varphi_0 + \lambda_1, \quad \dot{\varphi}_1 = (\lambda_2 + \varphi_1^2)\varphi_1,$$

which general solution has the form

$$\begin{aligned} &\varphi_0 = -\frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_2}(e^{2\lambda_2 t} - 1)^{-1/2}\operatorname{arctg}(e^{-2\lambda_2 t} - 1)^{1/2}, \\ &\varphi_1 = \lambda_2^{1/2} e^{\lambda_2 t}(1 - e^{2\lambda_2 t})^{-1/2}. \end{aligned} \tag{12}$$

Substituting the obtained formulae into (11) we get the exact solution of the nonlinear heat equation (7). Since the maximal in Lie's sense invariance group of Eq. (7) is the two-parameter group of translations with respect to $t$, $x$, solution (11), (12) can not be obtained by the symmetry reduction procedure. Consequently, it is essentially new.

In the same way we construct $Q$-conditionally invariant ansatzes for Eqs. (5), (6b) and (5), (6c). They are of the form

$$\theta(u) = \varphi_0(t) + \varphi_1(t)x. \tag{13}$$

Substituting (13) into the corresponding nonlinear equations we get the following systems of ODE:

$$\dot{\varphi}_0 = \lambda_2\varphi_0 + \varphi_1^2 + \lambda_1, \quad \dot{\varphi}_1 = \lambda_2\varphi_1$$

and

$$\dot{\varphi}_0 = \lambda_2\varphi_0 + \lambda_1, \quad \dot{\varphi}_1 = \lambda_2\varphi_1.$$

Provided the functions $a(u)$, $F(u)$ take a more specific form, it is possible to construct ansatzes reducing Eq. (1) to three, four and even five ODE. Corresponding results are listed below

$$
\begin{aligned}
&1)\quad a(u)=\lambda u^k,\quad F(u)=\lambda_1 u+\lambda_2 u^{1-k},\quad u^k=\varphi_0(t)+\varphi_1(t)x+\varphi_2(t)x^2;\\
&2)\quad a(u)=\lambda e^u,\quad F(u)=\lambda_1+\lambda_2 e^{-u},\quad e^u=\varphi_0(t)+\varphi_1(t)x+\varphi_2(t)x^2;\\
&3)\quad a(u)=\lambda u^{-3/2},\quad F(u)=\lambda_1 u+\lambda_2 u^{5/2},\\
&\qquad u^{-3/2}=\varphi_0(t)+\varphi_1(t)x+\varphi_2(t)x^2+\varphi_3(t)x^3;\\
&4)\quad a(u)=\lambda u^{-4/3},\quad F(u)=\lambda_1 u+\lambda_2 u^{7/3},\\
&\qquad u^{-4/3}=\varphi_0(t)+\varphi_1(t)x+\varphi_2(t)x^2+\varphi_3(t)x^3+\varphi_4(t)x^4.
\end{aligned}
\tag{14}
$$

(the formulae 4 from the above list were obtained by Galaktionov [6]).

Here $\lambda$, $\lambda_1$, $\lambda_2$ are arbitrary real constants; $\varphi_0(t)$, $\varphi_1(t)$, $\ldots$, $\varphi_4(t)$ are arbitrary smooth functions.

It is interesting to note that the cases 1, 2, 4 exhaust all possible nonlinearities $a(u)$ such that invariance group of Eq. (1) is wider than two-parameter translation group [7].

Besides the above mentioned cases, we established that Eq. (1) with $a(u)=1$, $F(u)=\frac{2}{3}\left(C_3+C_2u-\frac{1}{3}C_1^2u^3\right)$, $C_i\in\mathbb{R}^1$ is $Q$-conditionally invariant under the Lie–Bäcklund vector field (3) with $\eta=u_{xx}-C_1uu_x-\frac{1}{3}\left(C_3+C_2u-\frac{1}{3}C_1^2u^3\right)$. This fact can also be used for antyreduction of the corresponding nonlinear heat equation.

In conclusion let us mention another important point. It is well-known that Eq. (1) admits the Lie–Bäcklund vector field only if it is equivalent to the linear heat equation or to the Burgers equation [8]. Consequently, the conception of conditional invariance widens essentially our possibilities to use a non-Lie symmetry to solve nonlinear partial differential equations.


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