Antireduction and exact solutions of nonlinear heat equations

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We construct a number of ansatzes that reduce one-dimensional nonlinear heat equations to systems of ordinary differential equations. Integrating these, we obtain new exact solution of nonlinear heat equations with various nonlinearities.

By the term antireduction for a partial differential equation (PDE) we mean the construction of an ansatz which transforms the PDE to a system of differential equations for several unknown differentiable functions. As a rule, such procedure reduces the PDE under consideration to a system of PDE with fewer numbers of independent variables and greater number of dependent variables [1–4].

Antireduction of the nonlinear acoustics equation

$$u_{x_0x_1} - (u_{x_1}u)_{x_1} - u_{x_2x_2} - u_{x_3x_3} = 0 \tag{1}$$

is carried out in the paper [2] with the use of the ansatz

$$u = \frac{1}{3}x_1\varphi_1(x_0, x_2, x_3) - \frac{1}{6}x_1^2\varphi_2(x_0, x_2, x_3) + \varphi_3(x_0, x_2, x_3). \tag{2}$$

In [3] antireduction of the equation for short waves in gas dynamics

$$2u_{x_0x_1} - 2(2x_1 + u_{x_1})u_{x_1x_1} + u_{x_2x_2} + 2\lambda u_{x_1} = 0 \tag{3}$$

is carried out via the following ansatz:

$$u = x_1\varphi_1 + x_1^2\varphi_2 + x_1^{3/2}\varphi_3 + \varphi_4, \qquad \varphi_i = \varphi_i(x_0, x_2). \tag{4}$$

Ansatzes (2), (4) reduce equations (1), (3) to system of PDE for three and four functions, respectively.

In the present paper we adduce some new results on antireduction for the nonlinear heat equations of the form

$$u_t = \big(a(u)u_x\big)_x + F(u). \tag{5}$$

The antireduction of equation (5) is performed by means of the ansatz

$$h\big(t, x, u, \varphi_1(\omega), \varphi_2(\omega),\ \ldots,\ \varphi_N(\omega)\big) = 0 \tag{6}$$

where $\omega = \omega(t, x, u)$ is a new independent variable. Ansatz (6) reduces equation (5) to a system of ordinary differential equations (ODE) for the unknown functions $\varphi_i(\omega)$, $i = \overline{1, N}$.

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Below we list, without derivation, explicit forms of $a(u)$ and $F(u)$, such that equation (5) admits an antireduction of the form (6). For each case the reduced ODE are given.

1. $a(u) = \ddot\theta(u)\theta(u), \quad F(u) = \big(\lambda_1 + \lambda_2\dot\theta(u)\big)\big(\ddot\theta(u)\big)^{-1},$
   $\dot\theta(u) = \varphi_1(t) + \varphi_2(t)x, \quad \dot\varphi_1 = (\lambda_2 + \varphi_2^2)\varphi_1 + \lambda_1, \quad \dot\varphi_2 = (\lambda_2 + \varphi_2^2)\varphi_2;$
2. $a(u) = u\dot\theta(u), \quad F(u) = \big(\lambda_1 + \lambda_2\theta(u)\big)\big(\dot\theta(u)\big)^{-1},$
   $\theta(u) = \varphi_1(t) + \varphi_2(t)x, \quad \dot\varphi_1 = \lambda_2\varphi_1 + \varphi_2^2 + \lambda_1, \quad \dot\varphi_2 = \lambda_2\varphi_2;$
3. $a(u) = \dot\theta(u), \quad F(u) = \big(\lambda_1 + \lambda_2\theta(u)\big)\big(\dot\theta(u)\big)^{-1},$
   $\theta(u) = \varphi_1(t) + \varphi_2(t)x, \quad \dot\varphi_1 = \lambda_2\varphi_1 + \lambda_1, \quad \dot\varphi_2 = \lambda_2\varphi_2;$
4. $a(u) = \lambda u^k, \quad F(u) = \lambda_1 u + \lambda_2 u^{1-k}, \quad u^k = \varphi_1(t) + \varphi_2(t)x + \varphi_3(t)x^2,$
   $\dot\varphi_1 = 2\lambda\varphi_1\varphi_3 + \lambda k^{-1}\varphi_2^2 + k\lambda_2, \quad \dot\varphi_2 = 2\lambda(1 + 2k^{-1})\varphi_2\varphi_3 + k\lambda_1\varphi_2,$
   $\dot\varphi_3 = 2\lambda(1 + 2k^{-1})\varphi_3^2 + k\lambda_1\varphi_3;$
5. $a(u) = \lambda e^u, \quad F(u) = \lambda_1 + \lambda_2 e^{-u}, \quad e^u = \varphi_1(t) + \varphi_2(t)x + \varphi_3(t)x^2,$
   $\dot\varphi_1 = 2\lambda\varphi_1\varphi_3 + \lambda_1\varphi_1 + \lambda_2, \quad \dot\varphi_2 = 2\lambda\varphi_2\varphi_3 + \lambda_1\varphi_2, \quad \dot\varphi_3 = 2\lambda\varphi_3^2 + \lambda_1\varphi_3;$
6. $a(u) = \lambda u^{-3/2}, \quad F(u) = \lambda_1 u + \lambda_2 u^{5/2},$
   $u^{-3/2} = \varphi_1(t) + \varphi_2(t)x + \varphi_3(t)x^2 + \varphi_4(t)x^3,$
   $\dot\varphi_1 = 2\lambda\varphi_1\varphi_3 - \frac{2}{3}\lambda\varphi_2^2 - \frac{3}{2}\lambda_1\varphi_1 - \frac{3}{2}\lambda_2,$
   $\dot\varphi_2 = -\frac{2}{3}\lambda\varphi_2\varphi_3 + 6\lambda\varphi_1\varphi_4 - \frac{3}{2}\lambda_1\varphi_2,$
   $\dot\varphi_3 = -\frac{2}{3}\lambda\varphi_3^2 + 2\lambda\varphi_2\varphi_4 - \frac{3}{2}\lambda_1\varphi_3, \quad \dot\varphi_4 = -\frac{3}{2}\lambda_1\varphi_4;$
7. $a(u) = 1, \quad F(u) = (\alpha + \beta\ln u)u, \quad \ln u = \varphi_1(t) + \varphi_2(t)x,$
   $\dot\varphi_1 = \beta\varphi_1 + \varphi_2^2 + \alpha, \quad \dot\varphi_2 = \alpha\varphi_2;$
8. $a(u) = 1, \quad F(u) = \big(\alpha + \beta\ln u - \gamma^2(\ln u)^2\big)u, \quad \ln u = \varphi_1(t) + \varphi_2(t)e^{\gamma x},$
   $\dot\varphi_1 = \alpha + \beta\varphi_1 - \gamma^2\varphi_1^2, \quad \dot\varphi_2 = (\beta + \gamma^2 - 2\gamma^2\varphi_1)\varphi_2;$
9. $a(u) = 1, \quad F(u) = -u(1 + \ln u^2)\big(\alpha + \beta(\ln u)^{-1/2}\big),$
   $\displaystyle\int^{\ln\,u}\Big(2\alpha\tau + 4\beta\tau^{1/2} + \varphi_2(t)\Big)^{-1/2}d\tau = x + \varphi_1(t),$
   $\dot\varphi_1 = 0, \quad \dot\varphi_2 = 4\beta^2 - 2\alpha\varphi_2;$
10. $a(u) = 1, \quad F(u) = -2(u^3 + \alpha u^2 + \beta u),$
    (a) $\alpha = \beta = 0$
    $u = \big(\varphi_1(t) + 2\varphi_2(t)x\big)\big(1 + \varphi_1(t)x + \varphi_2(t)x^2\big)^{-1},$
    $\dot\varphi_1 = -6\varphi_1\varphi_2, \quad \dot\varphi_2 = -6\varphi_2^2;$
    (b) $\alpha^2 = 4\beta \neq 0$
    $u = \left(-\frac{\alpha}{2}\varphi_1(t) + \left(1 - \frac{\alpha}{2}x\right)\varphi_2(t)\right)\left(e^{\alpha x/2} + \varphi_1(t) + \varphi_2(t)x\right)^{-1},$
    $\dot\varphi_1 = -\frac{\alpha^2}{4}\varphi_1 - \alpha\varphi_2, \quad \dot\varphi_2 = -\frac{\alpha^2}{4}\varphi_2;$
    (c) $\alpha^2 > 4\beta$
    $u = \big((A + B)\varphi_1(t)e^{Bx} + (A - B)\varphi_2(t)e^{-Bx}\big)\times$

$$\times \left(e^{-Ax} + \varphi_1(t)e^{Bx} + \varphi_2(t)e^{-Bx}\right)^{-1},$$
$$A = -\frac{\alpha}{2}, \quad B = \frac{1}{2}(\alpha^2 - 4\beta)^{1/2},$$
$$\dot{\varphi}_1 = \left(\frac{\alpha^2}{2} - 3\beta - \frac{\alpha}{2}(\alpha^2 - 4\beta)^{1/2}\right)\varphi_1,$$
$$\dot{\varphi}_2 = \left(\frac{\alpha^2}{2} - 3\beta + \frac{\alpha}{2}(\alpha^2 - 4\beta)^{1/2}\right)\varphi_2;$$

(d) $\alpha^2 < 4\beta$

$$u = \big(\varphi_1(t)(A\cos Bx - B\sin Bx) + \varphi_2(t)(A\sin Bx +$$
$$+ B\cos Bx)\big)\left(e^{-Ax} + \varphi_1(t)\cos Bx + \varphi_2(t)\sin Bx\right)^{-1},$$
$$\dot{\varphi}_1 = \left(\frac{\alpha^2}{2} - 3\beta\right)\varphi_1 - \frac{\alpha}{2}(4\beta - \alpha^2)^{1/2}\varphi_2,$$
$$\dot{\varphi}_2 = \left(\frac{\alpha^2}{2} - 3\beta\right)\varphi_2 + \frac{\alpha}{2}(4\beta - \alpha^2)^{1/2}\varphi_1.$$

In the above formulae $\theta = \theta(u) \in C^2(\mathbb{R}^1, \mathbb{R}^1)$ is an arbitrary function; λ, λ_1, λ_2, α, β, γ are arbitrary real constants; overdot means differentiation with respect to the corresponding argument.

Most of above adduced system of ODE can be integrated. As a result, one obtains a number of new exact solutions of the nonlinear heat equation (5). Detailed study of reduced systems of ODE and construction of exact solutions of equation (5) will be a topic of our future paper. Here we present some exact solutions of the nonlinear heat equation

$$u_t = u_{xx} + F(u)$$

obtained with the help of ansatzes 7–10 which are listed above.

1) $F(u) = \big(\alpha + \beta\ln u - \gamma^2(\ln u)^2\big)u$,

(a) $\Delta = \beta^2 + 4\alpha\gamma^2 > 0$

$$u = C\left(\cos\frac{\Delta^{1/2}t}{2}\right)^{-2} e^{\gamma x + \gamma^2 t} + \frac{1}{2\gamma^2}\left(\beta - \Delta^{1/2}\mathrm{tg}\,\frac{\Delta^{1/2}t}{2}\right);$$

(b) $\Delta = -\beta^2 - 4\alpha\gamma^2 > 0$

$$u = C\left(\mathrm{ch}\,\frac{\Delta^{1/2}t}{2}\right)^{-2} e^{\gamma x + \gamma^2 t} + \frac{1}{2\gamma^2}\left(\beta + \Delta^{1/2}\mathrm{th}\,\frac{\Delta^{1/2}t}{2}\right);$$

(c) $\Delta = \beta^2 + 4\alpha\gamma^2 = 0$

$$u = Ct^{-2}e^{\gamma x + \gamma^2 t} + \frac{1}{2\gamma^2 t}(\beta t + 2);$$

2) $F(u) = -u(1 + \ln\, u^2)\big(\alpha + \beta(\ln\, u)^{-1/2}\big)$,

(a) $\alpha \neq 0$

$$\int^{\ln\, u}\left(2\alpha\tau + 4\beta\tau^{1/2} + Ce^{-2\alpha t} + 2\beta^2\alpha^{-1}\right)^{-1/2} d\tau = x;$$

(b) $\alpha = 0$

$$\int^{\ln u}\left(4\beta\tau^{1/2} + 4\beta^2 t\right)^{-1/2} d\tau = x;$$

3) $F(u) = -2u(u^2 + \alpha u + \beta)$,

(a) $\alpha^2 = 4\beta$

$$u = \left(1 - \frac{\alpha}{2}(x - \alpha t)\right)\left(x - \alpha t + C\exp\left(\frac{\alpha}{2}\Big(x + \frac{\alpha t}{2}\Big)\right)\right)^{-1};$$

(b) $\alpha^2 > 4\beta$

$$u = \Big((A+B)C_1\exp\big((A+B)(x-\alpha t)\big) + (A-B)C_2 \times$$
$$\times \exp\big((A-B)(x-\alpha t)\big)\Big)\Big(\exp(3\beta t) + C_1\exp\big((A+B)(x-\alpha t)\big) +$$
$$+ C_2\exp\big((A-B)(x-\alpha t)\big)\Big)^{-1},$$
$$A = -\frac{\alpha}{2}, \quad B = \frac{1}{2}(\alpha^2 - 4\beta)^{1/2};$$

(c) $\alpha^2 < 4\beta$

$$u = \big((\alpha A C_1 - BC_2)\cos B(x-\alpha t) + (AC_2 + BC_1) \times$$
$$\times \sin B(x-\alpha t)\big)\Big(\exp\big(3\beta t - A(x-\alpha t)\big) +$$
$$+ C_1\cos B(x-\alpha t) + C_2\sin B(x-\alpha t)\Big)^{-1},$$
$$A = -\frac{\alpha}{2}, \quad B = \frac{1}{2}(4\beta - \alpha^2)^{1/2}.$$

In the above formulae C, C_1, C_2 are arbitrary constants.

It is worth noting that the above solutions can not be obtained with the use of the classical Lie symmetry reduction technique [6]. That is why they are essentially new. Another impotant feature is that solutions 3(a) and 3(c) are soliton-like solutions. Consequently, nonlinear heat equation with cubic nonlinearity admits soliton-like solutions.

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