Ansätze of codimension one for the Navier–Stokes field and reduction of the Navier–Stokes equation

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Використовуючи максимальну в сенсі Лі (нескінченновимірну) алгебру інваріантності рівнянь Нав'є–Стокса, побудований повний набір нееквівалентних лієвських анзаців корозмірності один для поля Нав'є–Стокса. З їх допомогою проведено редукцію рівнянь Нав'є–Стокса до систем ДРЧП з трьома незалежними змінними. Вивчені симетрійні властивості редукованих систем.

Finding exact solutions of the Navier–Stokes equations (NSEs) for an incompressible viscous fluid is an actual problem of mathematical physics and hydrodynamics. There are some ways to solve this problem. One of them is a usage of symmetry analysis [1–8]. In this article we construct a complete set of inequivalent ansatze of codimension one for the Navier–Stokes field. Using them, we reduce the NSEs to systems of partial differential equations in three independent variables and study their symmetry properties.

It is known that the NSEs

$$\vec{u}_t + (\vec{u}\cdot\vec{\nabla})\vec{u} - \Delta\vec{u} + \vec{\nabla}p = \vec{0}, \quad \operatorname{div}\vec{u} = 0 \tag{1}$$

are invariant under the infinite dimensional algebra A(NS) with basic elements

$$\begin{aligned}
&\partial_t = \partial/\partial t, \quad D = 2t\partial_t + x_a\partial_a - u^a\partial_{u^a} - 2p\partial_p,\\
&J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a}, \quad a\neq b,\\
&R(\vec{m}(t)) = m^a(t)\partial_a + m^a_t(t)\partial_{u^a} - m^a_{tt}(t)x_a\partial_p, \quad Z(\chi(t)) = \chi(t)\partial_p.
\end{aligned} \tag{2}$$

Here and from now on $\vec{u} = \vec{u}(t,\vec{x}) = \{u^a\}$ is the velocity field of a fluid, $p = p(t,\vec{x})$ is the pressure, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial_t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\Delta = \vec{\nabla}\cdot\vec{\nabla}$, $m^a = m^a(t)$, $\chi = \chi(t)$ are arbitrary smooth functions of t (for example, from $C^\infty((t_0,t_1),\mathbb{R})$), $a,b = \overline{1,3}$, $i,j = 1,2$, repetition of an index signifies a sum.

The set of operators (2) determines the maximal, in the sense of Lie, invariance algebra of the NSEs [9–11].

Theorem 1. *A complete set of A(NS)-inequivalent one-dimensional subalgebras of A(NS) is exhausted by such algebras:*

1) $A^1_1(\varkappa) = \langle D + 2\varkappa J_{12}\rangle, \quad \varkappa \geq 0;$
2) $A^1_2(\varkappa) = \langle \partial_t + \varkappa J_{12}\rangle, \quad \varkappa \in \{0;1\};$
3) $A^1_3(\eta,\chi) = \langle J_{12} + R(0,0,\eta(t)) + Z(\chi(t))\rangle,$

where algebras $A^1_3(\eta,\chi)$ and $A^1_3(\tilde\eta,\tilde\chi)$ are equivalent if $\exists\,\varepsilon,\delta\in\mathbb{R}$, $\exists\,\lambda\in C^\infty((t_0,t_1),\mathbb{R})$:

$$(\tilde\eta,\tilde\chi)(t) = (e^{-\varepsilon}\eta, e^{2\varepsilon}(\chi + \ddot\lambda\eta - \ddot\eta\lambda))(te^{2\varepsilon}+\delta); \tag{3}$$

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4) $A_4^1(\vec{m},\chi) = \langle R(\vec{m}) + Z(\chi)\rangle, \quad (\vec{m},\chi) \neq (\vec{0},0),$

where algebras $A_4^1(\vec{m},\chi)$ *and* $A_4^1(\tilde{\vec{m}},\tilde{\chi})$ *are equivalent if* $\exists\,\varepsilon,\delta \in \mathbb{R}$, $\exists\, c \neq 0$, $\exists\, B \in O(3)$, $\exists\,\vec{l} \in C^\infty((t_0,t_1),\mathbb{R}^3)$*:*

$$(\tilde{\vec{m}},\chi)(t) = (ce^{-\varepsilon}B\vec{m}, \quad ce^{2\varepsilon}(\chi + \dot{\vec{l}}\cdot\vec{m} - \dot{\vec{m}}\cdot\vec{l}))(te^{2\varepsilon}+\delta). \tag{4}$$

Theorem 1 is proved by the method described in [12, 13].

With the algebras A_1^1–A_3^1 from theorem 1 and with the algebra A_4^1 (if some additional demands are satisfied) one can construct such a set of inequivalent ansätze of codimension one for the Navier–Stokes field:

$$\begin{aligned}
1.\quad & u^1 = |t|^{-1/2}(v^1\cos\tau - v^2\sin\tau) + \tfrac12 x_1 t^{-1} - \varkappa x_2 t^{-1},\\
& u^2 = |t|^{-1/2}(v^1\sin\tau + v^2\cos\tau) + \tfrac12 x_2 t^{-1} + \varkappa x_1 t^{-1},\\
& u^3 = |t|^{-1/2}v^3 + \tfrac12 x_3 t^{-1},\\
& p = t^{-1}q + \tfrac18 t^{-2}x_ax_a + \tfrac12\varkappa^2 t^{-2}r^2,
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
&\tau = \varkappa\ln|t|, \quad r = (x_1^2+x_2^2)^{1/2}, \quad y_1 = |t|^{-1/2}(x_1\cos\tau + x_2\sin\tau),\\
&y_2 = |t|^{-1/2}(-x_1\sin\tau + x_2\cos\tau), \quad y_3 = |t|^{-1/2}x_3;
\end{aligned}$$

here and from now on $v^a = v^a(y_1,y_2,y_3)$, $q = q(y_1,y_2,y_3)$, numeration of ansätze corresponds to that of algebras in theorem 1.

$$\begin{aligned}
2.\quad & u^1 = v^1\cos\varkappa t - v^2\sin\varkappa t - \varkappa x_2,\\
& u^2 = v^1\sin\varkappa t + v^2\cos\varkappa t + \varkappa x_1,\\
& u^3 = v^3, \quad p = q + \tfrac12\varkappa^2 r^2,
\end{aligned} \tag{6}$$

where $y_1 = x_1\cos\varkappa t + x_2\sin\varkappa t$, $y_2 = -x_1\sin\varkappa t + x_2\cos\varkappa t$, $y_3 = x_3$.

$$\begin{aligned}
3.\quad & u^1 = x_1r^{-1}v^1 - x_2r^{-1}v^2 + x_1r^{-2},\\
& u^2 = x_2r^{-1}v^1 + x_1r^{-1}v^2 + x_2r^{-2},\\
& u^3 = v^3 + \eta(t)r^{-1}v^2 + \dot\eta(t)\,\mathrm{arctg}\,x_2/x_1,\\
& p = q - \tfrac12\ddot\eta(t)(\eta(t))^{-1}x_3^2 - \tfrac12 r^{-2} + \chi(t)\,\mathrm{arctg}\,x_2/x_1,
\end{aligned} \tag{7}$$

where $y_1 = t$, $y_2 = r$, $y_3 = x_3 - \eta(t)\,\mathrm{arctg}\,x_2/x_1$.

Remark 1. The expression for the pressure p from the ansatz (7) is indeterminate in points $t \in \{t_0,t_1\}$, where $\eta(t) = 0$. If there are such points t, we will consider the ansatz (7) in intervals (t_0^n,t_1^n) that are contained by the interval (t_0,t_1) and for which one from the conditions

a) $\forall\, t \in (t_0^n,t_1^n):\ \eta(t) \neq 0;$

b) $\eta(t) \equiv 0$ in (t_0^n,t_1^n)

is satisfied. In the last case we consider that $\ddot\eta/\eta := 0$.

4. With the algebra $A_4^1(\vec{m},\chi)$, an ansatz can be constructed only for such a t wherefor $\vec{m}(t) \neq \vec{0}$. If this condition is satisfied, it follows from (2) that the algebra

$A_4^1(\vec{m},\chi)$ is equivalent to the algebra $A_4^1(\vec{m},0)$. An ansatz constructed with the algebra $A_4^1(\vec{m},0)$ is

$$\begin{aligned}
&\vec{u} = v^i\vec{n}^i + (\vec{m}\cdot\vec{m})^{-1}v^3\vec{m} + (\vec{m}\cdot\vec{x})(\vec{m}\cdot\vec{m})^{-1}\dot{\vec{m}} - y_i\dot{\vec{n}}^i,\\
&p = q - \tfrac{3}{2}(\vec{m}\cdot\vec{m})^{-1}((\dot{\vec{m}}\cdot\vec{n}^i)y_i)^2 - (\vec{m}\cdot\vec{m})^{-1}(\ddot{\vec{m}}\cdot\vec{x})(\vec{m}\cdot\vec{x}) +\\
&\qquad + (\ddot{\vec{m}}\cdot\vec{m})(\vec{m}\cdot\vec{m})^{-2}(\vec{m}\cdot\vec{x})^2,
\end{aligned} \tag{8}$$

where $y_i = \vec{n}^i\cdot\vec{x}$, $y_3 = t$,

$$\vec{n}^i = \vec{n}^i(t), \quad \vec{n}^i\cdot\vec{m} = \vec{n}^1\cdot\vec{n}^2 = 0, \quad |\vec{n}^i| = 1, \quad \dot{\vec{n}}^1\cdot\vec{n}^2 = 0. \tag{9}$$

Remark 2. Vector-functions $\vec{n}^i$ satisfying conditions (9) exist. They can be constructed in such a way: let us fix vector-functions $k^i = k^i(t)$ for which $\vec{k}^i\cdot\vec{m} = \vec{k}^1\cdot\vec{k}^2 = 0, |\vec{k}^i| = 1$ and set

$$\vec{n}^1 = \vec{k}^1\cos\psi(t) - \vec{k}^2\sin\psi(t), \quad \vec{n}^2 = \vec{k}^1\sin\psi(t) + \vec{k}^2\cos\psi(t). \tag{10}$$

Then $\dot{\vec{n}}^1\cdot\vec{n}^2 = \dot{\vec{k}}^1\cdot\vec{k}^2 - \dot{\psi} = 0$ if $\int(\dot{\vec{k}}^1\cdot\vec{k}^2)dt$.

Substituting the ansätze (5), (6) to the NSEs (1), we obtain reduced systems of PDEs that have the same general form

$$\begin{aligned}
&v^a v^1_a - v^1_{aa} + q_1 + \gamma_1 v^2 = 0,\\
&v^a v^2_a - v^2_{aa} + q_2 - \gamma_1 v^1 = 0,\\
&v^a v^3_a - v^3_{aa} + q_3 = 0,\\
&v^a_a = \gamma_2,
\end{aligned} \tag{11}$$

where the constant γ_i, takes the values

1. $\gamma_1 = -2\varkappa, \quad \gamma_2 = -\frac{3}{2}, \quad \text{if} \quad t>0, \quad \gamma_1 = 2\varkappa, \quad \gamma_2 = \frac{3}{2}, \quad \text{if} \quad t<0.$
2. $\gamma_1 = -2\varkappa, \quad \gamma_2 = 0.$

For the ansätze (7), (8) reduced equations have the form

3. $$\begin{aligned}
&v^1_1 + v^1v^1_2 + v^3v^1_3 - y_2^{-1}v^2v^2 - [v^1_{22} + (1+\eta^2y_2^{-2})v^1_{33} + 2\eta y_2^{-2}v^2_3] + q_2 = 0,\\
&v^2_1 + v^1v^2_2 + v^3v^2_3 + y_2^{-1}v^1v^2 - [v^2_{22} + (1+\eta^2y_2^{-2})v^2_{33} - 2\eta y_2^{-2}v^1_3] +\\
&\qquad + 2y_2^{-2}v^2 - \eta y_2^{-1}q_3 + \chi y_2^{-1} = 0,\\
&v^3_1 + v^1v^3_2 + v^2v^3_3 - [v^3_{22} + (1+\eta^2y_2^{-2})v^3_{33}] - 2\eta^2y_2^{-3}v^1_3 + 2\dot{\eta}y_2^{-1}v^2 +\\
&\qquad + 2\eta y_2^{-1}(y_2^{-1}v^2)_2 + (1+\eta^2y_2^{-2})q_3 - \ddot{\eta}\eta^{-1}y_3 - \chi\eta y_2^{-2} = 0,\\
&y_2^{-1}v^1 + v^1_2 + v^3_3 = 0.
\end{aligned} \tag{12}$$

4. $$\begin{aligned}
&v^i_3 + v^iv^i_j - v^i_{jj} + q_i + \rho^i(y_3)v^3 = 0,\\
&v^3_3 + v^jv^3_j - v^3_{jj} = 0,\\
&v^i_j + \rho^3(y_3) = 0,
\end{aligned} \tag{13}$$

where

$$\rho^i = \rho^i(y_3) = 2(\vec{m}\cdot\vec{m})^{-1}(\dot{\vec{m}}\cdot\vec{n}^i), \quad \rho^3 = \rho^3(\dot{y}_3) = (\vec{m}\cdot\vec{m})^{-1}(\dot{\vec{m}}\cdot\vec{m}). \tag{14}$$

Let us study symmetry properties of the systems (11)–(13). All following results are obtained with the standard Lie algorithm [11, 12]. At first consider the sustem (11).

Theorem 2. *The maximal, in the sense of Lie, invariance algebra of (11) is the algebra*

$$
\begin{aligned}
&a)\quad \langle \partial_a, \partial_q, J^1_{12}\rangle \quad \textit{if} \quad \gamma_1 \neq 0;\\
&b)\quad \langle \partial_a, \partial_q, J^1_{ab}\rangle \quad \textit{if} \quad \gamma_1 = 0, \quad \gamma_2 \neq 0;\\
&c)\quad \langle \partial_a, \partial_q, J^1_{ab}, D^1\rangle \quad \textit{if} \quad \gamma_1 = \gamma_2 = 0.
\end{aligned}
$$

Here

$$J^1_{ab} = y_a\partial_b - y_b\partial_a + v^a\partial_{v^b} - v^b\partial_{v^a}, \quad D^1 = y_a\partial_a - v^a\partial_{v^a} - 2q\partial_q.$$

All Lie symmetry operators of (11) are induced by operators from A(NS). Namely, the operators J^1_{ab}, D^1 are induced by J_{ab}, D and the operators $c_a\partial_a$ ($c_a = \mathrm{const}$), ∂_q is done by

$$R(|t|^{1/2}(c_1\cos\tau - c_2\sin\tau, c_1\sin\tau + c_2\cos\tau, c_3)), \quad Z(|t|^{-1})$$

for the anzatz (5) and by

$$R(c_1\cos\varkappa t - c_2\sin\varkappa t, c_1\sin\varkappa t + c_2\cos\varkappa t, c_3), \quad Z(1)$$

for the anzatz (6) respectively. Therefore, Lie reduction of the system (11) gives only solutions that can be obtained by reducing the NSEs with two- and three-dimensional subalgebras of A(NS). Let us proceed to the system (12). Let $A^{\max}$ be the maximal, in the sense of Lie, invariance algebra of (12). Studying symmetry properties of (12), one has to consider the following cases.

A. $\eta, \chi \equiv 0$. Then

$$A^{\max} = \langle \partial_1, D^1_2, R_1(\psi(y_1)), Z^1(\lambda(y_1))\rangle,$$

where $D^1_2 = 2y_1\partial_1 + y_2\partial_2 + y_3\partial_3 - v^a\partial_{v^a} - 2q\partial_q$, $Z^1(\lambda(y_1)) = \lambda(y_1)\partial_q$, $R_1(\psi(y_1)) = \psi\partial_3 + \psi_1\partial_{v^2} - \psi_{11}y_3\partial_q$; here and from now on $\psi = \psi(y_1)$, $\lambda = \lambda(y_1)$ are arbitrary smooth functions of $y_1 = t$.

B. $\eta \equiv 0$, $\chi \neq 0$. In this case expansion of $A^{\max}$ is for $\chi = (C_1y_1 + C_2)^{-1}$, where $C_1, C_2 = \mathrm{const}$. Let $C_1 \neq 0$. It can be done with the equivalence transformation (3) so that the constant C_2 will vanish, i.e. $\chi = Cy^{-1}$ where $C = \mathrm{const}$. Then

$$A^{\max} = \langle D^1_2, R_1(\psi(y_1)), Z^1(\lambda(y_1))\rangle.$$

If $C_1 = 0$, $\chi = C = \mathrm{const}$ and

$$A^{\max} = \langle \partial_1, R_1(\psi(y_1)), Z^1(\lambda(y_1))\rangle.$$

For other values of χ, i.e. when $\chi_{11}\chi \neq \chi_1\chi_1$,

$$A^{\max} = \langle R_1(\psi(y_1)), Z^1(\lambda(y_1))\rangle.$$

C. $\eta \neq 0$. With the equivalence transformation (3), we do $\chi = 0$. In this case expansion of $A^{\max}$ is for $\eta = \pm|C_1y_1 + C_2|^{1/2}$, where $C_1, C_2 = \mathrm{const}$. Let $C_1 \neq 0$.

It can be done with the equivalence transformation (3) so that the constant C_2 will vanish, i.e. $\eta = C|y_1|^{1/2}$, where $C = \mathrm{const}$. Then

$$A^{\max} = \langle D_2^1, Z^1(\lambda(y_1)), R_2(|y_1|^{1/2}), R_2(|y_1|^{1/2}\ln|y_1|)\rangle,$$

where $R_2(\psi(y_1)) = \psi\partial_3 + \psi_1\partial_{v^3}$. If $C_1 = 0$, i.e. $\eta = C = \mathrm{const}$;

$$A^{\max} = \langle \partial_1, Z^1(\lambda(y_1)), \partial_3, y_1\partial_3 + \partial_{v^3}\rangle.$$

For other values of η, i.e. when $(\eta^2)_{11} \neq 0$,

$$A^{\max} = \langle Z^1(\lambda(y_1)), R_2(\eta(y_1)), R_2\left(\eta(y_1)\int(\eta(y_1))^{-2}dy_1\right)\rangle.$$

In all cases considered above, Lie symmetry operators of (12) are induced by operators from A(NS). Namely, the operators ∂_1, D_2^1, $Z^1(\lambda(y_1))$ are induced by ∂_t, D, $Z(\lambda(t))$ respectively. In case $\eta \equiv 0$ the operator $R_1(\psi(y_1))$ and in case $\eta \neq 0$ the operator $R_1(\psi(y_1))$ where $\psi\ddot{\eta} - \psi\ddot{\eta} = 0$ are done by $R(0, 0, \psi(t))$. Therefore, Lie reduction of the system (12) gives only solutions that can be obtained by reducing the NSEs with two- and three-dimensional subalgebras of A(NS).

When $\eta = \chi = 0$ the system (12) describes axisymmetric motion of a fluid and can be transformed into a system of two equations for a stream function Ψ^1 and a function Ψ^2 that are determined by

$$\Psi_3^1 = y_2v^1, \quad \Psi_2^1 = -y_2v^3, \quad \Psi^2 = y_2v^2.$$

The transformed system has been studied by L.V. Kapitanskiy [8].

Consider the system (13). Let us introduce the notations

$$\begin{aligned}
&t = y_3, \quad \tilde{\rho} = \int\rho^3(t)dt, \quad R_3(\psi^1(t), \psi^2(t)) = \psi^i\partial_i + \psi_t^i\partial_{v^i} - \psi_{tt}^i y_i\partial_q,\\
&Z^1(\lambda(t)) = \lambda(t)\partial_q, \quad S = \partial_{v^3} - \rho^i(t)y_i\partial_q,\\
&E(\chi(t)) = 2_\chi\partial_t + \chi_t y_i\partial y_i + (\chi_{tt}y_i - \chi_t v^i)\partial_{v^i} - \left(2_{\chi_t q} + \frac{1}{2}\chi_{ttt}y_jy_j\right)\partial_q,\\
&J_{12}^1 = y_1\partial_2 - y_2\partial_1 + v^1\partial_{v^2} - v^2\partial_{v^1}.
\end{aligned}$$

Theorem 3. *The maximal, in the sense of Lie, invariance algebra of (13) is the algebra*

$$1)\quad \langle R_3(\psi^1(t),\, \psi^2(t)), Z^1(\lambda(t)), S, E(\chi^1(t)), E(\chi^2(t)), v^3\partial_{v^3}, J_{12}^1\rangle,$$

where $\chi^1 = e^{-\tilde{\rho}(t)}\int e^{\tilde{\rho}(t)}dt$, $\chi^2 = e^{-\tilde{\rho}(t)}$, *if* $\rho^1 = \rho^2 = 0$,

$$2)\quad \langle R_3(\psi^1(t),\, \psi^2(t)), Z^1(\lambda(t)), S, E(\chi(t)) + 2a_1v^3\partial_{v^3} + 2a_2J_{12}^1\rangle,$$

where a_1, a_2, a_3 *are fixed constants,* $\chi = e^{-\tilde{\rho}(t)}\left(\int e^{\tilde{\rho}(t)}dt + a_3\right)$ *if*

$$\begin{aligned}
\rho^1 &= e^{\frac{3}{2}\tilde{\rho}(t)}(\rho(t))^{-\frac{3}{2}-a_1}(C_1\cos(a_2\ln\hat{\rho}(t)) - C_2\sin(a_2\ln\hat{\rho}(t))),\\
\rho^2 &= e^{\frac{3}{2}\tilde{\rho}(t)}(\rho(t))^{-\frac{3}{2}-a_1}(C_1\sin(a_2\ln\hat{\rho}(t)) + C_2\cos(a_2\ln\hat{\rho}(t))),
\end{aligned} \tag{15}$$

where $\hat{\rho}(t) = \left|\int e^{\tilde{\rho}(t)}dt + a_3\right|$, $C_1, C_2 = \mathrm{const}$, $(C_1, C_2) \neq (0, 0)$;

$$3)\quad \langle R_3(\psi^1(t), \psi^2(t)), Z^1(\lambda(t)), S, E(\chi(t)) + 2a_1v^3\partial_{v^3} + 2a_2J_{12}^1\rangle,$$

where a_1, a_2 *are fixed constants,* $\chi = e^{-\tilde{\rho}(t)}$ *if*

$$\rho^1 = e^{\frac{3}{2}\tilde{\rho}(t)-a_1\hat{\rho}(t)}(C_1\cos(a_2\hat{\rho}(t)) - C_2\sin(a_2\hat{\rho}(t))),$$
$$\rho^2 = e^{\frac{3}{2}\tilde{\rho}(t)-a_1\hat{\rho}(t)}(C_1\sin(a_2\hat{\rho}(t)) + C_2\cos(a_2\hat{\rho}(t))),$$

where $\hat{\rho}(t) = \int e^{\tilde{\rho}(t)}dt$, $C_1, C_2 = \mathrm{const}$, $(C_1, C_2) \neq (0,0)$;

4) $\langle R_3(\psi^1(t),\, \psi^2(t)), Z^1(\lambda(t)), S\rangle$ *in all other cases.*

Here $\psi^i = \psi^i(t)$, $\lambda = \lambda(t)$ *are arbitrary smooth functions of* $t = y_3$.

Remark 3. If functions $\rho^b = \rho^b(t)$ are determined by (14), $e^{\tilde{\rho}(t)} = C|\tilde{m}(t)|$, where $C = \mathrm{const}$ and it follows from the condition $\rho^1 = \rho^2 = 0$ that $\vec{m} = |\vec{m}(t)|\vec{e}$, where $|\vec{e}| = 1$, $\vec{e} = \mathrm{const}$.

Remark 4. Vector-functions $\vec{n}^i$ from remark 2 are determined up to the transformation

$$\vec{n}^1 = \vec{n}^1\cos\delta - \vec{n}^2\sin\delta, \quad \vec{n}^2 = \vec{n}^1\sin\delta + \vec{n}^2\cos\delta,$$

where $\delta = \mathrm{const}$. Therefore, choosing δ, we can do so that $C_2 = 0$ (then $C_1 \neq 0$).

The operators $R_3(\psi^1, \psi^2) + \alpha S$, $Z^1(\lambda)$ are induced by $R(\vec{l}) + Z(\chi)$, $Z(\lambda)$ respectively, where $\vec{l} = \psi^i\vec{n}^i + \psi^3\vec{m}$, $\psi^3_t(\vec{m}\cdot\vec{m}) + 2\psi^i(\vec{n}^i_t\cdot\vec{m}) = \alpha$, $\chi - \frac{3}{2}(\vec{m}\cdot\vec{m})^{-1}(\psi^i(\vec{m}_t\cdot\vec{n}^i))^2 - \frac{1}{2}\psi^3\psi^i(\vec{m}_{tt}\cdot\vec{n}^i) + \frac{1}{2}\psi^i(\vec{l}_{tt}\cdot\vec{n}^i) = 0$,

If $\vec{m} = |\vec{m}(t)|\vec{e}$, where $\vec{e} = \mathrm{const}$, $|\vec{e}| = 1$, the operator J^1_{12} is induced by $e^1J_{23} + eJ_{31} + e^3J_{12}$. For

$$\vec{m} = \beta_3 e^{\sigma t}(\beta_2\cos\tau, \beta_2\sin\tau, \beta_1)^T, \quad \beta_1^2 + \beta_2^2 = 1, \quad \tau = \varkappa t + \delta,$$

the operator $\partial_t + \varkappa J_{12}$ induces the operator $\partial_{y_3} - \beta_1\varkappa J_{12} + \sigma v^3\partial_{v^2}$ if such vector-functions $\vec{n}^i$ are chosen:

$$\vec{n}^1 = \vec{k}^1\cos\beta_1\tau + \vec{k}^2\sin\beta_1\tau, \quad \vec{n}^2 = -\vec{k}^1\sin\beta_1\tau + \vec{k}^2\cos\beta_1\tau, \tag{16}$$

where $\vec{k}^1 = (-\sin\tau, \cos\tau, 0)^T$, $\vec{k}^2 = (\beta_1\cos\tau, \beta_1\sin\tau, -\beta_2)^T$. For

$$\vec{m} = \beta_3|t+\beta_4|^{\sigma+1/2}(\beta_2\cos\tau, \beta_2\sin\tau, \beta_1)^T, \quad \beta_1^2 + \beta_2^2 = 1,$$
$$\tau = \varkappa\ln|t+\beta_4| + \delta,$$

the operator $D + 2\beta_4\partial_t + 2\varkappa J_{12}$ induces the operator

$$D^1_3 + 2\beta_4\partial_{y_3} - 2\beta\varkappa J_{12} + 2\sigma v^3\partial_{v^3}$$

if vector-functions $\vec{n}^i$ are chosen in the form (15). In all other cases the basis elements of the maximal, in the sense of Lie, invariance algebra of (13) are not induced by operators from A(NS).

Remark 5. The invariance algebra of a system of the form (13) with a parameter-function $\rho^3 = \rho^3(t)$ is like one with a different parameter-function $\tilde{\rho}^3 = \rho^3(t)$. It suggest an idea that there is a local transformation of variables with which one can make ρ^3 to vanish. Indeed, let us transform variables in the way

$$\tilde{y}_i = y_i e^{\frac{1}{2}\tilde{\rho}(t)}, \quad \tilde{y}_3 = \int e^{\tilde{\rho}(t)}dt, \quad \tilde{v}^i = \left(v^i + \frac{1}{2}y_i\rho^3(t)\right)e^{-\frac{1}{2}\tilde{\rho}(t)}, \quad \tilde{v}^3 = v^3,$$
$$\tilde{q} = qe^{-\tilde{\rho}(t)} + \frac{1}{8}y_iy_i[(\rho^3(t))^2 - 2\dot{\rho}^3(t)]e^{-\tilde{\rho}(t)}.$$

As a result, we obtain the system

$$\tilde v^i_3 + \tilde v^j \tilde v^i_{jj} + \tilde q_i + \tilde\rho^i(\tilde y_3)\tilde v^3 = 0,$$
$$\tilde v^3_3 + \tilde v^j \tilde v^3_j - \tilde v^3_{jj} = 0,$$
$$\tilde v^i_j = 0,$$

for functions $\tilde v^a = \tilde v^a(\tilde y_1, \tilde y_2, \tilde y_3)$, $\tilde q = \tilde q(\tilde y_1, \tilde y_2, \tilde y_3)$, where $\tilde\rho^i(\tilde y_3) = \rho^i(t)e^{-\frac{3}{2}\tilde\rho(t)}$, subscripts 1, 2, 3 mean differentiation with respect to $\tilde y_1$, $\tilde y_2$, $\tilde y_3$ accordingly.

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