Symmetry reduction and exact solutions of the Navier–Stokes equations

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Ansatzes for the Navier–Stokes field are described. These ansatzes reduce the Navier–Stokes equations to system of differential equations in three, two, and one independent variables. The large sets of exact solutions of the Navier–Stokes equations are constructed.

1 Introduction

The Navier–Stokes equations (NSEs)

$$\begin{aligned} &\vec{u}_t + (\vec{u}\cdot\vec{\nabla})\vec{u} - \triangle\vec{u} + \vec{\nabla}p = \vec{0}, \\ &\operatorname{div}\vec{u} = 0 \end{aligned} \tag{1.1}$$

which describe the motion of an incompressible viscous fluid are the basic equations of modern hydrodynamics. In (1.1) and below $\vec{u} = \{u^a(t,\vec{x})\}$ denotes the velocity field of a fluid, $p = p(t,\vec{x})$ denotes the pressure, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\triangle = \vec{\nabla}\cdot\vec{\nabla}$ is the Laplacian, the kinematic coefficient of viscosity and fluid density are set equal to unity. Repead indices denote summation whereby we consider the indices a, b to take on values in $\{1,2,3\}$ and the indices i, j to take on values in $\{1,2\}$.

The problem of finding exact solutions of non-linear equations (1.1) is an important but rather complicated one. There are some ways to solve it. Considerable progress in this field can be achieved by means of making use of a symmetry approach. Equations (1.1) have non-trivial symmetry properties. It was known long ago [37, 2] that they are invariant under the eleven-parametric extended Galilei group. Let us denote it by $G_1(1,3)$. This group includes the Galilei group and scale transformations. The Lie algebra $AG_1(1,3)$ of $G_1(1,3)$ is generated by the operators

$$P_0, \quad J_{ab}, \quad D, \quad P_a, \quad G_a,$$

where

$$\begin{aligned} &P_0 = \partial_t, \quad D = 2t\partial_t + x_a\partial_a - u^a\partial_{u^a} - 2p\partial_p, \\ &J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a}, \quad a \neq b, \\ &G_a = t\partial_a + \partial_{u^a}, \quad P_a = \partial_a. \end{aligned}$$

Relatively recently it was found by means of the Lie method [8, 5, 26] that the maximal Lie invariance algebra (MIA) of the NSEs (1.1) is the infinite-dimensional algebra $A(NS)$ with the basis elements

$$\partial_t, \quad D, \quad J_{ab}, \quad R(\vec{m}), \quad Z(\chi), \tag{1.2}$$

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where

$$R(\vec{m}) = R(\vec{m}(t)) = m^a(t)\partial_a + m^a_t(t)\partial_{u^a} - m^a_{tt}(t)x_a\partial_p, \tag{1.3}$$

$$Z(\chi) = Z(\chi(t)) = \chi(t)\partial_p, \tag{1.4}$$

$m^a = m^a(t)$ and $\chi = \chi(t)$ are arbitrary smooth functions of t (degree of their smoothness is discussed in Note A.1).

The algebra $AG_1(1,3)$ is a subalgebra of $A(NS)$. Indeed, setting $m^a = \delta_{ab}$, where b is fixed, we obtain $R(\vec{m}) = \partial_b$, and if $m^a = \delta_{ab}t$ then $R(\vec{m}) = G_b$. Here δ_{ab} is the Kronecker symbol ($\delta_{ab} = 1$ if $a = b$, $\delta_{ab} = 0$ if $a \neq b$).

Operators (1.2) generate the following invariance transformations of system (1.1):

$$\begin{aligned}
&\partial_t: && \tilde{\vec{u}}(t,\vec{x}) = \vec{u}(t+\varepsilon,\vec{x}), \quad \tilde{p}(t,\vec{x}) = p(t+\varepsilon,\vec{x}) \\
& && \text{(translations with respect to } t\text{)}, \\
&J_{ab}: && \tilde{\vec{u}}(t,\vec{x}) = B\vec{u}(t,B^T\vec{x}), \quad \tilde{p}(t,\vec{x}) = p(t,B^T\vec{x}) \\
& && \text{(space rotations)}, \\
&D: && \tilde{\vec{u}}(t,\vec{x}) = e^{\varepsilon}\vec{u}(e^{2\varepsilon}t,e^{\varepsilon}\vec{x}), \quad \tilde{p}(t,\vec{x}) = e^{2\varepsilon}p(e^{2\varepsilon}t,e^{\varepsilon}\vec{x}) \\
& && \text{(scale transformations)}, \\
&R(\vec{m}): && \tilde{\vec{u}}(t,\vec{x}) = \vec{u}(t,\vec{x}-\vec{m}(t)) + \vec{m}_t(t), \\
& && \tilde{p}(t,\vec{x}) = p(t,\vec{x}-\vec{m}(t)) - \vec{m}_{tt}\cdot\vec{x} - \tfrac{1}{2}\vec{m}\cdot\vec{m}_{tt} \\
& && \text{(these transformations include the space translations} \\
& && \text{and the Galilei transformations)}, \\
&Z(\chi): && \tilde{\vec{u}}(t,\vec{x}) = \vec{u}(t,\vec{x}), \quad \tilde{p}(t,\vec{x}) = p(t,\vec{x}) + \chi(t).
\end{aligned} \tag{1.5}$$

Here $\varepsilon \in \mathbb{R}$, $B = \{\beta_{ab}\} \in O(3)$, i.e. $BB^T = \{\delta_{ab}\}$, B^T is the transposed matrix.

Besides continuous transformations (1.5) the NSEs admit discrete transformations of the form

$$\begin{aligned}
&\tilde{t} = t, \quad \tilde{x}_a = x_a,\ a \neq b, \quad \tilde{x}_b = -x_b, \\
&\tilde{p} = p, \quad \tilde{u}^a = u^a,\ a \neq b, \quad \tilde{u}^b = -u^b,
\end{aligned} \tag{1.6}$$

where b is fixed. Invariance under transformations (1.5) and (1.6) means that $(\tilde{\vec{u}},\tilde{p})$ is a solution of (1.1) if $(\vec{u},p)$ is a solution of (1.1).

A complete review of exact solutions found for the NSEs before 1963 is contained in [1]. We should like also to mark more modern reviews [16, 7, 36] despite their subjects slightly differ from subjects of our investigations. To find exact solutions of (1.1), symmetry approach in explicit form was used in [2, 31, 32, 6, 20, 21, 4, 17, 15, 12, 10, 11, 30]. This article is a continuation and a extention of our works [15, 12, 10, 11, 30]. In it we make symmetry reduction of the NSEs to systems of PDEs in three and two independent variables and to systems of ODEs, using subalgebraic structure of $A(NS)$. We investigate symmetry properties of the reduced systems of PDEs and construct exact solutions of the reduced systems of ODEs when it is possible. As a result, large classes of exact solutions of the NSEs are obtained.

The reduction problem for the NSEs is to describe ansatzes of the form [9]:

$$u^a = f^{ab}(t,\vec{x})v^b(\omega) + g^a(t,\vec{x}), \quad p = f^0(t,\vec{x})q(\omega) + g^0(t,\vec{x}) \tag{1.7}$$

that reduce system (1.1) in four independent variables to systems of differential equations in the functions v^a and q depending on the variables $\omega = \{\omega_n\}$ $(n = \overline{1,N})$, where N takes on a fixed value from the set $\{1,2,3\}$. In formulas (1.7) f^{ab}, g^a, f^0, g^0, and ω_n are smooth functions to be described. In such a general formulation the reduction problem is too complex to solve. But using Lie symmetry, some ansatzes (1.7) reducing the NSEs can be obtained. According to the Lie method, first a complete set of $A(NS)$-inequivalent subalgebras of dimension $M = 4 - N$ is to be constructed. For $N = 3$, $N = 2$, and $N = 1$ such sets are given in Subsections A.2, A.3, and A.4, correspondingly. Knowing subalgebraic structure of $A(NS)$, one can find explicit forms for the functions f^{ab}, g^a, f^0, g^0, and ω_n and obtain reduced systems in the functions v^k and q. This is made in Section 2 ($N = 3$), Section 3 ($N = 2$) and Section 4 ($N = 1$). Moreover, in Subsection 2.3 symmetry properties of all reduced systems of PDEs in three independent variables are investigated, and in Subsection 4.3 exact solutions of the reduced systems of ODEs are constructed. Symmetry properties and exact solutions of some reduced systems of PDEs in two independent variables are discussed in Sections 5 and 6. In Section 7 we make symmetry reduction of a some reduced system of PDEs in three independent variables.

In conclusion of the section, for convenience, we give some abbreviations, notations, and default rules used in this article.

Abbreviations:

the NSEs: the Navier–Stokes equations

the MIA: the maximal Lie invariance algebra (of either a some equation or a some system of equations)

a ODE: a ordinary differential equation

a PDE: a partial differential equation

Notations:

$C^\infty((t_0,t_1),\mathbb{R})$**:** the set of infinite-differentiable functions from (t_0,t_1) into $\mathbb{R}$, where $-\infty \le t_0 < t_1 \le +\infty$

$C^\infty((t_0,t_1),\mathbb{R}^3)$**:** the set of infinite-differentiable vector-functions from (t_0,t_1) into $\mathbb{R}^3$, where $-\infty \le t_0 < t_1 \le +\infty$

$\partial_t = \partial/\partial_t$, $\partial_a = \partial/\partial_{x_a}$, $\partial_{u^a} = \partial/\partial_{u^a}$, ...

Default rules:

Repead indices denote summation whereby we consider the indices a, b to take on values in $\{1,2,3\}$ and the indices i, j to take on values in $\{1,2\}$.

All theorems on the MIAs of PDEs are proved by means of the standard Lie algorithm.

Subscripts of functions denote differentiation.

2 Reduction of the Navier–Stokes equations to systems of PDEs in three independent variables

2.1 Ansatzes of codimension one

In this subsection we give ansatzes that reduce the NSEs to systems of PDEs in three independent variables. The ansatzes are constructed with the subalgebraic analysis of $A(NS)$ (see Subsection A.2) by means of the method discribed in Section B.

$$\begin{aligned} 1.\quad & u^1 = |t|^{-1/2}(v^1\cos\tau - v^2\sin\tau) + \tfrac{1}{2}x_1t^{-1} - \varkappa x_2t^{-1}, \\ & u^2 = |t|^{-1/2}(v^1\sin\tau + v^2\cos\tau) + \tfrac{1}{2}x_2t^{-1} + \varkappa x_1t^{-1}, \\ & u^3 = |t|^{-1/2}v^3 + \tfrac{1}{2}x_3t^{-1}, \\ & p = |t|^{-1}q + \tfrac{1}{2}\varkappa^2t^{-2}r^2 + \tfrac{1}{8}t^{-2}x_ax_a, \end{aligned} \tag{2.1}$$

where

$$y_1 = |t|^{-1/2}(x_1\cos\tau + x_2\sin\tau), \quad y_2 = |t|^{-1/2}(-x_1\sin\tau + x_2\cos\tau),$$
$$y_3 = |t|^{-1/2}x_3, \quad \varkappa \geq 0, \quad \tau = \varkappa\ln|t|.$$

Here and below $v^a = v^a(y_1,y_2,y_3)$, $q = q(y_1,y_2,y_3)$, $r = (x_1^2+x_2^2)^{1/2}$.

$$\begin{aligned} 2.\quad & u^1 = v^1\cos\varkappa t - v^2\sin\varkappa t - \varkappa x_2, \\ & u^2 = v^1\sin\varkappa t + v^2\cos\varkappa t + \varkappa x_1, \\ & u^3 = v^3, \\ & p = q + \tfrac{1}{2}\varkappa^2r^2, \end{aligned} \tag{2.2}$$

where

$$y_1 = x_1\cos\varkappa t + x_2\sin\varkappa t, \quad y_2 = -x_1\sin\varkappa t + x_2\cos\varkappa t,$$
$$y_3 = x_3, \quad \varkappa \in \{0;1\}.$$

$$\begin{aligned} 3.\quad & u^1 = x_1r^{-1}v^1 - x_2r^{-1}v^2 + x_1r^{-2}, \\ & u^2 = x_2r^{-1}v^1 + x_1r^{-1}v^2 + x_2r^{-2}, \\ & u^3 = v^3 + \eta(t)r^{-1}v^2 + \eta_t(t)\arctan x_2/x_1, \\ & p = q - \tfrac{1}{2}\eta_{tt}(t)(\eta(t))^{-1}x_3^2 - \tfrac{1}{2}r^{-2} + \chi(t)\arctan x_2/x_1, \end{aligned} \tag{2.3}$$

where

$$y_1 = t, \quad y_2 = r, \quad y_3 = x_3 - \eta(t)\arctan x_2/x_1, \quad \eta,\chi \in C^\infty((t_0,t_1),\mathbb{R}).$$

Note 2.1 The expression for the pressure p from ansatz (2.3) is indeterminate in the points $t \in (t_0,t_1)$ where $\eta(t) = 0$. If there are such points t, we will consider ansatz (2.3) on the intervals (t_0^n,t_1^n) that are contained in the interval (t_0,t_1) and that satisfy one of the conditions:

a) $\eta(t) \neq 0 \quad \forall\, t \in (t_0^n,t_1^n)$;

b) $\eta(t) = 0 \quad \forall\, t \in (t_0^n,t_1^n)$.

In the last case we consider $\eta_{tt}/\eta := 0$.

$$\begin{aligned} 4.\quad & \vec u = v^i\vec n^i + (\vec m\cdot\vec m)^{-1}v^3\vec m + (\vec m\cdot\vec m)^{-1}(\vec m\cdot\vec x)\vec m_t - y_i\vec n^i_t,\\ & p = q - \tfrac32(\vec m\cdot\vec m)^{-1}((\vec m_t\cdot\vec n^i)y_i)^2 - (\vec m\cdot\vec m)^{-1}(\vec m_{tt}\cdot\vec x)(\vec m\cdot\vec x) + \\ & \qquad + \tfrac12(\vec m_{tt}\cdot\vec m)(\vec m\cdot\vec m)^{-2}(\vec m\cdot\vec x)^2, \end{aligned} \tag{2.4}$$

where

$$y_i = \vec n^i\cdot\vec x,\quad y_3 = t,\quad \vec m, \vec n^i \in C^\infty((t_0,t_1),\mathbb{R}^3).$$

$$\vec n^i\cdot\vec m = \vec n^1\cdot\vec n^2 = \vec n^1_t\cdot\vec n^2 = 0,\quad |\vec n^i| = 1. \tag{2.5}$$

Note 2.2 There exist vector-functions $\vec n^i$ which satisfy conditions (2.5). They can be constructed in the following way: let us fix the vector-functions $\vec k^i = \vec k^i(t)$ such that $\vec k^i\cdot\vec m = \vec k^1\cdot\vec k^2 = 0$, $|\vec k^i| = 1$, and set

$$\begin{aligned} &\vec n^1 = \vec k^1\cos\psi(t) - \vec k^2\sin\psi(t),\\ &\vec n^2 = \vec k^1\sin\psi(t) + \vec k^2\cos\psi(t). \end{aligned} \tag{2.6}$$

Then $\quad \vec n^1_t\cdot\vec n^2 = \vec k^1_t\cdot\vec k^2 - \psi_t = 0$ if $\psi = \int(\vec k^1_t\cdot\vec k^2)dt$.

2.2 Reduced systems

1–2. Substituting ansatzes (2.1) and (2.2) into the NSEs (1.1), we obtain reduced systems of PDEs with the same general form

$$\begin{aligned} &v^av^1_a - v^1_{aa} + q_1 + \gamma_1 v^2 = 0,\\ &v^av^2_a - v^2_{aa} + q_2 - \gamma_1 v^1 = 0,\\ &v^av^3_a - v^3_{aa} + q_3 = 0,\\ &v^a_a = \gamma_2. \end{aligned} \tag{2.7}$$

Hereafter subscripts 1, 2, and 3 of functions denote differentiation with respect to y_1, y_2, and y_3, accordingly. The constants γ_i take the values

1. $\gamma_1 = -2\varkappa,\quad \gamma_2 = -\frac32\quad$ if $\quad t>0,\quad \gamma_1 = 2\varkappa,\quad \gamma_2 = \frac32\quad$ if $\quad t<0.$
2. $\gamma_1 = -2\varkappa,\quad \gamma_2 = 0.$

For ansatzes (2.3) and (2.4) the reduced equations have the form

$$\begin{aligned} 3.\quad & v^1_1 + v^1v^1_2 + v^3v^1_3 - y_2^{-1}v^2v^2 - \left(v^1_{22} + (1+\eta^2y_2^{-2})v^1_{33}\right) - 2\eta y_2^{-2}v^2_3 + q_2 = 0,\\ & v^2_1 + v^1v^2_2 + v^3v^2_3 + y_2^{-1}v^1v^2 - \left(v^2_{22} + (1+\eta^2y_2^{-2})v^2_{33}\right) + \\ & \quad + 2\eta y_2^{-2}v^1_3 + 2y_2^{-2}v^2 - \eta y_2^{-1}q_3 + \chi y_2^{-1} = 0,\\ & v^3_1 + v^1v^3_2 + v^3v^3_3 - \left(v^3_{22} + (1+\eta^2y_2^{-2})v^3_{33}\right) - 2\eta^2y_2^{-3}v^1_3 + 2\eta_1y_2^{-1}v^2 + \\ & \quad + 2\eta y_2^{-1}(y_2^{-1}v^2)_2 + (1+\eta^2y_2^{-2})q_3 - \eta_{11}\eta^{-1}y_3 - \chi\eta y_2^{-2} = 0,\\ & y_2^{-1}v^1 + v^1_2 + v^3_3 = 0. \end{aligned} \tag{2.8}$$

$$\begin{aligned} 4.\quad & v^i_3 + v^jv^i_j - v^i_{jj} + q_i + \rho^i(y_3)v^3 = 0,\\ & v^3_3 + v^jv^3_j - v^3_{jj} = 0,\\ & v^i_i + \rho^3(y_3) = 0, \end{aligned} \tag{2.9}$$

where

$$\rho^i = \rho^i(y_3) = 2(\vec{m}\cdot\vec{m})^{-1}(\vec{m}_t\cdot\vec{n}^i),$$
$$\rho^3 = \rho^3(y_3) = (\vec{m}\cdot\vec{m})^{-1}(\vec{m}_t\cdot\vec{m}). \qquad (2.10)$$

2.3 Symmetry of reduced systems

Let us study symmetry properties of systems (2.7), (2.8), and (2.9). All results of this subsection are obtained by means of the standard Lie algorithm [28, 27]. First, let us consider system (2.7).

Theorem 2.1 *The MIA of system (2.7) is the algebra*

$$\begin{aligned} &a)\quad \langle\partial_a,\partial_q,J^1_{12}\rangle \quad \textit{if}\quad \gamma_1\neq 0;\\ &b)\quad \langle\partial_a,\partial_q,J^1_{ab}\rangle \quad \textit{if}\quad \gamma_1=0,\ \gamma_2\neq 0;\\ &c)\quad \langle\partial_a,\partial_q,J^1_{ab},D^1_1\rangle \quad \textit{if}\quad \gamma_1=\gamma_2=0.\end{aligned}$$

Here $J^1_{ab} = y_a\partial_b - y_b\partial_a + v^a\partial_{v^b} - v^b\partial_{v^a}$, $D^1_1 = y_a\partial_a - v^a\partial_{v^a} - 2q\partial_q$.

Note 2.3 All Lie symmetry operators of (2.7) are induced by operators from $A(NS)$: The operators J^1_{ab} and D^1_1 are induced by J_{ab} and D. The operators $c_a\partial_a$ ($c_a = \mathrm{const}$) and ∂_q are induced by either

$$R(|t|^{1/2}(c_1\cos\tau - c_2\sin\tau, c_1\sin\tau + c_2\cos\tau, c_3)),\quad Z(|t|^{-1}),$$

where $\tau = \varkappa\ln|t|$, for ansats (2.1) or

$$R(c_1\cos\varkappa t - c_2\sin\varkappa t, c_1\sin\varkappa t + c_2\cos\varkappa t, c_3),\quad Z(1)$$

for ansatz (2.2), respectively. Therefore, Lie reductions of system (2.7) give only solutions that can be obtained by reducing the NSEs with two- and three-dimensional subalgebras of $A(NS)$.

Let us continue to system (2.8). We denote $A^{\max}$ as the MIA of (2.8). Studying symmetry properties of (2.8), one has to consider the following cases:

A. $\eta,\chi\equiv 0$. Then

$$A^{\max} = \langle\partial^1, D^1_2, R_1(\psi(y_1)), Z^1(\lambda(y_1))\rangle,$$

where

$$D^1_2 = 2y_1\partial_1 + y_2\partial_2 + y_3\partial_3 - v^a\partial_{v^a} - 2q\partial_q,$$
$$R_1(\psi(y_1)) = \psi\partial_3 + \psi_1\partial_{v^3} - \psi_{11}y_3\partial_q,\quad Z^1(\lambda(y_1)) = \lambda(y_1)\partial_q.$$

Here and below $\psi = \psi(y_1)$ and $\lambda = \lambda(y_1)$ are arbitrary smooth functions of $y_1 = t$.

B. $\eta\equiv 0$, $\chi\not\equiv 0$. In this case an extension of $A^{\max}$ exists for $\chi = (C_1y_1 + C_2)^{-1}$, where $C_1, C_2 = \mathrm{const}$. Let $C_1\neq 0$. We can make C_2 vanish by means of equivalence transformation (A.6), i.e., $\chi = Cy_1^{-1}$, where $C = \mathrm{const}$. Then

$$A^{\max} = \langle D^1_2, R_1(\psi(y_1)), Z^1(\lambda(y_1))\rangle.$$

If $C_1 = 0$, $\chi = C = \text{const}$ and

$$A^{\max} = \langle \partial_1, R_1(\psi(y_1)), Z^1(\lambda(y_1)) \rangle.$$

For other values of χ, i.e., when $\chi_{11}\chi \neq \chi_1\chi_1$,

$$A^{\max} = \langle R_1(\psi(y_1)), Z^1(\lambda(y_1)) \rangle.$$

C. $\eta \neq 0$. By means of equivalence transformation (A.6) we make $\chi = 0$. In this case an extension of $A^{\max}$ exists for $\eta = \pm |C_1 y_1 + C_2|^{1/2}$, where $C_1, C_2 = \text{const}$. Let $C_1 \neq 0$. We can make C_2 vanish by means of equivalence transformation (A.6), i.e., $\eta = C|y_1|^{1/2}$, where $C = \text{const}$. Then

$$A^{\max} = \langle D_2^1, R_2(|y_1|^{1/2}), R_2(|y_1|^{1/2} \ln |y_1|), Z^1(\lambda(y_1)) \rangle,$$

where $R_2(\psi(y_1)) = \psi\partial_3 + \psi_1\partial_{v^3}$. If $C_1 = 0$, i.e., $\eta = C = \text{const}$,

$$A^{\max} = \langle \partial^1, \partial_3, y_1\partial_3 + \partial_{v^3} Z^1(\lambda(y_1)) \rangle.$$

For other values of η, i.e., when $(\eta^2)_{11} \neq 0$,

$$A^{\max} = \langle R_2(\eta(y_1)), R_2(\eta(y_1) \textstyle\int (\eta(y_1))^{-2} dy_1), Z^1(\lambda(y_1)) \rangle.$$

Note 2.4 In all cases considered above the Lie symmetry operators of (2.8) are induced by operators from $A(NS)$: The operators ∂_1, D_2^1, and $Z^1(\lambda(y_1))$ are induced by ∂_t, D, and $Z(\lambda(t))$, respectively. The operator $R(0,0,\psi(t))$ induces the operator $R_1(\psi(y_1))$ for $\eta \equiv 0$ and the operator $R_2(\psi(y_1))$ (if $\psi_{11}\eta - \psi\eta_{11} = 0$) for $\eta \neq 0$. Therefore, the Lie reduction of system (2.8) gives only solutions that can be obtained by reducing the NSEs with two- and three-dimentional subalgebras of $A(NS)$.

When $\eta = \chi = 0$, system (2.8) describes axially symmetric motion of a fluid and can be transformed into a system of two equations for a stream function Ψ^1 and a function Ψ^2 that are determined by

$$\Psi^1_3 = y_2 v^1, \quad \Psi^1_2 = -y_2 v^3, \quad \Psi^2 = y_2 v^2.$$

The transformed system was studied by L.V. Kapitanskiy [20, 21].

Consider system (2.9). Let us introduce the notations

$$\begin{aligned}
&t = y_3, \quad \rho = \rho(t) = \textstyle\int \rho^3(t)dt, \\
&R_3(\psi^1(t), \psi^2(t)) = \psi^i\partial_{y_i} + \psi^i_t\partial_{v^i} - \psi^i_{tt} y_i \partial_q, \\
&Z^1(\lambda(t)) = \lambda(t)\partial_q, \quad S = \partial_{v^3} - \rho^i(t) y_i \partial_q, \\
&E(\chi(t)) = 2\chi\partial_t + \chi_t y_i \partial_{y_i} + (\chi_{tt} y_i - \chi_t v^i)\partial_{v^i} - (2\chi_t q + \tfrac{1}{2}\chi_{ttt} y_j y_j)\partial_q, \\
&J^1_{12} = y_1\partial_2 - y_2\partial_1 + v^1\partial_{v^2} - v^2\partial_{v^1}.
\end{aligned}$$

Theorem 2.2 *The MIA of(2.9) is the algebra*

$$1)\ \langle R_3(\psi^1(t),\, \psi^2(t)),\, Z^1(\lambda(t)),\, S,\, E(\chi^1(t)),\, E(\chi^2(t)),\, v^3\partial_{v^3},\, J^1_{12} \rangle,$$

where $\chi^1 = e^{-\rho(t)} \int e^{\rho(t)} dt$ *and* $\chi^2 = e^{-\rho(t)}$, *if* $\rho^i = 0$;

$$2)\ \langle R_3(\psi^1(t),\, \psi^2(t)),\, Z^1(\lambda(t)),\, S,\, E(\chi(t)) + 2a_1 v^3\partial_{v^3} + 2a_2 J^1_{12} \rangle,$$

where a_1, a_2, *and* a_3 *are fixed constants,* $\chi = e^{-\rho(t)}\Big(\int e^{\rho(t)}dt + a_3\Big)$, *if*

$$\rho^1 = e^{\frac{3}{2}\rho}\hat\rho^{-\frac{3}{2}-a_1}\big(C_1\cos(a_2\ln\hat\rho) - C_2\sin(a_2\ln\hat\rho)\big),$$
$$\rho^2 = e^{\frac{3}{2}\rho}\hat\rho^{-\frac{3}{2}-a_1}\big(C_1\sin(a_2\ln\hat\rho) + C_2\cos(a_2\ln\hat\rho)\big)$$

with $\hat\rho = \hat\rho(t) = |\int e^{\rho(t)}dt + a_3|$, $C_1, C_2 = \mathrm{const}$, $(C_1, C_2) \neq (0,0)$;

3) $\langle R_3(\psi^1(t), \psi^2(t)),\ Z^1(\lambda(t)),\ S,\ E(\chi(t)) + 2a_1v^3\partial_{v^3} + 2a_2J^1_{12}\rangle$,

where a_1 *and* a_2 *are fixed constants,* $\chi = e^{-\rho(t)}$, *if*

$$\rho^1 = e^{\frac{3}{2}\rho - a_1\hat\rho}\big(C_1\cos(a_2\hat\rho) - C_2\sin(a_2\hat\rho)\big),$$
$$\rho^2 = e^{\frac{3}{2}\rho - a_1\hat\rho}\big(C_1\sin(a_2\hat\rho) + C_2\cos(a_2\hat\rho)\big)$$

with $\hat\rho = \hat\rho(t) = \int e^{\rho(t)}dt$, $C_1, C_2 = \mathrm{const}$, $(C_1, C_2) \neq (0,0)$;

4) $\langle R_3(\psi^1(t),\ \psi^2(t)),\ Z^1(\lambda(t)),\ S\rangle$

in all other cases.

Here $\psi^i = \psi^i(t)$, $\lambda = \lambda(t)$ *are arbitrary smooth function of* $t = y_3$.

Note 2.5 If functions ρ^b are determined by (2.10), then $e^{\rho(t)} = C|\vec m(t)|$, where $C = \mathrm{const}$, and the condition $\rho^i = 0$ implies that $\vec m = |\vec m(t)|\vec e$, where $\vec e = \mathrm{const}$ and $|\vec e| = 1$.

Note 2.6 The vector-functions $\vec n^i$ from Note 2.2 are determined up to the transformation

$$\vec n^1 = \vec n^1\cos\delta - \vec n^2\sin\delta, \quad \vec n^2 = \vec n^1\sin\delta + \vec n^2\cos\delta,$$

where $\delta = \mathrm{const}$. Therefore, δ can be chosen such that $C_2 = 0$ (then $C_1 \neq 0$).

Note 2.7 The operators $R_3(\psi^1, \psi^2) + \alpha S$ and $Z^1(\lambda)$ are induced by $R(\vec l) + Z(\chi)$ and $Z(\lambda)$, respectively. Here $\vec l = \psi^i\vec n^i + \psi^3\vec m$, $\psi^3_t(\vec m\cdot\vec m) + 2\psi^i(\vec n^i_t\cdot\vec m) = \alpha$,

$$\chi - \tfrac{3}{2}(\vec m\cdot\vec m)^{-1}((\vec m_t\cdot\vec n^i)\psi^i)^2 - \tfrac{1}{2}(\vec m_{tt}\cdot\vec n^i)\psi^3\psi^i + \tfrac{1}{2}(\vec l_{tt}\cdot\vec n^i)\psi^i = 0.$$

If $\vec m = |\vec m|\vec e$, where $\vec e = \mathrm{const}$ and $|\vec e| = 1$, the operator J^1_{12} is induced by $e^1J_{23} + e^2J_{31} + e^3J_{12}$.

For

$$\vec m = \beta_3 e^{\sigma t}(\beta_2\cos\tau, \beta_2\sin\tau, \beta_1)^T$$

with $\tau = \varkappa t + \delta$ and $\beta_a = \mathrm{const}$, where $\beta_1^2 + \beta_2^2 = 1$, the operator $\partial_t + \varkappa J_{12}$ induces the operator $\partial_{y_3} - \beta_1\varkappa J^1_{12} + \sigma v^3\partial_{v^3}$ if the following vector-functions $\vec n^i$ are chosen:

$$\vec n^1 = \vec k^1\cos\beta_1\tau + \vec k^2\sin\beta_1\tau, \quad \vec n^2 = -\vec k^1\sin\beta_1\tau + \vec k^2\cos\beta_1\tau, \tag{2.11}$$

where $\vec k^1 = (-\sin\tau, \cos\tau, 0)^T$ and $\vec k^2 = (\beta_1\cos\tau, \beta_1\sin\tau, -\beta_2)^T$.

For

$$\vec m = \beta_3|t + \beta_4|^{\sigma+1/2}(\beta_2\cos\tau, \beta_2\sin\tau, \beta_1)^T$$

with $\tau = \varkappa \ln|t+\beta_4| + \delta$ and $\beta_a, \beta_4 = \text{const}$, where $\beta_1^2 + \beta_2^2 = 1$, the operator $D + 2\beta_4\partial_t + 2\varkappa J_{12}$ induces the operator

$$D_3^1 + 2\beta_4\partial_{y_3} - 2\beta_1\varkappa J_{12}^1 + 2\sigma v^3\partial_{v^3},$$

where $D_3^1 = y_i\partial_{y_i} + 2y_3\partial_{y_3} - v^i\partial_{v^i} - 2q\partial_q$, if the vector-functions $\vec{n}^i$ are chosen in form (2.11). In all other cases the basis elements of the MIA of (2.9) are not induced by operators from $A(NS)$.

Note 2.8 The invariance algebras of systems of form (2.9) with different parameter-functions $\rho^3 = \rho^3(t)$ and $\tilde{\rho}^3 = \tilde{\rho}^3(t)$ are similar . It suggests that there exists a local transformation of variables which make ρ^3 vanish. So, let us transform variables in the following way:

$$\begin{aligned}
&\tilde{y}_i = y_i e^{\frac{1}{2}\rho(t)}, \quad \tilde{y}_3 = \textstyle\int e^{\rho(t)}dt,\\
&\tilde{v}^i = \big(v^i + \tfrac{1}{2}y_i\rho^3(t)\big)e^{-\frac{1}{2}\rho(t)}, \quad \tilde{v}^3 = v^3,\\
&\tilde{q} = qe^{-\rho(t)} + \tfrac{1}{8}y_iy_i\big((\rho^3(t)^2) - 2\rho^3_t(t)\big)e^{-\rho(t)}.
\end{aligned} \tag{2.12}$$

As a result, we obtain the system

$$\begin{aligned}
&\tilde{v}^i_3 + \tilde{v}^j\tilde{v}^i_j - \tilde{v}^i_{jj} + \tilde{q}_i + \tilde{\rho}^i(\tilde{y}_3)\tilde{v}^3 = 0,\\
&\tilde{v}^3_3 + \tilde{v}^j v^3_j - \tilde{v}^3_{jj} = 0,\\
&\tilde{v}^i_i = 0
\end{aligned}$$

for the functions $\tilde{v}^a = \tilde{v}^a(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ and $\tilde{q} = \tilde{q}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$. Here subscripts 1, 2, and 3 denote differentiation with respect to $\tilde{y}_1$, $\tilde{y}_2$, and $\tilde{y}_3$, accordingly. Also $\tilde{\rho}^i(\tilde{y}_3) = \rho^i(t)e^{-\frac{3}{2}\rho(t)}$.

3 Reduction of the Navier–Stokes equations to systems of PDEs in two independent variables

3.1 Ansatzes of codimension two

In this subsection we give ansatzes that reduce the NSEs to systems of PDEs in two independent variables. The ansatzes are constructed with the subalgebrical analysis of $A(NS)$ (see Subsection A.3) by means of the method discribed in Section B.

1. $$\begin{aligned}
&u^1 = (rR)^{-1}((x_1 - \varkappa x_2)w^1 - x_2w^2 + x_1x_3r^{-1}w^3),\\
&u^2 = (rR)^{-1}((x_2 + \varkappa x_1)w^1 + x_1w^2 + x_2x_3r^{-1}w^3),\\
&u^3 = x_3(rR)^{-1}w^1 - R^{-1}w^3,\\
&p = R^{-2}s,
\end{aligned} \tag{3.1}$$

where $z_1 = \arctan x_2/x_1 - \varkappa \ln R$, $z_2 = \arctan r/x_3$, $\varkappa \geq 0$.

Here and below $w^a = w^a(z_1, z_2)$, $s = s(z_1, z_2)$, $r = (x_1^2 + x_2^2)^{1/2}$, $R = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, $\varkappa$, ε, σ, μ, and ν are real constants.

$$\begin{aligned} 2.\quad & u^1 = |t|^{-1/2} r^{-1}(x_1 w^1 - x_2 w^2) + \tfrac{1}{2} t^{-1} x_1 + x_1 r^{-2}, \\ & u^2 = |t|^{-1/2} r^{-1}(x_2 w^1 + x_1 w^2) + \tfrac{1}{2} t^{-1} x_2 + x_2 r^{-2}, \\ & u^3 = |t|^{-1/2} w^3 + \varkappa r^{-1} w^2 + \tfrac{1}{2} t^{-1} x_3, \\ & p = |t|^{-1} s - \tfrac{1}{2} r^{-2} + \tfrac{1}{8} t^{-2} R^2 + \varepsilon |t|^{-1} \arctan x_2/x_1, \end{aligned} \tag{3.2}$$

where $z_1 = |t|^{-1/2} r$, $z_2 = |t|^{-1/2} x_3 - \varkappa \arctan x_2/x_1$, $\varkappa \geq 0$, $\varepsilon \geq 0$.

$$\begin{aligned} 3.\quad & u^1 = r^{-1}(x_1 w^1 - x_2 w^2) + x_1 r^{-2}, \\ & u^2 = r^{-1}(x_2 w^1 + x_1 w^2) + x_2 r^{-2}, \\ & u^3 = w^3 + \varkappa r^{-1} w^2, \\ & p = s - \tfrac{1}{2} r^{-2} + \varepsilon \arctan x_2/x_1, \end{aligned} \tag{3.3}$$

where $z_1 = r$, $z_2 = x_3 - \varkappa \arctan x_2/x_1$, $\varkappa \in \{0; 1\}$, $\varepsilon \geq 0$ if $\varkappa = 1$ and $\varepsilon \in \{0; 1\}$ if $\varkappa = 0$.

$$\begin{aligned} 4.\quad & u^1 = |t|^{-1/2}(\mu w^1 + \nu w^3)\cos\tau - |t|^{-1/2} w^2 \sin\tau + \\ & \qquad + \nu\xi t^{-1}\cos\tau + \tfrac{1}{2} t^{-1} x_1 - \varkappa t^{-1} x_2, \\ & u^2 = |t|^{-1/2}(\mu w^1 + \nu w^3)\sin\tau + |t|^{-1/2} w^2 \cos\tau + \\ & \qquad + \nu\xi t^{-1}\sin\tau + \tfrac{1}{2} t^{-1} x_2 + \varkappa t^{-1} x_1, \\ & u^3 = |t|^{-1/2}(-\nu w^1 + \mu w^3) + \mu\xi t^{-1} + \tfrac{1}{2} t^{-1} x_3, \\ & p = |t|^{-1} s - \tfrac{1}{2} t^{-2}\xi^2 + \tfrac{1}{8} t^{-2} R^2 + \tfrac{1}{2}\varkappa^2 t^{-2} r^2 + \\ & \qquad + \varepsilon |t|^{-3/2}(\nu x_1 \cos\tau + \nu x_2 \sin\tau + \mu x_3), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} & z_1 = |t|^{-1/2}(\mu x_1 \cos\tau + \mu x_2 \sin\tau - \nu x_3), \\ & z_2 = |t|^{-1/2}(x_2 \cos\tau - x_1 \sin\tau), \\ & \xi = \sigma(\nu x_1 \cos\tau + \nu x_2 \sin\tau + \mu x_3) + 2\varkappa\nu(x_2 \cos\tau - x_1 \sin\tau), \\ & \tau = \varkappa \ln|t|, \quad \varkappa > 0, \quad \mu \geq 0, \quad \nu \geq 0, \quad \mu^2 + \nu^2 = 1, \quad \sigma\varepsilon = 0, \quad \varepsilon \geq 0. \end{aligned}$$

$$\begin{aligned} 5.\quad & u^1 = |t|^{-1/2} w^1 + \tfrac{1}{2} t^{-1} x_1, \\ & u^2 = |t|^{-1/2} w^2 + \tfrac{1}{2} t^{-1} x_2, \\ & u^3 = |t|^{-1/2} w^3 + (\sigma + \tfrac{1}{2}) t^{-1} x_3, \\ & p = |t|^{-1} s - \tfrac{1}{2}\sigma^2 t^{-2} x_3^2 + \tfrac{1}{8} t^{-2} R^2 + \varepsilon |t|^{-3/2} x_3, \end{aligned} \tag{3.5}$$

where

$$z_1 = |t|^{-1/2} x_1, \quad z_2 = |t|^{-1/2} x_2, \quad \sigma\varepsilon = 0, \quad \varepsilon \geq 0.$$

6. $$u^1 = (\mu w^1 + \nu w^3)\cos t - w^2 \sin t + \nu\xi\cos t - x_2,$$
$$u^2 = (\mu w^1 + \nu w^3)\sin t + w^2 \cos t + \nu\xi\sin t + x_1,$$
$$u^3 = (-\nu w^1 + \mu w^3) + \mu\xi,$$
$$p = s - \tfrac{1}{2}\xi^2 + \tfrac{1}{2}r^2 + \varepsilon(\nu x_1 \cos t + \nu x_2 \sin t + \mu x_3), \tag{3.6}$$

where

$$z_1 = (\mu x_1 \cos t + \mu x_2 \sin t - \nu x_3),$$
$$z_2 = (x_2 \cos t - x_1 \sin t),$$
$$\xi = \sigma(\nu x_1 \cos t + \nu x_2 \sin t + \mu x_3) + 2\nu(x_2 \cos t - x_1 \sin t),$$
$$\mu \ge 0, \quad \nu \ge 0, \quad \mu^2 + \nu^2 = 1, \quad \sigma\varepsilon = 0, \quad \varepsilon \ge 0.$$

7. $$u^1 = w^1, \quad u^2 = w^2, \quad u^3 = w^3 + \sigma x_3,$$
$$p = s - \tfrac{1}{2}\sigma^2 x_3^2 + \varepsilon x_3, \tag{3.7}$$

where

$$z_1 = x_1, \quad z_2 = x_2, \quad \sigma\varepsilon = 0, \quad \varepsilon \in \{0; 1\}.$$

8. $$u^1 = x_1 w^1 - x_2 r^{-2}(w^2 - \chi(t)),$$
$$u^2 = x_2 w^1 + x_1 r^{-2}(w^2 - \chi(t)),$$
$$u^3 = (\rho(t))^{-1}(w^3 + \rho_t(t)x_3 + \varepsilon \arctan x_2/x_1),$$
$$p = s - \tfrac{1}{2}\rho_{tt}(t)(\rho(t))^{-1}x_3^2 + \chi_t(t)\arctan x_2/x_1, \tag{3.8}$$

where

$$z_1 = t, \quad z_2 = r, \quad \varepsilon \in \{0; 1\}, \quad \chi, \rho \in C^\infty((t_0, t_1), \mathbb{R}).$$

9. $$\vec{u} = \vec{w} + \lambda^{-1}(\vec{n}^i \cdot \vec{x})\vec{m}^i_t - \lambda^{-1}(\vec{k}\cdot\vec{x})\vec{k}_t,$$
$$p = s - \tfrac{1}{2}\lambda^{-1}(\vec{m}^i_{tt}\cdot\vec{x})(\vec{n}^i\cdot\vec{x}) - \tfrac{1}{2}\lambda^{-2}(m^i_{tt}\cdot\vec{k})(\vec{n}^i\cdot\vec{x})(\vec{k}\cdot\vec{x}), \tag{3.9}$$

where

$$z_1 = t, \quad z_2 = (\vec{k}\cdot\vec{x}), \quad \vec{m}^i \in C^\infty((t_0, t_1), \mathbb{R}^3),$$
$$\vec{m}^1_{tt}\cdot\vec{m}^2 - \vec{m}^1\cdot\vec{m}^2_{tt} = 0, \quad \vec{k} = \vec{m}^1 \times \vec{m}^2, \quad \vec{n}^1 = \vec{m}^2 \times \vec{k},$$
$$\vec{n}^2 = \vec{k} \times \vec{m}^1, \quad \lambda = \lambda(t) = \vec{k}\cdot\vec{k} \ne 0 \quad \forall\, t \in (t_0, t_1).$$

3.2 Reduced systems

Substituting ansatzes (3.1)–(3.9) into the NSEs (1.1), we obtain the following systems of reduced equations:

1. $$w^2 w^1_1 + w^3 w^1_2 - w^1 w^3 \cot z_2 - (w^1)^2 - (w^2 + \varkappa w^1)^2 \sin^2 z_2 -$$
$$- (w^3)^2 - \big((\varkappa^2 + \sin^{-2} z_2)w^1_{11} + w^1_{22} - \varkappa w^1_1 - 2w^3_2 - 2w^2_1 -$$
$$- 2w^1\big)\sin z_2 + w^1_2 \cos z_2 - w^1 \sin^{-1} z_2 - (2s + \varkappa s_1)\sin^2 z_2 = 0,$$

$$\begin{aligned}
&w^2w^2_1 + w^3w^2_2 + w^3(w^2 + 2\varkappa w^1)\cot z_2 - \\
&- \varkappa\big((w^1)^2 + (w^3)^2 + (w^2 + \varkappa w^1)^2\sin^2 z_2\big) - \\
&- \big((\varkappa^2 + \sin^{-2} z_2)w^2_{11} + w^2_{22} + 3\varkappa w^2_1 + 2\varkappa(w^3_2 + \varkappa w^1_1 + w^1)\big)\sin z_2 + \\
&+ (2w^1_1 + 2w^3_1\cot z_2 - w^2 - 2\varkappa w^1)\sin^{-1} z_2 - \\
&- (w^2_2 + 2\varkappa w^1_2)\cos z_2 + 2\varkappa s\sin^2 z_2 + (1 + \varkappa^2\sin^2 z_2)s_1 = 0, \qquad (3.10)\\
&w^2w^3_1 + w^3w^3_2 - (w^3)^2\cot z_2 - (w^2 + \varkappa w^1)^2\sin z_2\cos z_2 - \\
&- \big((\varkappa^2 + \sin^{-2} z_2)w^3_{11} + w^3_{22} + \varkappa w^3_1 + 2w^1_2\big)\sin z_2 + \\
&+ (2w^1 + w^3_2 + w^2_1 + \varkappa w^1_1)\cos z_2 + s_2\sin^2 z_2 = 0, \\
&w^1 + w^2_1 + w^3_2 = 0.
\end{aligned}$$

Hereafter numeration of the reduced systems corresponds to that of the ansatzes in Subsection 3.1. Subscripts 1 and 2 denote differentiation with respect to the variables z_1 and z_2, accordingly.

2–3. $$\begin{aligned}
&w^1w^1_1 + w^3w^1_2 - z_1^{-1}w^2w^2 - \big(w^1_{11} + (1 + \varkappa^2 z_1^{-2})w^1_{22}\big) - \\
&- 2\varkappa z_1^{-2}w^2_2 + s_1 = 0, \\
&w^1w^2_1 + w^3w^2_2 + z_1^{-1}w^1w^2 - \big(w^2_{11} + (1 + \varkappa^2 z_1^{-2})w^2_{22}\big) + \\
&+ 2\varkappa z_1^{-2}w^1_2 + 2z_1^{-2}w^2 - \varkappa z_1^{-1}s_2 + \varepsilon z_1^{-1} = 0, \qquad (3.11)\\
&w^1w^3_1 + w^3w^3_2 - 2\varkappa z_1^{-2}w^1w^2 - \big(w^3_{11} + (1 + \varkappa^2 z_1^{-2})w^3_{22}\big) + \\
&+ 2\varkappa(z_1^{-2}w^2)_1 - 2\varkappa^2 z_1^{-3}w^1_2 + (1 + \varkappa^2 z_1^{-2})s_2 - \varepsilon\varkappa z_1^{-2} = 0, \\
&w^1_1 + w^3_2 + z_1^{-1}w^1 + \gamma = 0,
\end{aligned}$$

where $\gamma = \pm 3/2$ for ansatz (3.2) and $\gamma = 0$ for ansatz (3.3). Here and below the upper and lower sign in the symbols "$\pm$" and "$\mp$" are associated with $t > 0$ and $t < 0$, respectively.

4–7. For ansatzes (3.4)–(3.7) the reduced equations can be written in the form

$$\begin{aligned}
&w^iw^1_i - w^1_{ii} + s_1 + \alpha_2 w^2 = 0, \\
&w^iw^2_i - w^2_{ii} + s_2 - \alpha_2 w^1 + \alpha_1 w^3 = 0, \\
&w^iw^3_i - w^3_{ii} + \alpha_4 w^3 + \alpha_5 = 0, \\
&w^i_i = \alpha_3
\end{aligned} \qquad (3.12)$$

where the constants α_n $(n = \overline{1,5})$, take on the values

4.	$\alpha_1 = \pm 2\varkappa\nu$,	$\alpha_2 = \mp 2\varkappa\mu$,	$\alpha_3 = \mp(\sigma + 3/2)$,	$\alpha_4 = \pm\sigma$,	$\alpha_5 = \varepsilon$.
5.	$\alpha_1 = 0$,	$\alpha_2 = 0$,	$\alpha_3 = \mp(\sigma + 3/2)$,	$\alpha_4 = \pm\sigma$,	$\alpha_5 = \varepsilon$.
6.	$\alpha_1 = 2\nu$,	$\alpha_2 = -2\mu$,	$\alpha_3 = -\sigma$,	$\alpha_4 = \sigma$,	$\alpha_5 = \varepsilon$.
7.	$\alpha_1 = 0$,	$\alpha_2 = 0$,	$\alpha_3 = -\sigma$,	$\alpha_4 = \sigma$,	$\alpha_5 = \varepsilon$.

8. $$\begin{aligned}
&w^1_1 + (w^1)^2 - z_2^{-4}(w^2 - \chi)^2 + z_2 w^1 w^1_2 - w^1_{22} - \\
&- 3z_2 w^1_2 + z_2^{-1}s_2 = 0,
\end{aligned} \qquad (3.13)$$

$$w^2_1 + z_2 w^1 w^2_2 - w^2_{22} + z_2^{-1}w^2_2 = 0, \qquad (3.14)$$

$$w^3_1 + z_2 w^1 w^3_2 - w^3_{22} - z_2^{-1}w^3_2 + z_2^{-2}(w^2 - \chi) = 0, \qquad (3.15)$$

$$2w^1+z_2w^1_2+\rho_1/\rho=0. \tag{3.16}$$

9. $$\vec w_1-\lambda\vec w_{22}+s_2\vec k+\lambda^{-1}(\vec n^i\cdot\vec w)\vec m^i_t+z_2\vec e=\vec 0, \tag{3.17}$$

$$\vec k\cdot\vec w_2=0, \tag{3.18}$$

where $y_1=t$ and

$$\vec e=\vec e(t)=2\lambda^{-2}(\vec m^1_t\cdot\vec m^2-\vec m^1\cdot\vec m^2_t)\vec k_t\times\vec k+\lambda^{-2}(2\vec k_t\cdot\vec k_t-\vec k_{tt}\cdot\vec k).$$

Let us study symmetry properties of reduced systems (3.10) and (3.11).

Theorem 3.1 *The MIA of (3.10) is given by the algebra* $\langle\partial_1\rangle$.

Theorem 3.2 *The MIA of (3.11) is given by the following algebras:*

a) $\langle\partial_2,\partial_s,D^2_1=z_i\partial_i-w^a\partial_{w^a}-2s\partial_s\rangle$ *if* $\gamma=\varkappa=\varepsilon=0$;

b) $\langle\partial_2,\partial_s\rangle$ *if* $(\gamma,\varkappa,\varepsilon)\neq(0,0,0)$.

All the Lie symmetry operators of systems (3.10) and (3.11) are induced by elements of $A(NS)$. So, for system (3.10) the operator ∂_1 is induced by J_{12}. For system (3.11), when $\gamma=0$ ($\gamma=\pm3/2$), the operators D^2_1, ∂_2, and ∂_s (∂_2 and ∂_s) are induced by D, $R(0,0,1)$, and $Z(1)$ ($R(0,0,|t|^{-1/2})$ and $Z(|t|^{-1})$), accordingly. Therefore, the Lie reductions of systems (3.10) and (3.11) give only solutions that can be obtained by reducing the NSEs with three-dimensional subalgebras of $A(NS)$ immediately to ODEs.

Investigation of reduced systems (3.13)–(3.16), (3.17)–(3.18), and (3.12) is given in Sections 5 and 6.

4 Reduction of the Navier–Stokes equations to ordinary differential equations

4.1 Ansatzes of codimension three

By means of subalgebraic analysis of $A(NS)$ (see Subsection A.3) and the method described in Section B one can obtain the following ansatzes that reduce the NSEs to ODEs:

1. $$\begin{aligned}&u^1=x_1R^{-2}\varphi^1-x_2(Rr)^{-1}\varphi^2+x_1x_3r^{-1}R^{-2}\varphi^3,\\&u^2=x_2R^{-2}\varphi^1+x_1(Rr)^{-1}\varphi^2+x_2x_3r^{-1}R^{-2}\varphi^3,\\&u^3=x_3R^{-2}\varphi^1-rR^{-2}\varphi^3,\\&p=R^{-2}h,\end{aligned} \tag{4.1}$$

where $\omega=\arctan r/x_3$. Here and below $\varphi^a=\varphi^a(\omega)$, $h=h(\omega)$, $r=(x_1^2+x_2^2)^{1/2}$, $R=(x_1^2+x_2^2+x_3^2)^{1/2}$.

2. $$\begin{aligned}&u^1=r^{-2}(x_1\varphi^1-x_2\varphi^2),\quad u^2=r^{-2}(x_2\varphi^1+x_1\varphi^2),\\&u^3=r^{-1}\varphi^3,\quad p=r^{-2}h,\end{aligned} \tag{4.2}$$

where $\omega = \arctan x_2/x_1 - \varkappa \ln r$, $\varkappa \geq 0$.

$$\begin{aligned} 3.\quad & u^1 = x_1|t|^{-1}\varphi^1 - x_2 r^{-2}\varphi^2 + \tfrac{1}{2}x_1 t^{-1}, \\ & u^2 = x_2|t|^{-1}\varphi^1 + x_1 r^{-2}\varphi^2 + \tfrac{1}{2}x_2 t^{-1}, \\ & u^3 = |t|^{-1/2}\varphi^3 + (\sigma + \tfrac{1}{2})x_3 t^{-1} + \nu|t|^{1/2}t^{-1}\arctan x_2/x_1, \\ & p = |t|^{-1}h + \tfrac{1}{8}t^{-2}R^2 - \tfrac{1}{2}\sigma^2 x_3^2 t^{-2} + \\ & \qquad + \varepsilon_1|t|^{-1}\arctan x_2/x_1 + \varepsilon_2 x_3|t|^{-3/2}, \end{aligned} \tag{4.3}$$

where $\omega = |t|^{-1/2}r$, $\nu\sigma = 0$, $\varepsilon_2\sigma = 0$, $\varepsilon_1 \geq 0$, $\nu \geq 0$.

$$\begin{aligned} 4.\quad & u^1 = x_1\varphi^1 - x_2 r^{-2}\varphi^2, \\ & u^2 = x_2\varphi^1 + x_1 r^{-2}\varphi^2, \\ & u^3 = \varphi^3 + \sigma x_3 + \nu\arctan x_2/x_1, \\ & p = h - \tfrac{1}{2}\sigma^2 x_3^2 + \varepsilon_1\arctan x_2/x_1 + \varepsilon_2 x_3, \end{aligned} \tag{4.4}$$

where $\omega = r$, $\nu\sigma = 0$, $\varepsilon_2\sigma = 0$, and for $\sigma = 0$ one of the conditions

$$\nu = 1,\ \varepsilon_1 \geq 0; \quad \nu = 0,\ \varepsilon_1 = 1,\ \varepsilon_2 \geq 0; \quad \nu = \varepsilon_1 = 0,\ \varepsilon_2 \in \{0;1\}$$

is satisfied.

Two ansatzes are described better in the following way:

5. The expressions for u^a and p are determined by (2.1), where

$$\begin{aligned} & v^1 = a_1\varphi^1 + a_2\varphi^3 + b_{1i}\omega_i, \\ & v^2 = \varphi^2 + b_{2i}\omega_i, \\ & v^3 = a_2\varphi^1 - a_1\varphi^3 + b_{3i}\omega_i, \\ & p = h + c_{1i}\omega_i + c_{2i}\omega\omega_i + \tfrac{1}{2}d_{ij}\omega_i\omega_j. \end{aligned} \tag{4.5}$$

In formulas (4.5) we use the following definitions:

$$\begin{aligned} & \omega_1 = a_1 y_1 + a_2 y_3, \quad \omega_2 = y_2, \quad \omega = \omega_3 = a_2 y_1 - a_1 y_3; \\ & a_i = \text{const}, \quad a_1^2 + a_2^2 = 1; \quad a_2 = 0 \text{ if } \gamma_1 = 0; \\ & \gamma_1 = -2\varkappa,\ \gamma_2 = -\tfrac{3}{2} \quad \text{if} \quad t > 0 \quad \text{and} \quad \gamma_1 = 2\varkappa,\ \gamma_2 = \tfrac{3}{2} \quad \text{if} \quad t < 0. \end{aligned}$$

b_{ai}, B_i, c_{ij}, and d_{ij} are real constants that satisfy the equations

$$\begin{aligned} & b_{1i} = a_1 B_i, \quad b_{3i} = a_2 B_i, \quad c_{2i} + a_2\gamma_1 b_{2i} = 0, \\ & b_{21}B_i + b_{22}b_{2i} - \gamma_1 a_1 B_i + d_{2i} = 0, \\ & B_1 B_i + B_2 b_{2i} + \gamma_1 a_1 B_i + d_{1i} = 0, \\ & (B_1 + b_{22})(B_2 + a_1\gamma_1 - b_{21}) = 0. \end{aligned} \tag{4.6}$$

6. The expressions for u^a and p have form (2.2), where v^a and q are determined by (4.5), (4.6), and $\gamma_1 = -2\varkappa$, $\gamma_2 = 0$.

Note 4.1 Formulas (4.5) and (4.6) determine an ansatz for system (2.7), where equations (4.6) are the necessary and sufficient condition to reduce system (2.7) by means of an ansatz of form (4.5).

$$\begin{aligned}
7.\quad & u^1=\varphi^1\cos x_3/\eta^3-\varphi^2\sin x_3/\eta^3+x_1\theta^1(t)+x_2\theta^2(t),\\
& u^2=\varphi^1\sin x_3/\eta^3+\varphi^2\cos x_3/\eta^3-x_1\theta^2(t)+x_2\theta^1(t),\\
& u^3=\varphi^3+\eta^3_t(\eta^3)^{-1}x_3,\\
& p=h-\tfrac12\eta^3_{tt}(\eta^3)^{-1}x_3^2-\tfrac12\eta^j_{tt}\eta^j(\eta^i\eta^i)^{-1}r^2,
\end{aligned}\qquad (4.7)$$

where $\omega=t$,

$$\eta^a\in C^\infty((t_0,t_1),\mathbb{R}),\quad \eta^3\neq0,\quad \eta^i\eta^i\neq0,\quad \eta^1_t\eta^2-\eta^1\eta^2_t\in\{0;\tfrac12\},$$
$$\theta^1=\eta^i_t\eta^i(\eta^j\eta^j)^{-1},\quad \theta^2=(\eta^1_t\eta^2-\eta^1\eta^2_t)(\eta^j\eta^j)^{-1}.$$

$$\begin{aligned}
8.\quad & \vec u=\vec\varphi+\lambda^{-1}(\vec n^a\cdot\vec x)\vec m^a_t,\\
& p=h-\lambda^{-1}(\vec m^a_{tt}\cdot\vec x)(\vec n^a\cdot\vec x)+\tfrac12\lambda^{-2}(\vec m^b_{tt}\cdot\vec m^a)(\vec n^a\cdot\vec x)(\vec n^b\cdot\vec x),
\end{aligned}\qquad (4.8)$$

where $\omega=t$, $\vec m^a\in C^\infty((t_0,t_1),\mathbb{R})$, $\vec m^a_{tt}\cdot\vec m^b-\vec m^a\cdot\vec m^b_{tt}=0$,

$$\lambda=\lambda(t)=(\vec m^1\times\vec m^2)\cdot\vec m^3\neq0\quad\forall t\in(t_0,t_1),$$
$$\vec n^1=\vec m^2\times\vec m^3,\quad \vec n^2=\vec m^3\times\vec m^1,\quad \vec n^3=\vec m^1\times\vec m^2.$$

4.2 Reduced systems

Substituting the ansatzes 1–8 into the NSEs (1.1), we obtain the following systems of ODE in the functions φ^a and h:

$$\begin{aligned}
1.\quad & \varphi^3\varphi^1_\omega-\varphi^a\varphi^a-\varphi^1_{\omega\omega}-\varphi^1_\omega\cot\omega-2h=0,\\
& \varphi^3\varphi^2_\omega+\varphi^2\varphi^3\cot\omega-\varphi^2_{\omega\omega}-\varphi^2_\omega\cot\omega+\varphi^2\sin^{-2}\omega=0,\\
& \varphi^3\varphi^3_\omega-\varphi^2\varphi^2\cot\omega-\varphi^3_{\omega\omega}-\varphi^3_\omega\cot\omega+\varphi^3\sin^{-2}\omega-2\varphi^1_\omega+h_\omega=0,\\
& \varphi^1+\varphi^3_\omega+\varphi^3\cot\omega=0.
\end{aligned}\qquad (4.9)$$

$$\begin{aligned}
2.\quad & (\varphi^2-\varkappa\varphi^1)\varphi^1_\omega-(1+\varkappa^2)\varphi^1_{\omega\omega}-\varphi^1\varphi^1-\varphi^2\varphi^2-\varkappa h_\omega-2h=0,\\
& (\varphi^2-\varkappa\varphi^1)\varphi^2_\omega-(1+\varkappa^2)\varphi^2_{\omega\omega}-2(\varkappa\varphi^2_\omega+\varphi^1_\omega)+h_\omega=0,\\
& (\varphi^2-\varkappa\varphi^1)\varphi^3_\omega-(1+\varkappa^2)\varphi^3_{\omega\omega}-\varphi^1\varphi^3-\varphi^3-2\varkappa\varphi^3_\omega=0,\\
& \varphi^2_\omega-\varkappa\varphi^1_\omega=0.
\end{aligned}\qquad (4.10)$$

$$\begin{aligned}
3\text{–}4.\quad & \varphi^1\varphi^1-\omega^{-4}\varphi^2\varphi^2+\omega\varphi^1\varphi^1_\omega-\varphi^1_{\omega\omega}-3\omega^{-1}\varphi^1_\omega+\omega^{-1}h_\omega=0,\\
& \omega\varphi^1\varphi^2_\omega-\varphi^2_{\omega\omega}+\omega^{-1}\varphi^2_\omega+\varepsilon_1=0,\\
& \omega\varphi^1\varphi^3_\omega+\sigma_1\varphi^3+\nu\omega^{-2}\varphi^2-\varphi^3_{\omega\omega}-\omega^{-1}\varphi^3_\omega+\varepsilon_2=0,\\
& 2\varphi^1+\omega\varphi^1_\omega+\sigma_2=0,
\end{aligned}\qquad (4.11)$$

where

3. $\sigma_1=\sigma$, $\quad\sigma_2=(\sigma+\frac32)\quad$ if $\quad t>0$,
   $\sigma_1=-\sigma$, $\quad\sigma_2=-(\sigma+\frac32)\quad$ if $\quad t<0$.
4. $\sigma_1=\sigma_2=\sigma$.

$$\begin{aligned}
5\text{–}6.\quad & \varphi^3\varphi^1_\omega-\varphi^1_{\omega\omega}-\mu_{1i}\varphi^i+c_{11}+c_{21}\omega=0,\\
& \varphi^3\varphi^2_\omega-\varphi^2_{\omega\omega}-\mu_{2i}\varphi^i+c_{12}+c_{22}\omega+\gamma_2a_2\varphi^3=0,\\
& \varphi^3\varphi^3_\omega-\varphi^3_{\omega\omega}+\gamma_1a_2\varphi^2+h_\omega=0,\\
& \varphi^3_\omega=\sigma,
\end{aligned}\qquad (4.12)$$

where $\mu_{11}=-B_1$, $\mu_{12}=-B_2-\gamma_1 a_1$, $\mu_{21}=-b_{21}+\gamma_1 a_1$, $\mu_{22}=-b_{22}$, $\sigma=\gamma_1-B_1-b_{22}$.

7. $$\begin{aligned}&\varphi^1_\omega+\theta^1\varphi^1+\theta^2\varphi^2-(\eta^3)^{-1}\varphi^3\varphi^2+(\eta^3)^{-2}\varphi^1=0,\\&\varphi^2_\omega-\theta^2\varphi^1+\theta^1\varphi^2+(\eta^3)^{-1}\varphi^3\varphi^1+(\eta^3)^{-2}\varphi^2=0,\\&\varphi^3_\omega+\eta^3_t(\eta^3)^{-1}\varphi^3=0,\\&2\theta^1+\eta^3_t(\eta^3)^{-1}=0.\end{aligned}\tag{4.13}$$

8. $$\begin{aligned}&\vec\varphi_\omega+\lambda^{-1}(\vec n^b\cdot\vec\varphi)\vec m^b_t=0,\\&\vec n^a\cdot\vec m^a_t=0.\end{aligned}\tag{4.14}$$

4.3 Exact solutions of the reduced systems

1. Ansatz (4.1) and system (4.9) determine the class of solutions of the NSEs (1.1) that are called the steady axially symmetric conically similar flows of a viscous fluid in hydrodynamics. This class of solutions was studied in a number of works (for example, see references in [16]). For $\varphi^2=0$ it was shown, by N.A. Slezkin [34], that system (4.9) is reduced to a Riccati equation. The general solution of this equation was expressed in terms of hypergeometric functions. Later similar calculations were made by V.I. Yatseev [38] and H.B. Squire [35]. The particular case in the class of solutions with $\varphi^2=0$ is formed by the Landau jets [24]. For swirling flows, where $\varphi^2\neq 0$, the order of system (4.9) can be reduced too. For example [33], an arbitrary solution of (4.9) satisfies the equation

$$\varphi^2\varphi^2\sin^2\omega-\sin\omega(\Phi_\omega\sin^{-1}\omega)_\omega+2\Phi_\omega\cot\omega+2\Phi=\mathrm{const},$$

where $\Phi=(\varphi^3_\omega-\frac12\varphi^3\varphi^3)\sin^2\omega-\varphi^3\cos\omega\sin\omega$, and the Yatseev results [38] are completely extended to the case $\varphi^2\sin\omega=\mathrm{const}$.

2. System (4.10) implies that

$$\begin{aligned}&\varphi^2=\varkappa\varphi^1+C_1,\\&h=\varkappa(1+\varkappa^2)\varphi^1_\omega+(2\varkappa^2+2-\varkappa C_1)\varphi^1+C_2,\\&(1+\varkappa^2)\varphi^1_{\omega\omega}+(4\varkappa-C_1)\varphi^1_\omega+\varphi^1\varphi^1+4\varphi^1+\\&\ +(1+\varkappa^2)^{-1}(C_1^2+2C_2)=0,\\&(1+\varkappa^2)\varphi^3_{\omega\omega}-(C_1-2\varkappa)\varphi^3_\omega+(1+\varphi^1)\varphi^3=0.\end{aligned}\tag{4.15}$$

If $\varphi^3=0$, the solution determined by ansatz (4.10) and formulas (4.15) coincides with the Hamel solution [18, 23]. In Section 6 we consider system (6.14) which is more general than system (4.10).

3–4. Let us integrate the last equation of system (4.11), i.e.,

$$\varphi^1=C_1\omega^{-2}-\tfrac12\sigma_2.\tag{4.16}$$

Taking into account the integration result, the other equations of system (4.11) can be written in the form

$$h_\omega=\omega^{-3}\varphi^2\varphi^2+C_1^2\omega^{-3}-\tfrac14\sigma_2^2\omega,$$

$$\varphi^2_{\omega\omega}-((C_1+1)\omega^{-1}-\tfrac12\sigma_2\omega)\varphi^2_\omega=\varepsilon_1,$$

$$\varphi^3_{\omega\omega} - ((C_1-1)\omega^{-1} - \tfrac{1}{2}\sigma_2\omega)\varphi^3_\omega - \sigma_1\varphi^3 = \nu\omega^{-2}\varphi^2 + \varepsilon_2. \tag{4.17}$$

Therefore,

$$h = \int \omega^{-3}\varphi^2\varphi^2 d\omega - \tfrac{1}{2}C_1^2\omega^{-2} - \tfrac{1}{8}\sigma_2^2\omega^2, \tag{4.18}$$

$$\begin{aligned}\varphi^2 = {} & C_2 + C_3\int |\omega|^{C_1+1}e^{-\frac{1}{4}\sigma_2\omega^2}d\omega + \\ & + \varepsilon_1\int |\omega|^{C_1+1}e^{-\frac{1}{4}\sigma_2\omega^2}\Big(\int |\omega|^{-C_1-1}e^{\frac{1}{4}\sigma_2\omega^2}d\omega\Big)d\omega.\end{aligned} \tag{4.19}$$

If $\sigma_1 = 0$, it follows that

$$\begin{aligned}\varphi^3 = {} & C_4 + C_5\int |\omega|^{C_1-1}e^{-\frac{1}{4}\sigma_2\omega^2}d\omega + \\ & + \int |\omega|^{C_1-1}e^{-\frac{1}{4}\sigma_2\omega^2}\Big(\int |\omega|^{-C_1+1}e^{\frac{1}{4}\sigma_2\omega^2}(\varepsilon_2 + \nu\omega^{-2}\varphi^2)d\omega\Big)d\omega.\end{aligned} \tag{4.20}$$

Let $\sigma_1 \neq 0$ (and, therefore, $\nu = 0$). Then, if $\sigma_2 \neq 0$, the general solution of equation (4.17) is expressed in terms of Whittaker functions:

$$\varphi^3 = |\omega|^{\frac{1}{2}C_1-1}e^{-\frac{1}{8}\sigma_2\omega^2}W(-\sigma_1\sigma_2^{-1} + \tfrac{1}{4}C_1 - \tfrac{1}{2}, \tfrac{1}{4}C_1, \tfrac{1}{4}\sigma_2\omega^2),$$

where $W(\varkappa,\mu,\tau)$ is the general solution of the Whittaker equation

$$4\tau^2 W_{\tau\tau} = (\tau^2 - 4\varkappa\tau + 4\mu^2 - 1)W. \tag{4.21}$$

If $\sigma_2 = 0$, the general solution of equation (4.16) is expressed in terms of Bessel functions:

$$\varphi^3 = |\omega|^{\frac{1}{2}C_1}Z_{\frac{1}{2}C_1}\big((-\sigma_1)^{1/2}\omega\big),$$

where $Z_\nu(\tau)$ is the general solution of the Bessel equation

$$\tau^2 Z_{\tau\tau} + \tau Z_\tau + (\tau^2 - \nu^2)Z = 0. \tag{4.22}$$

Note 4.2 If $\sigma_2 = 0$, all quadratures in formulas (4.18)–(4.20) are easily integrated. For example,

$$\varphi^2 = \begin{cases} C_2 + C_3\ln|\omega| + \frac{1}{4}\varepsilon_1\omega^2 & \text{if} \quad C_1 = -2, \\ C_2 + C_3\frac{1}{2}\omega^2 + \frac{1}{2}\varepsilon_1\omega^2(\ln\omega - \frac{1}{2}) & \text{if} \quad C_1 = 0, \\ C_2 + C_3(C_1+2)^{-1}|\omega|^{C_1+2} - \frac{1}{2}\varepsilon_1 C_1^{-1}\omega^2 & \text{if} \quad C_1 \neq -2, 0. \end{cases}$$

5–6. Let $\sigma = 0$. Then the last equation of system (4.12) implies that $\varphi^3 = C_0 =$ const. The other equations of system (4.12) can be written in the form

$$\begin{aligned} & h = -\gamma_1 a_2\int \varphi^2(\omega)d\omega, \\ & \varphi^i_{\omega\omega} - C_0\varphi^i_\omega + \mu_{ij}\varphi^j = \nu_{1i} + \nu_{2i}\omega, \end{aligned} \tag{4.23}$$

where $\nu_{11} = c_{11}$, $\nu_{21} = c_{21}$, $\nu_{12} = c_{12} + \gamma_2 a_2 C_0$, $\nu_{22} = c_{22}$. System (4.23) is a linear nonhomogeneous system of ODEs with constant coefficients. The form of its general solution depends on the Jordan form of the matrix $\mathrm{M} = \{\mu_{ij}\}$. Now let us transform the dependent variables

$$\varphi^i = e_{ij}\psi^j,$$

where the constants e_{ij} are determined by means of the system of linear algebraic equations

$$e_{ij}\tilde{\mu}_{jk} = \mu_{ij}e_{jk} \quad (i,j,k=1,2)$$

with the condition $\det\{e_{ij}\} \neq 0$. Here $\tilde{\mathrm{M}} = \{\tilde{\mu}_{ij}\}$ is the real Jordan form of the matrix M. The new unknown functions ψ^i have to satisfy the following system

$$\psi^i_{\omega\omega} - C_0\psi^i_\omega + \tilde{\mu}_{ij}\psi^j = \tilde{\nu}_{1i} + \tilde{\nu}_{2i}\omega, \tag{4.24}$$

where $\nu_{1i} = e_{ij}\tilde{\nu}_{1j}$, $\nu_{2i} = e_{ij}\tilde{\nu}_{2j}$. Depending on the form of $\tilde{\mathrm{M}}$, we consider the following cases:

A. $\det\tilde{\mathrm{M}} = 0$ (this is equivalent to the condition $\det\mathrm{M} = 0$).

i. $\tilde{M} = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$, where $\varepsilon \in \{0;1\}$. Then

$$\begin{aligned}
\psi^2 &= C_1 + C_2e^{C_0\omega} - \tfrac{1}{2}\tilde{\nu}_{22}C_0^{-1}\omega^2 - (\tilde{\nu}_{12} - \tilde{\nu}_{22}C_0^{-1})C_0^{-1}\omega,\\
\psi^1 &= C_3 + C_4e^{C_0\omega} - \tfrac{1}{2}\tilde{\nu}_{21}C_0^{-1}\omega^2 - (\tilde{\nu}_{11} - \tilde{\nu}_{21}C_0^{-1})C_0^{-1}\omega +\\
&\quad + \varepsilon\Big(-\tfrac{1}{6}\tilde{\nu}_{22}C_0^{-2}\omega^3 - \tfrac{1}{2}(\tilde{\nu}_{12} - 2\tilde{\nu}_{22}C_0^{-1})C_0^{-2}\omega^2 +\\
&\quad + \big(C_1 + (\tilde{\nu}_{21} - 2\tilde{\nu}_{22}C_0^{-1})C_0^{-2}\big)C_0^{-1}\omega - C_2C_0^{-1}\omega e^{C_0\omega}\Big)
\end{aligned} \tag{4.25}$$

for $C_0 \neq 0$, and

$$\begin{aligned}
\psi^2 &= C_1 + C_2\omega + \tfrac{1}{6}\tilde{\nu}_{22}\omega^3 + \tfrac{1}{2}\tilde{\nu}_{12}\omega^2,\\
\psi^1 &= C_3 + C_4\omega + \tfrac{1}{6}(\tilde{\nu}_{21} - C_2)\omega^3 + \tfrac{1}{2}(\tilde{\nu}_{11} - C_1)\omega^2 - \tfrac{1}{120}\tilde{\nu}_{22}\omega^5 - \tfrac{1}{24}\tilde{\nu}_{12}\omega^4
\end{aligned} \tag{4.26}$$

for $C_0 = 0$.

ii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & 0 \end{pmatrix}$, where $\varkappa_1 \in \mathbb{R}\backslash\{0\}$. Then the form of ψ^2 is given either by formula (4.25) for $C_0 \neq 0$ or by formula (4.26) for $C_0 = 0$. The form of ψ^1 is given by formula (4.28) (see below).

B. $\det\tilde{\mathrm{M}} \neq 0$ (this is equivalent to the condition $\det\mathrm{M} \neq 0$).

i. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & \varkappa_2 \end{pmatrix}$, where $\varkappa_i \in \mathbb{R}\backslash\{0\}$. Then

$$\psi^2 = \tilde{\nu}_{22}\varkappa_2^{-1}\omega + (\tilde{\nu}_{12} - C_0\tilde{\nu}_{22}\varkappa_2^{-1})\varkappa_2^{-1} + C_1\theta^{21}(\omega) + C_2\theta^{22}(\omega), \tag{4.27}$$

$$\psi^1 = \tilde{\nu}_{21}\varkappa_1^{-1}\omega + (\tilde{\nu}_{11} - C_0\tilde{\nu}_{21}\varkappa_1^{-1})\varkappa_1^{-1} + C_3\theta^{11}(\omega) + C_4\theta^{12}(\omega), \tag{4.28}$$

where

$$\theta^{i1}(\omega) = \exp\big(\tfrac{1}{2}(C_0 - \sqrt{D_i})\omega\big), \quad \theta^{i2}(\omega) = \exp\big(\tfrac{1}{2}(C_0 + \sqrt{D_i})\omega\big)$$

if $D_i = C_0^2 - 4\varkappa_i > 0$,

$$\theta^{i1}(\omega) = e^{\frac{1}{2}C_0\omega}\cos\big(\tfrac{1}{2}\sqrt{-D_i}\omega\big), \quad \theta^{i2}(\omega) = e^{\frac{1}{2}C_0\omega}\sin\big(\tfrac{1}{2}\sqrt{-D_i}\omega\big)$$

if $D_i < 0$,

$$\theta^{i1}(\omega) = e^{\frac{1}{2}C_0\omega}, \qquad \theta^{i2}(\omega) = \omega e^{\frac{1}{2}C_0\omega}$$

if $D_i = 0$.

ii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_2 & 1 \\ 0 & \varkappa_2 \end{pmatrix}$, where $\varkappa_2 \in \mathbb{R}\backslash\{0\}$. Then the form of ψ^2 is given by formula (4.27), and

$$\begin{aligned} \psi^1 = {} & \big(\tilde\nu_{11} - (\tilde\nu_{12} - C_0\tilde\nu_{22}\varkappa_2^{-1})\varkappa_2^{-1} - C_0(\tilde\nu_{21} - \tilde\nu_{22}\varkappa_2^{-1})\varkappa_2^{-1}\big)\varkappa_2^{-1} + \\ & + (\tilde\nu_{21} - \tilde\nu_{22}\varkappa_2^{-1})\varkappa_2^{-1}\omega + C_3\theta^{21}(\omega) + C_4\theta^{22}(\omega) - C_i\eta^i(\omega), \end{aligned}$$

where

$$\begin{aligned} &\eta^j(\omega) = D_2^{-1}\omega\big(2\theta^{2j}_\omega(\omega) - C_0\theta^{2j}(\omega)\big) && \text{if} \quad D_2 \neq 0, \\ &\eta^1(\omega) = \tfrac12\omega^2 e^{\frac12 C_0\omega}, \quad \eta^2(\omega) = \tfrac16\omega^3 e^{\frac12 C_0\omega} && \text{if} \quad D_2 = 0. \end{aligned}$$

iii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & -\varkappa_2 \\ \varkappa_2 & \varkappa_1 \end{pmatrix}$, where $\varkappa_i \in \mathbb{R}$, $\varkappa_2 \neq 0$. Then

$$\begin{aligned} \psi^1 = {} & (\varkappa_i\varkappa_i)^{-1}(\tilde\nu_{21}\varkappa_1 + \tilde\nu_{22}\varkappa_2)\omega + (\varkappa_i\varkappa_i)^{-1}(\tilde\nu_{11}\varkappa_1 + \tilde\nu_{12}\varkappa_2) - \\ & - C_0(\varkappa_i\varkappa_i)^{-2}\big(\tilde\nu_{21}(\varkappa_2^2 - \varkappa_1^2) - \tilde\nu_{22}2\varkappa_1\varkappa_2\big) + C_n\theta^{1n}(\omega), \\ \psi^2 = {} & (\varkappa_i\varkappa_i)^{-1}(-\tilde\nu_{21}\varkappa_2 + \tilde\nu_{22}\varkappa_1)\omega + (\varkappa_i\varkappa_i)^{-1}(-\tilde\nu_{11}\varkappa_2 + \tilde\nu_{12}\varkappa_1) - \\ & - C_0(\varkappa_i\varkappa_i)^{-2}\big(\tilde\nu_{21}2\varkappa_1\varkappa_2 + \tilde\nu_{22}(\varkappa_2^2 - \varkappa_1^2)\big) + C_n\theta^{2n}(\omega), \end{aligned}$$

where $n = \overline{1,4}$,

$$\begin{aligned} &\gamma = \sqrt{(C_0^2 - 4\varkappa_1)^2 + (4\varkappa_2)^2}, \\ &\beta_1 = \tfrac14\sqrt{2(\gamma + C_0^2 - 4\varkappa_1)}, \quad \beta_2 = \tfrac14\tfrac{|\varkappa_2|}{\varkappa_2}\sqrt{2(\gamma - C_0^2 + 4\varkappa_1)}, \\ &\theta^{11}(\omega) = \theta^{22}(\omega) = \exp\big((\tfrac12 C_0 - \beta_1)\omega\big)\cos\beta_2\omega, \\ &-\theta^{21}(\omega) = \theta^{12}(\omega) = \exp\big((\tfrac12 C_0 - \beta_1)\omega\big)\sin\beta_2\omega, \\ &\theta^{13}(\omega) = \theta^{24}(\omega) = \exp\big((\tfrac12 C_0 + \beta_1)\omega\big)\cos\beta_2\omega, \\ &\theta^{23}(\omega) = -\theta^{14}(\omega) = \exp\big((\tfrac12 C_0 + \beta_1)\omega\big)\sin\beta_2\omega. \end{aligned}$$

If $\sigma \neq 0$, the last equation of system (4.12) implies that $\psi^3 = \sigma\omega$ (translating ω, the integration constant can be made to vanish). The other equations of system (4.12) can be written in the form

$$\begin{aligned} &h = -\gamma_1 a_2 \textstyle\int \varphi^2(\omega)d\omega - \tfrac12\sigma^2\omega^2, \\ &\varphi^i_{\omega\omega} - \sigma\omega\varphi^i_\omega + \mu_{ij}\varphi^j = \nu_{1i} + \nu_{2i}\omega, \end{aligned} \tag{4.29}$$

where $\nu_{11} = c_{11}$, $\nu_{21} = c_{21}$, $\nu_{12} = c_{12}$, $\nu_{22} = c_{22} + \gamma_2 a_2\sigma$. The form of the general solution of system (4.29) depends on the Jordan form of the matrix $\mathrm{M} = \{\mu_{ij}\}$. Now, let us transform the dependent variables

$$\varphi^i = e_{ij}\psi^j,$$

where the constants e_{ij} are determined by means of the system of linear algebraic equations

$$e_{ij}\tilde\mu_{jk} = \mu_{ij}e_{jk} \quad (i,j,k = 1,2)$$

with the condition $\det\{e_{ij}\} \neq 0$. Here $\tilde{\mathrm{M}} = \{\tilde\mu_{ij}\}$ is the real Jordan form of the matrix M. The new unknown functions ψ^i have to satisfy the following system

$$\psi^i_{\omega\omega} - \sigma\omega\psi^i_\omega + \tilde\mu_{ij}\psi^j = \tilde\nu_{1i} + \tilde\nu_{2i}\omega, \tag{4.30}$$

where $\nu_{1i} = e_{ij}\tilde{\nu}_{1j}$, $\nu_{2i} = e_{ij}\tilde{\nu}_{2j}$. Depending on the form of $\tilde{\mathrm{M}}$, we consider the following cases:

A. $\det\tilde{\mathrm{M}} = 0$ (this is equivalent to the condition $\det\mathrm{M} = 0$).

i. $\tilde{M} = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$, where $\varepsilon \in \{0;1\}$. Then

$$\psi^2 = C_1 + C_2\int e^{\frac12\sigma\omega^2}d\omega - \sigma^{-1}\tilde{\nu}_{22}\omega + \tilde{\nu}_{12}\int e^{\frac12\sigma\omega^2}\bigl(\int e^{-\frac12\sigma\omega^2}d\omega\bigr)d\omega, \qquad (4.31)$$

$$\psi^1 = C_3 + C_4\int e^{\frac12\sigma\omega^2}d\omega - \sigma^{-1}\tilde{\nu}_{21}\omega + \int e^{\frac12\sigma\omega^2}\bigl(\int e^{-\frac12\sigma\omega^2}(\tilde{\nu}_{11} - \varepsilon\psi^2)d\omega\bigr)d\omega.$$

ii. $\tilde{\mathrm{M}} = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$. Then the form of ψ^2 is given by formula (4.31), and

$$\psi^1 = C_3\omega + C_4\bigl(\omega\int e^{\frac12\sigma\omega^2}d\omega - \sigma^{-1}e^{\frac12\sigma\omega^2}\bigr) + \sigma^{-1}\tilde{\nu}_{11} + \\ + \sigma^{-1}\tilde{\nu}_{21}\bigl(\sigma\omega\int e^{\frac12\sigma\omega^2}\lambda^1(\omega)d\omega - e^{\frac12\sigma\omega^2}\lambda^1(\omega)\bigr),$$

where $\lambda^1(\omega) = \int e^{-\frac12\sigma\omega^2}d\omega$.

iii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & 0 \end{pmatrix}$, where $\varkappa_1 \in \mathbb{R}\backslash\{0;\sigma\}$. Then ψ^2 is determited by (4.31), and the form of ψ^1 is given by (4.33) (see below).

B. $\det\tilde{\mathrm{M}} \neq 0$, $\det\{\tilde{\mu}_{ij} - \sigma\delta_{ij}\} = 0$ (this is equivalent to the conditions $\det\mathrm{M} \neq 0$, $\det\{\mu_{ij} - \sigma\delta_{ij}\} = 0$; here δ_{ij} is the Kronecker symbol).

i. $\tilde{\mathrm{M}} = \begin{pmatrix} \sigma & \varepsilon \\ 0 & \sigma \end{pmatrix}$, where $\varepsilon \in \{0;1\}$. Then

$$\psi^2 = C_1\omega + C_2\bigl(\omega\int e^{\frac12\sigma\omega^2}d\omega - \sigma^{-1}e^{\frac12\sigma\omega^2}\bigr) + \sigma^{-1}\tilde{\nu}_{12} + vspace1mm \\ + \sigma^{-1}\tilde{\nu}_{22}\bigl(\sigma\omega\int e^{\frac12\sigma\omega^2}\lambda^1(\omega)d\omega - e^{\frac12\sigma\omega^2}\lambda^1(\omega)\bigr), \qquad (4.32)$$

$$\psi^1 = C_3\omega + C_4\bigl(\omega\int e^{\frac12\sigma\omega^2}d\omega - \sigma^{-1}e^{\frac12\sigma\omega^2}\bigr) + \sigma^{-1}\tilde{\nu}_{11} + \\ + \sigma\omega\int e^{\frac12\sigma\omega^2}\lambda^2(\omega)d\omega - e^{\frac12\sigma\omega^2}\lambda^2(\omega) + \sigma^{-1}(\tilde{\nu}_{21}\omega - \varepsilon\psi^2),$$

where $\lambda^1(\omega) = \int e^{-\frac12\sigma\omega^2}d\omega$, $\lambda^2(\omega) = \sigma^{-1}\int e^{-\frac12\sigma\omega^2}(\tilde{\nu}_{21} - \varepsilon\psi^2_\omega)d\omega$.

ii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & \sigma \end{pmatrix}$, where $\varkappa_1 \in \mathbb{R}\backslash\{0;\sigma\}$. In this case ψ^2 is determined by (4.32), and the form of ψ^1 is given by (4.33) (see below).

C. $\det\tilde{\mathrm{M}} \neq 0$, $\det\{\tilde{\mu}_{ij} - \sigma\delta_{ij}\} \neq 0$ (this is equivalent to the condition $\det\mathrm{M} \neq 0$, $\det\{\mu_{ij} - \sigma\delta_{ij}\} \neq 0$: here δ_{ij} is the Kronecker symbol).

i. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & \varkappa_2 \end{pmatrix}$, where $\varkappa_i \in \mathbb{R}\backslash\{0;\sigma\}$. Then

$$\psi^1 = \varkappa_1^{-1}\tilde{\nu}_{11} + (\varkappa_1 - \sigma)^{-1}\tilde{\nu}_{21}\omega + |\omega|^{-1/2}e^{\frac14\sigma\omega^2} \times \\ \times\Bigl(C_3M\bigl(\tfrac12\varkappa_1\sigma^{-1} + \tfrac14, \tfrac14, \tfrac12\sigma\omega^2\bigr) + C_4M\bigl(\tfrac12\varkappa_1\sigma^{-1} + \tfrac14, -\tfrac14, \tfrac12\sigma\omega^2\bigr)\Bigr), \qquad (4.33)$$

$$\psi^2 = \varkappa_2^{-1}\tilde\nu_{12} + (\varkappa_2-\sigma)^{-1}\tilde\nu_{22}\omega + |\omega|^{-1/2}e^{\frac14\sigma\omega^2}\times \\ \times\Big(C_1M\big(\tfrac12\varkappa_2\sigma^{-1}+\tfrac14,\tfrac14,\tfrac12\sigma\omega^2\big) + C_2M\big(\tfrac12\varkappa_2\sigma^{-1}+\tfrac14,-\tfrac14,\tfrac12\sigma\omega^2\big)\Big), \tag{4.34}$$

where $M(\varkappa,\mu,\tau)$ is the Whittaker function:

$$M(\varkappa,\mu,\tau) = \tau^{\frac12+\mu}e^{-\frac12\tau}{}_1F_1\big(\tfrac12+\mu-\varkappa, 2\mu+1,\tau\big), \tag{4.35}$$

and ${}_1F_1(a,b,\tau)$ is the degenerate hypergeometric function defined by means of the series:

$${}_1F_1(a,b,\tau) = 1 + \sum_{n=1}^{\infty}\frac{a(a+1)\dots(a+n-1)}{b(b+1)\dots(b+n-1)}\frac{\tau^n}{n!},$$

$b\neq 0,-1,-2,\dots$.

ii. $\tilde{\mathrm M} = \begin{pmatrix}\varkappa_1 & -\varkappa_2\\ \varkappa_2 & \varkappa_1\end{pmatrix}$, where $\varkappa_i\in\mathbb R$, $\varkappa_2\neq 0$. Then

$$\psi^1 = (\varkappa_j\varkappa_j)^{-1}(\varkappa_1\tilde\nu_{11}+\varkappa_2\tilde\nu_{12}) + ((\varkappa_1-\sigma)^2+\varkappa_2^2)^{-1}((\varkappa_1-\sigma)\tilde\nu_{21}+\varkappa_2\tilde\nu_{22})\omega + \\ + C_1\mathrm{Re}\,\eta^1(\omega) - C_2\mathrm{Im}\,\eta^1(\omega) + C_3\mathrm{Re}\,\eta^2(\omega) - C_4\mathrm{Im}\,\eta^2(\omega),$$

$$\psi^2 = (\varkappa_j\varkappa_j)^{-1}(-\varkappa_2\tilde\nu_{11}+\varkappa_1\tilde\nu_{12}) + \\ + ((\varkappa_1-\sigma)^2+\varkappa_2^2)^{-1}(-\varkappa_2\tilde\nu_{21}+(\varkappa_1-\sigma)\tilde\nu_{22})\omega + \\ + C_1\mathrm{Im}\,\eta^1(\omega) + C_2\mathrm{Re}\,\eta^1(\omega) + C_3\mathrm{Im}\,\eta^2(\omega) + C_4\mathrm{Re}\,\eta^2(\omega),$$

where

$$\eta^1(\omega) = M\big(\tfrac12(\varkappa_1+\varkappa_2\mathrm i)\sigma^{-1}+\tfrac14,\tfrac14,\tfrac12\sigma\omega^2\big), \\ \eta^2(\omega) = M\big(\tfrac12(\varkappa_1+\varkappa_2\mathrm i)\sigma^{-1}+\tfrac14,-\tfrac14,\tfrac12\sigma\omega^2\big), \quad \mathrm i^2=-1.$$

iii. $\tilde{\mathrm M} = \begin{pmatrix}\varkappa_2 & 1\\ 0 & \varkappa_2\end{pmatrix}$, where $\varkappa_2\in\mathbb R\backslash\{0;\sigma\}$. Here the form of ψ^2 is given by (4.34), and

$$\psi^1 = (\tilde\nu_{11}-\tilde\nu_{12}\varkappa_2^{-1})\varkappa_2^{-1} + \big(\tilde\nu_{21}-\tilde\nu_{22}(\varkappa_2-\sigma)^{-1}\big)(\varkappa_2-\sigma)^{-1}\omega + \\ + |\omega|^{-1/2}e^{\frac14\sigma\omega^2}\Big(C_3\theta^1(\tau)+C_4\theta^2(\tau) - \sigma^{-1}\theta^1(\tau)\int\tau^{-1}\theta^2(\tau)C_i\theta^i(\tau)d\tau + \\ + \sigma^{-1}\theta^2(\tau)\int\tau^{-1}\theta^1(\tau)C_i\theta^i(\tau)d\tau\Big),$$

where $\tau=\frac12\sigma\omega^2$,

$$\theta^1(\tau) = M\big(\tfrac12\varkappa_2\sigma^{-1}+\tfrac14,\tfrac14,\tau\big), \quad \theta^2(\tau) = M\big(\tfrac12\varkappa_2\sigma^{-1}+\tfrac14,-\tfrac14,\tau\big).$$

Note 4.3 The general solution of the equation

$$\psi_{\omega\omega} - \sigma\omega\psi_\omega - (n+1)\sigma\psi = 0,$$

where n is an integer and $n\geq 0$, is determined by the formula

$$\psi = \left(\frac{d^n}{d\omega^n}e^{\frac12\sigma\omega^2}\right)\left(C_1 + C_2\int e^{\frac12\sigma\omega^2}\left(\frac{d^n}{d\omega^n}e^{\frac12\sigma\omega^2}\right)^{-2}d\omega\right).$$

Note 4.4 If function ψ satisfies the equation

$$\psi_{\omega\omega} - \sigma\omega\psi_\omega + \varkappa\psi = 0 \quad (\varkappa \neq -\sigma),$$

then $\int \psi(\omega)d\omega = (\varkappa + \sigma)^{-1}(\sigma\omega\psi - \psi_\omega) + C_1$.

7. The last equation of system (4.13) is the compatibility condition of the NSEs (1.1) and ansatz (4.7). Integrating this equation, we obtain that

$$\eta^3 = C_0(\eta^i\eta^i)^{-1}, \quad C_0 \neq 0.$$

As $\varphi^3_\omega = -\eta^3_\omega(\eta^3)^{-1}\varphi^3 = 2\theta^1\varphi^3$, $\varphi^3 = C_3\eta^i\eta^i$. Then system (4.13) is reduced to the equations

$$\begin{aligned} &\varphi^1_\omega = \chi^1(\omega)\varphi^1 - \chi^2(\omega)\varphi^2, \\ &\varphi^2_\omega = \chi^2(\omega)\varphi^1 + \chi^1(\omega)\varphi^2, \end{aligned} \tag{4.36}$$

where $\chi^1 = -C_0^{-2}(\eta^i\eta^i)^2 - \theta^1$ and $\chi^2 = \theta^2 - C_3C_0^{-1}(\eta^i\eta^i)^2$. System (4.36) implies that

$$\begin{aligned} &\varphi^1 = \exp\bigl(\textstyle\int \chi^1(\omega)d\omega\bigr)\Bigl(C_1\cos\bigl(\int \chi^2(\omega)d\omega\bigr) - C_2\sin\bigl(\int \chi^2(\omega)d\omega\bigr)\Bigr), \\ &\varphi^2 = \exp\bigl(\textstyle\int \chi^1(\omega)d\omega\bigr)\Bigl(C_1\sin\bigl(\int \chi^2(\omega)d\omega\bigr) + C_2\cos\bigl(\int \chi^2(\omega)d\omega\bigr)\Bigr). \end{aligned}$$

8. Let us apply the trasformation generated by the operator $R(\vec{k}(t))$, where

$$\vec{k}_t = \lambda^{-1}(\vec{n}^b\cdot\vec{k})\vec{m}^b_t - \vec{\varphi},$$

to ansatz (4.8). As a result we obtain an ansatz of the same form, where the functions $\vec{\varphi}$ and h are replaced by the new functions $\tilde{\vec{\varphi}}$ and $\tilde{h}$:

$$\begin{aligned} &\tilde{\vec{\varphi}} = \vec{\varphi} - \lambda^{-1}(\vec{n}^a\cdot\vec{k})\vec{m}^a_t + \vec{k}_t = 0, \\ &\tilde{h} = h - \lambda^{-1}(\vec{m}^a_{tt}\cdot\vec{k})(\vec{n}^a\cdot\vec{k}) + \tfrac{1}{2}\lambda^{-2}(\vec{m}^b_{tt}\cdot\vec{m}^a)(\vec{n}^a\cdot\vec{k})(\vec{n}^b\cdot\vec{k}). \end{aligned}$$

Let us make $\tilde{h}$ vanish by means of the transformation generated by the operator $Z(-\tilde{h}(t))$. Therefore, the functions φ^a and h can be considered to vanish. The equation $(\vec{n}^a\cdot\vec{m}^a_t) = 0$ is the compatibility condition of ansatz (4.8) and the NSEs (1.1).

Note 4.5 The solutions of the NSEs obtained by means of ansatzes 5–8 are equivalent to either solutions (5.1) or solutions (5.5).

5 Reduction of the Navier–Stokes equations to linear systems of PDEs

Let us show that non-linear systems 8 and 9, from Subsection 3.2, are reduced to linear systems of PDEs.

5.1 Investigation of system (3.17)–(3.18)

Consider system 9 from Subsection 3.2, i.e., equations (3.17) and (3.18). Equation (3.18) integrates with respect to z_2 to the following expression:

$$\vec{k}\cdot\vec{w}=\psi(t).$$

Here $\psi=\psi(t)$ is an arbitrary smooth function of $z_1=t$. Let us make the transformation from the symmetry group of the NSEs:

$$\tilde{\vec{u}}(t,\vec{x})=\vec{u}(t,\vec{x}-\vec{l})+\vec{l}_t(t),$$
$$\tilde{p}(t,\vec{x})=p(t,\vec{x}-\vec{l})-\vec{l}_{tt}(t)\cdot\vec{x},$$

where $\vec{l}_{tt}\cdot\vec{m}^i-\vec{l}\cdot\vec{m}^i_{tt}=0$ and

$$\vec{k}\cdot\big(\vec{l}_t-\lambda^{-1}(\vec{n}^i\cdot\vec{l})m^i_t+\lambda^{-1}(\vec{k}\cdot\vec{l})\vec{k}_t\big)+\psi=0.$$

This transformation does not modify ansatz (3.9), but it makes the function $\psi(t)$ vanish, i.e., $\vec{k}\cdot\tilde{\vec{w}}=0$. Therefore, without loss of generality we may assume, at once, that $\vec{k}\cdot\vec{w}=0$.

Let $f^i=f^i(z_1,z_2)=\vec{m}^i\cdot\vec{w}$. Since $\vec{m}^1_{tt}\cdot\vec{m}^2-\vec{m}^1\cdot\vec{m}^2_{tt}=0$, it follows that $\vec{m}^1_t\cdot\vec{m}^2-\vec{m}^1\cdot\vec{m}^2_t=C=\mathrm{const}$. Let us multiply the scalar equation (3.17) by $\vec{m}^i$ and $\vec{k}$. As a result we obtain the linear system of PDEs with variable coefficients in the functions f^i and s:

$$f^i_1-\lambda f^i_{22}+C\lambda^{-1}\big((\vec{m}^i\cdot\vec{m}^2)f^1-(\vec{m}^i\cdot\vec{m}^1)f^2\big)-2C\lambda^{-2}\big((\vec{k}\times\vec{k}_t)\cdot\vec{m}^i\big)z_2=0,$$
$$s_2=2\lambda^{-2}(\vec{n}^i\cdot\vec{k}_t)f^i+\lambda^{-2}(\vec{k}_{tt}\cdot\vec{k}-2\vec{k}_t\cdot\vec{k}_t)z_2.$$

Consider two possible cases.

A. Let $C=0$. Then there exist functions $g^i=g^i(\tau,\omega)$, where $\tau=\int\lambda(t)dt$ and $\omega=z_2$, such that $f^i=g^i_\omega$ and $g^i_\tau-g^i_{\omega\omega}=0$. Therefore,

$$\begin{aligned}&\vec{u}=\lambda^{-1}(g^i_\omega(\tau,\omega)+\vec{m}^i_t\cdot\vec{x})\vec{n}^i-\lambda^{-1}(\vec{k}_t\cdot\vec{x})k,\\&p=2\lambda^{-2}(\vec{n}^i\cdot\vec{k}_t)g^i(\tau,\omega)+\tfrac12\lambda^{-2}(\vec{k}_{tt}\cdot\vec{k}-2\vec{k}_t\cdot\vec{k}_t)\omega^2-\\&\quad-\tfrac12\lambda^{-1}(\vec{n}^i\cdot\vec{x})(\vec{m}^i_{tt}\cdot\vec{x})-\tfrac12\lambda^{-2}(\vec{k}\cdot\vec{m}^i_{tt})(\vec{n}^i\cdot\vec{x})(\vec{k}\cdot\vec{x}),\end{aligned}\tag{5.1}$$

where $\vec{m}^1_t\cdot\vec{m}^2-\vec{m}^1\cdot\vec{m}^2_t=0$, $\vec{k}=\vec{m}^1\times\vec{m}^2$, $\vec{n}^1=\vec{m}^2\times\vec{k}$, $\vec{n}^2=\vec{k}\times\vec{m}^1$, $\lambda=|\vec{k}|^2$, $\omega=\vec{k}\cdot\vec{x}$, $\tau=\int\lambda(t)dt$, and $g^i_\tau-g^i_{\omega\omega}=0$.

For example, if $\vec{m}=(\eta^1(t),0,0)$ and $\vec{n}=(0,\eta^2(t),0)$ with $\eta^i(t)\neq0$, it follows that

$$\begin{aligned}&u^1=(\eta^1)^{-1}(f^1+\eta^1_tx_1),\quad u^2=(\eta^2)^{-1}(f^2+\eta^2_tx_2),\quad u^3=-(\eta^1\eta^2)_t(\eta^1\eta^2)^{-1}x_3,\\&p=-\tfrac12\eta^1_{tt}(\eta^1)^{-1}x_1^2-\tfrac12\eta^2_{tt}(\eta^2)^{-1}x_2^2+\\&\quad+\Big(\tfrac12(\eta^1\eta^2)_{tt}(\eta^1\eta^2)^{-1}-\big((\eta^1\eta^2)_t(\eta^1\eta^2)^{-1}\big)^2\Big)x_3^2,\end{aligned}$$

where $f^i=f^i(\tau,\omega)$, $f^i_\tau-f^i_{\omega\omega}=0$, $\tau=\int(\eta^1\eta^2)^2dt$, and $\omega=\eta^1\eta^2x_3$. If $\vec{m}^1=(\eta^1(t),\eta^2(t),0)$ and $\vec{m}^2=(0,0,\eta^3(t))$ with $\eta^3(t)\neq0$ and $\eta^i(t)\eta^i(t)\neq0$, we obtain that

$$
\begin{aligned}
&u^1 = (\eta^i\eta^i)^{-1}\Big\{\eta^1(g_\omega + \eta^i_t x_i) - \eta^2\big(\eta^3_t(\eta^3)^{-2}\omega + \eta^2_t x_1 - \eta^1_t x_2\big)\Big\},\\
&u^2 = (\eta^i\eta^i)^{-1}\Big\{\eta^2(g_\omega + \eta^i_t x_i) + \eta^1\big(\eta^3_t(\eta^3)^{-2}\omega + \eta^2_t x_1 - \eta^1_t x_2\big)\Big\},\\
&u^3 = (\eta^3)^{-1}(f + \eta^3_t x_3),\\
&p = 2(\eta^3)^{-1}(\eta^1\eta^2_t - \eta^1_t\eta^2)(\eta^i\eta^i)^{-2}g + \tfrac12\lambda^{-1}\times\\
&\qquad\times\Big\{\lambda^{-1}\big((\eta^3_{tt}\eta^3 - 2\eta^3_t\eta^3_t)\eta^i\eta^i - 2\eta^3\eta^3_t\eta^i\eta^i_t - 2(\eta^3)^2\eta^i_t\eta^i_t\big)\omega^2 +\\
&\qquad+(\eta^3)^2\big((\eta^2\eta^2_{tt} - \eta^1\eta^1_{tt})(x_1^2 - x_2^2) - 2(\eta^1_{tt}\eta^2 + \eta^1\eta^2_{tt})x_1x_2\big) - \eta^i\eta^i\eta^3\eta^3_{tt}x_3^2\Big\}.
\end{aligned}
$$

Here $f = f(\tau,\omega)$, $f_\tau - f_{\omega\omega} = 0$, $g = g(\tau,\omega)$, $g_\tau - g_{\omega\omega} = 0$, $\tau = \int(\eta^3)^2\eta^i\eta^i dt$, $\omega = \eta^3(\eta^2x_1 - \eta^1x_2)$, and $\lambda = (\eta^3)^2\eta^i\eta^i$.

Note 5.1 The equation

$$\vec m^1_t\cdot\vec m^2 - \vec m^1\cdot\vec m^2_t = 0 \tag{5.2}$$

can easily be solved in the following way: Let us fix arbitrary smooth vector-functions $\vec m^1, \vec l \in C^\infty((t_0,t_1),\mathbb{R}^3)$ such that $\vec m^1(t)\neq\vec 0$, $\vec l(t)\neq\vec 0$, and $\vec m^1(t)\cdot\vec l(t) = 0$ for all $t\in(t_0,t_1)$. Then the vector-function $\vec m^2 = \vec m^2(t)$ is taken in the form

$$\vec m^2(t) = \rho(t)\vec m^1 + \vec l(t). \tag{5.3}$$

Equation (5.2) implies

$$\rho(t) = \int(\vec m^1\cdot\vec m^1)^{-1}(\vec m^1_t\cdot\vec l - \vec m^1\cdot\vec l_t)dt. \tag{5.4}$$

B. Let $C\neq0$. By means of the transformation $\vec m^i\to a_{ij}\vec m^j$, where $a_{ij} = \mathrm{const}$ and $\det\{a_{ij}\} = C$, we make $C = 1$. Then we obtain the following solution of the NSEs (1.1)

$$
\begin{aligned}
&\vec u = \lambda^{-1}\Big(\theta^{ij}(t)g^j_\omega(\tau,\omega) + \theta^{i0}(t)\omega + \vec m^i_t\cdot\vec x - \lambda^{-1}((\vec k\times\vec m^i)\cdot\vec x)\Big)\vec n^i - \lambda^{-1}(\vec k_t\cdot\vec x)\vec k,\\
&p = 2\lambda^{-2}(\vec n^i\cdot\vec k_t)(\theta^{ij}(t)g^i(\tau,\omega) + \tfrac12\theta^{i0}(t)\omega^2) + \tfrac12\lambda^{-2}(\vec k_{tt}\cdot\vec k - 2\vec k_t\cdot\vec k_t)\omega^2 -\\
&\qquad-\tfrac12\lambda^{-1}(\vec n^i\cdot\vec x)(\vec m^i_{tt}\cdot\vec x) - \tfrac12\lambda^{-2}(\vec k\cdot\vec m^i_{tt})(\vec n^i\cdot\vec x)(\vec k\cdot\vec x).
\end{aligned}
\tag{5.5}
$$

Here $\vec m^1_t\cdot\vec m^2 - \vec m^1\cdot\vec m^2_t = 1$, $\vec k = \vec m^1\times\vec m^2$, $\vec n^1 = \vec m^2\times\vec k$, $\vec n^2 = \vec k\times\vec m^1$, $\lambda = |\vec k|^2$, $\omega = \vec k\cdot\vec x$, $\tau = \int\lambda(t)dt$, and $g^i_\tau - g^i_{\omega\omega} = 0$. $(\theta^{1i}(t),\theta^{2i}(t))$ $(i = 1,2)$ are linearly independent solutions of the system

$$\theta^i_t + \lambda^{-1}(\vec m^i\cdot\vec m^2)\theta^1 - \lambda^{-1}(\vec m^i\cdot\vec m^1)\theta^2 = 0, \tag{5.6}$$

and $(\theta^{10}(t),\theta^{20}(t))$ is a particular solution of the nonhomogeneous system

$$\theta^i_t + \lambda^{-1}(\vec m^i\cdot\vec m^2)\theta^1 - \lambda^{-1}(\vec m^i\cdot\vec m^1)\theta^2 = 2\lambda^{-2}\big((\vec k\times\vec k_t)\cdot\vec m^i\big). \tag{5.7}$$

For example, if $\vec m^1 = (\eta\cos\psi,\eta\sin\psi,0)$ and $\vec m^2 = (-\eta\sin\psi,\eta\cos\psi,0)$, where $\eta = \eta(t)\neq0$ and $\psi = -\frac12\int(\eta)^{-2}dt$ (therefore, $\vec m^1_t\cdot\vec m^2 - \vec m^1\cdot\vec m^2_t = 1$), we obtain

$$
\begin{aligned}
&u^1 = \eta^{-1}\big(f^1\cos\psi - f^2\sin\psi + \eta_t x_1 - \tfrac12\eta^{-1}x_2\big),\\
&u^2 = \eta^{-1}\big(f^1\sin\psi + f^2\cos\psi + \eta_t x_2 + \tfrac12\eta^{-1}x_1\big),\\
&u^3 = -2\eta_t\eta^{-1}x_3,\\
&p = (\eta_{tt}\eta - 3\eta_t\eta_t)\eta^{-2}x_3^2 - \tfrac12(\eta_{tt}\eta^{-1} - \tfrac14\eta^{-4})x_ix_i.
\end{aligned}
$$

Here $f^i = f^i(\tau,\omega)$, $f^i_\tau - f^i_{\omega\omega} = 0$, $\tau = \int(\eta)^4 dt$, and $\omega = (\eta)^2 x_3$.

Note 5.2 As in the case $C = 0$, the solutions of the equation

$$\vec m^1_t \cdot \vec m^2 - \vec m^1 \cdot \vec m^2_t = 1 \tag{5.8}$$

can be sought in form (5.3). As a result we obtain that

$$\rho(t) = \int |\vec m^1|^{-2}(\vec m^1_t \cdot \vec l - \vec m^1 \cdot \vec l_t - 1)dt. \tag{5.9}$$

Note 5.3 System (5.6) can be reduced to a second-order homogeneous differential equation either in θ^1, i.e.,

$$\left(\lambda|\vec m^1|^{-2}\theta^1_t\right)_t + \left(\left((\vec m^1\cdot\vec m^2)|\vec m^1|^{-2}\right)_t + |\vec m^1|^{-2}\right)\theta^1 = 0 \tag{5.10}$$

(then $\theta^2 = |\vec m^1|^{-2}(\lambda\theta^1_t + (\vec m^1\cdot\vec m^2)\theta^1)$), or in θ^2, i.e.,

$$\left(\lambda|\vec m^2|^{-2}\theta^2_t\right)_t + \left(-\left((\vec m^1\cdot\vec m^2)|\vec m^2|^{-2}\right)_t + |\vec m^2|^{-2}\right)\theta^2 = 0 \tag{5.11}$$

(then $\theta^1 = |\vec m^2|^{-2}(-\lambda\theta^2_t + (\vec m^1\cdot\vec m^2)\theta^2)$). Under the notation of Note 5.1 equation (5.10) has the form:

$$\left((\vec l\cdot\vec l)\theta^1_t\right)_t + |\vec m^1|^{-2}(\vec m^1_t\cdot\vec l - \vec m^1\cdot\vec l_t)\theta^1 = 0. \tag{5.12}$$

The vector-functions $\vec m^1$ and $\vec l$ are chosen in such a way that one can find a fundamental set of solutions for equation (5.12). For example, let $\vec m \times \vec m_t \neq 0\ \forall\, t \in (t_0,t_1)$. Let us introduce the notation $\vec m := \vec m^1$ and put $\vec l = \eta(t)\vec m\times\vec m_t$, where $\eta \in C^\infty((t_0,t_1),\mathbb{R})$, $\eta(t)\neq 0\ \forall t\in(t_0,t_1)$. Then

$$\begin{aligned}
&\vec m\cdot\vec l = 0,\quad \vec m_t\cdot\vec l - \vec m\cdot\vec l_t = 0,\quad \vec m^2 = -\left(\textstyle\int|\vec m|^{-2}dt\right)\vec m + \eta\vec m\times\vec m_t,\\
&\vec k = \eta\vec m\times(\vec m\times\vec m_t),\quad \lambda = (\eta)^2|\vec m|^2|\vec m\times\vec m_t|^{-2},\\
&\vec n^2 = \eta|\vec m|^2\vec m\times\vec m_t,\quad \vec n^1 = \left(\textstyle\int|\vec m|^{-2}dt\right)\vec n^2 + (\eta)^2|\vec m\times\vec m_t|^{-2}\vec m,\\
&\theta^{11}(t) = \textstyle\int(\eta)^{-2}|\vec m\times\vec m_t|^{-2}dt,\quad \theta^{21}(t) = 1-\theta^{11}\int|\vec m|^{-2}dt,\\
&\theta^{12}(t) = 1,\quad \theta^{22}(t) = -\textstyle\int|\vec m|^{-2}dt,\\
&\theta^{10}(t) = 2\textstyle\int\left(((\vec m\times\vec m_t)\cdot\vec m_{tt})|\vec m\times\vec m_t|^{-2} + \int\eta^{-1}|\vec m|^{-4}dt\right)\eta^{-2}|\vec m\times\vec m_t|^{-2}dt,\\
&\theta^{20}(t) = -\theta^{10}(t)\textstyle\int|\vec m|^{-2}dt + 2\int\eta^{-1}|\vec m|^{-4}dt.
\end{aligned}$$

Consider the following cases: $\vec m\times\vec m_t \equiv \vec 0$, i.e., $\vec m = \chi(t)\vec a$, where $\chi(t)\in C^\infty((t_0,t_1),\,\mathbb{R})$, $\chi(t)\neq 0\ \forall t\in(t_0,t_1)$, $\vec a = \mathrm{const}$, and $|\vec a| = 1$. Let us put

$$\vec l(t) = \eta^1(t)\vec b + \eta^2(t)\vec c,$$

where $\eta^1,\eta^2\in C^\infty((t_0,t_1),\mathbb{R})$, $(\eta^1(t),\eta^2(t))\neq(0,0)\ \forall\, t\in(t_0,t_1)$, $\vec b = \mathrm{const}$, $|\vec b| = 1$, $\vec a\cdot\vec b = 0$, and $\vec c = \vec a\times\vec b$. Then

$$\begin{aligned}
&\vec m^2 = -\left(\chi\textstyle\int\chi^{-2}dt\right)\vec a + \eta^1\vec b + \eta^2\vec c,\quad \vec k = \chi\eta^1\vec c - \chi\eta^2\vec b,\\
&\lambda = (\chi)^2\eta^i\eta^i,\quad \vec n^2 = (\chi)^2(\eta^1\vec b + \eta^2\vec c),\quad \vec n^1 = \left(\textstyle\int\chi^{-2}dt\right)\vec n^2 + \chi\eta^i\eta^i\vec a,\\
&\theta^{11} = \textstyle\int(\eta^i\eta^i)^{-1}dt,\quad \theta^{21} = 1-\theta^{11}\int\chi^{-2}dt,\quad \theta^{12} = 1,\quad \theta^{22} = -\int\chi^{-2}dt,\\
&\theta^{10} = 2\textstyle\int(\eta^2_t\eta^1 - \eta^2\eta^1_t)\chi^{-1}(\eta^i\eta^i)^{-1}dt,\quad \theta^{20} = -\theta^{10}\int\chi^{-2}dt.
\end{aligned}$$

Note 5.4 In formulas (5.1) and (5.5) solutions of the NSEs (1.1) are expressed in terms of solutions of the decomposed system of two linear one-dimensional heat equations (LOHEs) that have the form:

$$g^i_\tau = g^i_{\omega\omega}. \tag{5.13}$$

The Lie symmetry of the LOHE are known. Large sets of its exact solutions were constructed [27, 3]. The Q-conditional symmetries of LOHE were investigated in [14]. Moreover, being decomposed system (5.13) admits transformations of the form

$$\tilde g^1(\tau',\omega') = F^1(\tau,\omega,g^1(\tau,\omega)), \quad \tau' = G^1(\tau,\omega), \quad \omega' = H^1(\tau,\omega),$$
$$\tilde g^2(\tau'',\omega'') = F^2(\tau,\omega,g^2(\tau,\omega)), \quad \tau'' = G^2(\tau,\omega), \quad \omega'' = H^2(\tau,\omega),$$

where $(G^1,H^1) \neq (G^2,H^2)$, i.e. the independent variables can be transformed in the functions g^1 and g^2 in different ways. A similar statement is true for system (5.19)–(5.20) (see below) if $\varepsilon = 0$.

Note 5.5 It can be proved that an arbitrary Navier–Stokes field $(\vec u, p)$, where

$$\vec u = \vec w(t,\omega) + (\vec k^i(t)\cdot\vec x)\vec l^i(t)$$

with $\vec k^i, \vec l^i \in C^\infty((t_0,t_1),\mathbb{R}^3)$, $\vec k^1 \times \vec k^2 \neq 0$, and $\omega = (\vec k^1 \times \vec k^2)\cdot\vec x$, is equivalent to either a solution from family (5.1) or a solution from family (5.5). The equivalence transformation is generated by $R(\vec m)$ and $Z(\chi)$.

5.2 Investigation of system (3.13)–(3.16)

Consider system 8 from Subsection 3.2, i.e., equations (3.13)–(3.16). Equation (3.16) immediately gives

$$w^1 = -\tfrac12 \rho_t\rho^{-1} + (\eta - 1)z_2^{-2}, \tag{5.14}$$

where $\eta = \eta(t)$ is an arbitrary smooth function of $z_1 = t$. Substituting (5.14) into remaining equations (5.13)–(5.15), we get

$$q_2 = \tfrac12\big((\rho_t\rho^{-1})_t - \tfrac12(\rho_t\rho^{-1})^2\big)z_2 - \eta_t z_2^{-1} - (\eta-1)^2 z_2^{-3} + (w^2-\chi)^2 z_2^{-3}, \tag{5.15}$$

$$w^2_1 - w^2_{22} + \big(\eta z_2^{-1} - \tfrac12\rho_t\rho^{-1}z_2\big)w^2_2 = 0, \tag{5.16}$$

$$w^3_1 - w^3_{22} + \big(\eta z_2^{-1} - \tfrac12\rho_t\rho^{-1}z_2\big)w^3_2 + \varepsilon(w^2-\chi)z_2^{-2} = 0. \tag{5.17}$$

Recall that $\rho = \rho(t)$ and $\chi = \chi(t)$ are arbitrary smooth functions of t; $\varepsilon \in \{0;1\}$. After the change of the independent variables

$$\tau = \textstyle\int |\rho(t)|dt, \quad z = |\rho(t)|^{1/2} z_2 \tag{5.18}$$

in equations (5.16) and (5.17), we obtain a linear system of a simpler form:

$$w^2_\tau - w^2_{zz} + \hat\eta(\tau)z^{-1}w^2_z = 0, \tag{5.19}$$

$$w^3_\tau - w^3_{zz} + (\hat\eta(\tau) - 2)z^{-1}w^3_z + \varepsilon(w^2 - \hat\chi(\tau))z^{-2} = 0, \tag{5.20}$$

where $\hat\eta(\tau)=\eta(t)$ and $\hat\chi(\tau)=\chi(t)$. Equation (5.15) implies

$$q=\tfrac14\big((\rho_t\rho^{-1})_t-\tfrac12(\rho_t\rho^{-1})^2\big)z_2^2-\eta_t\ln|z_2|-\\ -\tfrac12(\eta-1)^2z_2^{-2}+\textstyle\int(w^2(\tau,z)-\hat\chi(\tau))^2z_2^{-3}dz_2. \tag{5.21}$$

Formulas (5.14), (5.18)–(5.21), and ansatz (3.8) determine a solution of the NSEs (1.1).

If $\varepsilon=0$ system (5.19)–(5.20) is decomposed and consists of two translational linear equations of the general form

$$f_\tau+\tilde\eta(\tau)z^{-1}f_z-f_{zz}=0, \tag{5.22}$$

where $\tilde\eta=\hat\eta$ ($\tilde\eta=\hat\eta-2$) for equation (5.19) ((5.20)). Tilde over η is omitted below. Let us investigate symmetry properties of equation (5.22) and construct some of its exact solutions.

Theorem 5.1 *The MIA of (5.22) is given by the following algebras*

a) $L_1=\langle f\partial_f,\ g(\tau,z)\partial_f\rangle$ *if* $\eta(\tau)\neq\mathrm{const}$;

b) $L_2=\langle\partial_\tau,\ \hat D,\ \Pi, f\partial_f,\ g(\tau,z)\partial_f\rangle$ *if* $\eta(\tau)=\mathrm{const}$, $\eta\notin\{0;-2\}$;

c) $L_3=\langle\partial_\tau,\ \hat D,\ \Pi,\ \partial_z+\frac12\eta z^{-1}f\partial_f,\ G=2\tau\partial_\tau-(z-\eta z^{-1}\tau)f\partial_f,\ f\partial_f,$ $g(\tau,z)\partial_f\rangle$ *if* $\eta\in\{0;-2\}$.

Here $\hat D=2\tau\partial_\tau+z\partial_z$, $\Pi=4\tau^2\partial_\tau+4\tau z\partial_z-(z^2+2(1-\eta)\tau)f\partial_f$; $g=g(\tau,z)$ *is an arbitrary solution of (5.22).*

When $\eta=0$, equation (5.22) is the heat equation, and, when $\eta=-2$, it is reduced to the heat equation by means of the change $\tilde f=zf$.

For the case $\eta=\mathrm{const}$ equation (5.22) can be reduced by inequivalent one-dimensional subalgebras of L_2. We construct the following solutions:

For the subalgebra $\langle\partial_\tau+af\partial_f\rangle$, where $a\in\{-1;0;1\}$, it follows that

$$\begin{aligned}&f=e^{-\tau}z^\nu(C_1J_\nu(z)+C_2Y_\nu(z))\quad\text{if}\quad a=-1,\\&f=e^{\tau}z^\nu(C_1I_\nu(z)+C_2K_\nu(z))\quad\text{if}\quad a=1,\\&f=C_1z^{\eta+1}+C_2\quad\text{if}\quad a=0\quad\text{and}\quad\eta\neq-1,\\&f=C_1\ln z+C_2\quad\text{if}\quad a=0\quad\text{and}\quad\eta=-1.\end{aligned}$$

Here J_ν and Y_ν are the Bessel functions of a real variable, whereas I_ν and K_ν are the Bessel functions of an imaginary variable, and $\nu=\frac12(\eta+1)$.

For the subalgebra $\langle\hat D+2af\partial_f\rangle$, where $a\in\mathbb R$, it follows that

$$f=|\tau|^ae^{-\frac12\omega}|\omega|^{\frac12(\eta-1)}W\big(\tfrac14(\eta-1)-a,\tfrac14(\eta+1),\omega\big)$$

with $\omega=\frac14z^2\tau^{-1}$. Here $W(\varkappa,\mu,\omega)$ is the general solution of the Whittaker equation

$$4\omega^2W_{\omega\omega}=(\omega^2-4\varkappa\omega+4\mu^2-1)W.$$

For the subalgebra $\langle\partial_\tau+\Pi+af\partial_f\rangle$, where $a\in\mathbb R$, it follows that

$$f=(4\tau^2+1)^{\frac14(\eta-1)}\exp(-\tau\omega+\tfrac12a\arctan2\tau)\varphi(\omega)$$

with $\omega = z^2(4\tau^2+1)^{-1}$. The function φ is a solution of the equation

$$4\omega\varphi_{\omega\omega} + 2(1-\eta)\varphi_\omega + (\omega - a)\varphi = 0.$$

For example if $a=0$, then $\varphi(\omega) = \omega^\mu\Big(C_1 J_\mu(\tfrac{1}{2}\omega) + C_2 Y_\mu(\tfrac{1}{2}\omega)\Big)$, where $\mu = \frac{1}{4}(\eta+1)$.

Consider equation (5.22), where η is an arbitrary smooth function of τ.

Theorem 5.2 *Equation (5.22) is Q-conditional invariant under the operators*

$$Q^1 = \partial_\tau + g^1(\tau,z)\partial_z + \big(g^2(\tau,z)f + g^3(\tau,z)\big)\partial_f \tag{5.23}$$

if and only if

$$\begin{gathered} g^1_\tau - \eta z^{-1} g^1_z + \eta z^{-2} g^1 - g^1_{zz} + 2g^1_z g^1 - \eta_\tau z^{-1} + 2g^2_z = 0, \\ g^k_\tau + \eta z^{-1} g^k_z - g^k_{zz} + 2g^1_z g^k = 0, \quad k = 2,3, \end{gathered} \tag{5.24}$$

and

$$Q^2 = \partial_z + B(\tau,z,f)\partial_f \tag{5.25}$$

if and only if

$$B_\tau - \eta z^{-2} B + \eta z^{-1} B_z - B_{zz} - 2BB_{zf} - B^2 B_{ff} = 0. \tag{5.26}$$

An arbitrary operator of Q-conditional symmetry of equation (5.22) is equivalent to either an operator of form (5.23) or an operator of form (5.25).

Theorem 5.2 is proved by means of the method described in [13].

Note 5.6 It can be shown (in a way analogous to one in [13]) that system (5.24) is reduced to the decomposed linear system

$$f^a_\tau + \eta z^{-1} f^a_z - f^a_{zz} = 0 \tag{5.27}$$

by means of the following non-local transformation

$$\begin{aligned} g^1 &= -\frac{f^1_{zz} f^2 - f^1 f^2_{zz}}{f^1_z f^2 - f^1 f^2_z} + \eta z^{-1}, \\ g^2 &= -\frac{f^1_{zz} f^2_z - f^1_z f^2_{zz}}{f^1_z f^2 - f^1 f^2_z}, \\ g^3 &= f^3_{zz} - \eta z^{-1} f^3_z + g^1 f^3_z - g^2 f^3. \end{aligned} \tag{5.28}$$

Equation (5.26) is reduced, by means of the change

$$B = -\Phi_\tau/\Phi_f, \quad \Phi = \Phi(\tau,z,f)$$

and the hodograph transformation

$$y_0 = \tau, \quad y_1 = z, \quad y_2 = \Phi, \quad \Psi = f,$$

to the following equation in the function $\Psi = \Psi(y_0,y_1,y_2)$:

$$\Psi_{y_0} + \eta(y_0) y_1^{-1} \Psi_{y_1} - \Psi_{y_1 y_1} = 0.$$

Therefore, unlike Lie symmetries Q-conditional symmetries of (5.22) are more extended for an arbitrary smooth function $\eta = \eta(\tau)$. Thus, Theorem 5.2 implies that equation (5.22) is Q-conditional invariant under the operators

$$\partial_z, \quad X = \partial_\tau + (\eta - 1)z^{-1}\partial_z, \quad G = (2\tau + C)\partial_z - zf\partial_f$$

with $C = \mathrm{const}$. Reducing equation (5.22) by means of the operator G, we obtain the following solution:

$$f = C_2\big(z^2 - 2\textstyle\int(\eta(\tau) - 1)d\tau\big) + C_1. \tag{5.29}$$

In generalizing this we can construct solutions of the form

$$f = \sum_{k=0}^{N} T^k(\tau) z^{2k}, \tag{5.30}$$

where the coefficients $T^k = T^k(\tau)$ $(k = \overline{0, N})$ satisfy the system of ODEs:

$$T^k_\tau + (2k + 2)(\eta(\tau) - 2k - 1)T^{k+1} = 0, \quad k = \overline{0, N-1}, \quad T^N_\tau = 0. \tag{5.31}$$

Equation (5.31) is easily integrated for arbitrary $N \in \mathbb{N}$. For example if $N = 2$, it follows that

$$f = C_3\Big\{z^4 - 4z^2\textstyle\int(\eta(\tau) - 3)d\tau + 8\int\Big((\eta(\tau) - 1)\int(\eta(\tau) - 3)d\tau\Big)d\tau\Big\} +$$
$$+ C_2\Big\{z^2 - 2\textstyle\int(\eta(\tau) - 1)d\tau\Big\} + C_1.$$

An explicit form for solution (5.30) with $N = 1$ is given by (5.29).

Generalizing the solution

$$f = C_0 \exp\big\{-z^2(4\tau + 2C)^{-1} + \textstyle\int(\eta(\tau) - 1)(2\tau + C)^{-1}d\tau\big\} \tag{5.32}$$

obtained by means of reduction of (5.22) by the operator G, we can construct solutions of the general form

$$\begin{aligned} f = {} & \sum_{k=0}^{N} S^k(\tau)\big(z(2\tau + C)^{-1}\big)^{2k} \times \\ & \times \exp\Big\{-z^2(4\tau + 2C)^{-1} + \textstyle\int(\eta(\tau) - 1)(2\tau + C)^{-1}d\tau\Big\}, \end{aligned} \tag{5.33}$$

where the coefficients $S^k = S^k(\tau)$ $(k = \overline{0, N})$ satisfy the system of ODEs:

$$\begin{aligned} & S^k_\tau + (2k + 2)(\eta(\tau) - 2k - 1)(2\tau + C)^{-2}S^{k+1} = 0, \\ & k = \overline{0, N-1}, \quad S^N_\tau = 0. \end{aligned} \tag{5.34}$$

For example if $N = 1$, then

$$f = \Big\{C_1\Big(z^2(2\tau + C)^{-2} - 2\textstyle\int(\eta(\tau) - 1)(2\tau + C)^{-2}d\tau\Big) + C_0\Big\} \times$$
$$\times \exp\Big\{-z^2(4\tau + 2C)^{-1} + \textstyle\int(\eta(\tau) - 1)(2\tau + C)^{-1}d\tau\Big\}.$$

Here we do not present results for arbitrary N as they are very cumbersome.

Putting $g^2 = g^3 = 0$ in system (5.24), we obtain one equation in the function g^1:

$$g^1_\tau - \eta z^{-1} g^1_z + \eta z^{-2} g^1 - g^1_{zz} + 2g^1_z g^1 - \eta_\tau z^{-1} = 0.$$

It follows that $g^1 = -g_z/g + (\eta - 1)/z$, where $g = g(\tau, z)$ is a solution of the equation

$$g_\tau + (\eta - 2)z^{-1} g_z - g_{zz} = 0. \tag{5.35}$$

Q-conditional symmetry of (5.22) under the operator

$$Q = \partial_\tau + \big(-g_z/g + (\eta - 1)/z\big)\partial_z \tag{5.36}$$

gives rise to the following

Theorem 5.3 *If g is a solution of equation (5.35) and*

$$\begin{aligned} f(\tau, z) = {} & \textstyle\int_{z_0}^{z} z' g(\tau, z')dz' + \\ & + \textstyle\int_{\tau_0}^{\tau} \Big(z_0 g_z(\tau', z_0) - (\eta(\tau') - 1)g(\tau', z_0)\Big)d\tau', \end{aligned} \tag{5.37}$$

where (τ_0, z_0) is a fixed point, then f is a solution of equation (5.22).

Proof. Equation (5.35) implies

$$(zg)_\tau = (zg_z - (\eta - 1)g)_z$$

Therefore, $f_z = zg, \quad f_\tau = zg_z - (\eta - 1)g$ and

$$f_\tau + \eta z^{-1} f_z - f_{zz} = zg_z - (\eta - 1)g + \eta g - (zg)_z = 0. \qquad \text{QED.}$$

The converse of Theorem 5.3 is the following obvious

Theorem 5.4 *If f is a solution of (5.22), the function*

$$g = z^{-1} f_z \tag{5.38}$$

satisfies (5.35).

Theorems 5.3 and 5.4 imply that, when $\eta = 2n$ ($n \in \mathbb{Z}$), solutions of (5.22) can be constructed from known solutions of the heat equation by means of applying either formula (5.37) (for $n > 0$) or formula (5.38) (for $n < 0$) $|n|$ times.

Let us investigate symmetry properties and construct some exact solutions of system (5.19)–(5.20) for $\varepsilon = 1$, i.e., the system

$$w^1_\tau - w^1_{zz} + \hat\eta(\tau) z^{-1} w^1_z = 0, \tag{5.39}$$

$$w^2_\tau - w^2_{zz} + (\hat\eta(\tau) - 2)z^{-1} w^2_z + (w^1 - \hat\chi(\tau))z^{-2} = 0. \tag{5.40}$$

If (w^1, w^2) is a solution of system (5.39)–(5.40), then $(w^1, w^2 + g)$ (where $g = g(\tau, z)$) is also a solution of (5.39)–(5.40) if and only if the function g satisfies the following equation

$$g_\tau - g_{zz} + (\hat\eta(\tau) - 2)z^{-1} g_z = 0 \tag{5.41}$$

System (5.39)–(5.40), for some $\hat\chi=\hat\chi(\tau)$, has particular solutions of the form

$$w^1=\sum_{k=0}^{N}T^k(\tau)z^{2k},\quad w^2=\sum_{k=0}^{N-1}S^k(\tau)z^{2k},$$

where $T^0(\tau)=\hat\chi(\tau)$. For example, if $\hat\chi(\tau)=-2C_1\int(\hat\eta(\tau)-1)d\tau+C_2$ and $N=1$, then

$$w^1=C_1\bigl(z^2-2\textstyle\int(\hat\eta(\tau)-1)d\tau\bigr)+C_2,\quad w^2=-C_1\tau.$$

Let $\hat\chi(\tau)=0$.

Theorem 5.5 *The MIA of system (5.39)–(5.40) with $\hat\chi(\tau)=0$ is given by the following algebras*

a) $\langle w^i\partial_{w^i},\ \tilde w^i(\tau,z)\partial_{w^i}\rangle$ *if* $\hat\eta(\tau)\neq\mathrm{const}$;
b) $\langle 2\tau\partial_\tau+z\partial_z,\ \partial_\tau,\ w^i\partial_{w^i},\ \tilde w^i(\tau,z)\partial_{w^i}\rangle$ *if* $\hat\eta(\tau)=\mathrm{const}$, $\hat\eta\neq0$;
c) $\langle 2\tau\partial_\tau+z\partial_z,\ \partial_\tau,\ w^1z^{-1}\partial_{w^2},\ w^i\partial_{w^i},\ \tilde w^i(\tau,z)\partial_{w^i}\rangle$ *if* $\hat\eta\equiv0$.

Here $(\tilde w^1,\tilde w^2)$ is an arbitrary solution of (5.39)–(5.40) with $\hat\chi(\tau)=0$.

For the case $\hat\chi(\tau)=0$ and $\hat\eta(\tau)=\mathrm{const}$ system (5.39)–(5.40) can be reduced by inequivalent one-dimensional subalgebras of its MIA. We obtain the following solutions:

For the subalgebra $\langle\partial_\tau\rangle$ it follows that

$$w^1=C_1\ln z+C_2,$$
$$w^2=\tfrac14C_1(\ln^2z-\ln z)+\tfrac12C_2\ln z+C_3z^{-2}+C_4$$

if $\hat\eta=-1$;

$$w^1=C_1z^2+C_2,$$
$$w^2=\tfrac14C_1z^2+\tfrac12C_2\ln^2z+C_3\ln z+C_4$$

if $\hat\eta=1$;

$$w^1=C_1z^{\hat\eta+1}+C_2,$$
$$w^2=\tfrac12C_1(\hat\eta+1)^{-1}z^{\hat\eta+1}+C_2(\hat\eta-1)^{-1}\ln z+C_3z^{\hat\eta-1}+C_4$$

if $\hat\eta\notin\{-1;1\}$.

For the subalgebra $\langle\partial_\tau-w^i\partial_{w^i}\rangle$ it follows that

$$w^1=e^{-\tau}z^{\frac12(\hat\eta+1)}\psi^1(z),\quad w^2=e^{-\tau}z^{\frac12(\hat\eta-1)}\psi^2(z),$$

where the functions ψ^1 and ψ^2 satisfy the system

$$z^2\psi^1_{zz}+z\psi^1_z+\bigl(z^2-\tfrac14(\hat\eta+1)^2\bigr)\psi^1=0,\tag{5.42}$$

$$z^2\psi^2_{zz}+z\psi^2_z+\bigl(z^2-\tfrac14(\hat\eta-1)^2\bigr)\psi^2=z\psi^1.\tag{5.43}$$

The general solution of system (5.42)–(5.43) can be expressed by quadratures in terms of the Bessel functions of a real variable $J_\nu(z)$ and $Y_\nu(z)$:

$$\psi^1 = C_1 J_{\nu+1}(z) + C_2 Y_{\nu+1}(z),$$
$$\psi^2 = C_3 J_\nu(z) + C_4 Y_\nu(z) + \tfrac{\pi}{2} Y_\nu(z) \int J_\nu(z)\psi^1(z)dz - \tfrac{\pi}{2} J_\nu(z) \int Y_\nu(z)\psi^1(z)dz$$

with $\nu = \frac{1}{2}(\hat{\eta}-1)$;

For the subalgebra $\langle \partial_\tau + w^i\partial_{w^i}\rangle$ it follows that

$$w^1 = e^\tau z^{\frac{1}{2}(\hat{\eta}+1)}\psi^1(z), \quad w^2 = e^\tau z^{\frac{1}{2}(\hat{\eta}-1)}\psi^2(z),$$

where the functions ψ^1 and ψ^2 satisfy the system

$$z^2\psi^1_{zz} + z\psi^1_z - \left(z^2 + \tfrac{1}{4}(\hat{\eta}+1)^2\right)\psi^1 = 0, \tag{5.44}$$

$$z^2\psi^2_{zz} + z\psi^2_z - \left(z^2 + \tfrac{1}{4}(\hat{\eta}-1)^2\right)\psi^2 = z\psi^1. \tag{5.45}$$

The general solution of system (5.44)–(5.45) can be expressed by quadratures in terms of the Bessel functions of an imaginary variable $I_\nu(z)$ and $K_\nu(z)$:

$$\psi^1 = C_1 I_{\nu+1}(z) + C_2 K_{\nu+1}(z),$$
$$\psi^2 = C_3 I_\nu(z) + C_4 K_\nu(z) + K_\nu(z) \int I_\nu(z)\psi^1(z)dz - I_\nu(z) \int K_\nu(z)\psi^1(z)dz$$

with $\nu = \frac{1}{2}(\hat{\eta}-1)$.

For the subalgebra $\langle 2\tau\partial_\tau + z\partial_z + aw^i\partial_{w^i}\rangle$ it follows that

$$w^1 = |\tau|^a e^{-\frac{1}{2}\omega}|\omega|^{\frac{1}{4}(\hat{\eta}-1)}\psi^1(\omega), \quad w^2 = |\tau|^a e^{-\frac{1}{2}\omega}|\omega|^{\frac{1}{4}(\hat{\eta}-3)}\psi^2(\omega)$$

with $\omega = \frac{1}{4}z^2\tau^{-1}$, where the functions ψ^1 and ψ^2 satisfy the system

$$4\omega^2\psi^1_{\omega\omega} = \Big(\omega^2 + \big(a - \tfrac{1}{4}(\hat{\eta}-1)\big)\omega + \tfrac{1}{4}(\hat{\eta}+1)^2 - 1\Big)\psi^1, \tag{5.46}$$

$$4\omega^2\psi^2_{\omega\omega} = \Big(\omega^2 + \big(a - \tfrac{1}{4}(\hat{\eta}-3)\big)\omega + \tfrac{1}{4}(\hat{\eta}-1)^2 - 1\Big)\psi^2 + 2|\omega|^{1/2}\psi^1. \tag{5.47}$$

The general solution of system (5.46)–(5.47) can be expressed by quadratures in terms of the Whittaker functions.

6 Symmetry properties and exact solutions of system (3.12)

As was mentioned in Section 3, ansatzes (3.4)–(3.7) reduce the NSEs (1.1) to the systems of PDEs of a similar structure that have the general form (see (3.12)):

$$\begin{aligned} &w^iw^1_i - w^1_{ii} + s_1 + \alpha_2 w^2 = 0,\\ &w^iw^2_i - w^2_{ii} + s_2 - \alpha_2 w^1 + \alpha_1 w^3 = 0,\\ &w^iw^3_i - w^3_{ii} + \alpha_4 w^3 + \alpha_5 = 0,\\ &w^i_i = \alpha_3, \end{aligned} \tag{6.1}$$

where α_n $(n = \overline{1,5})$ are real parameters.

Setting $\alpha_k = 0$ ($k = \overline{2,5}$) in (6.1), we obtain equations describing a plane convective flow that is brought about by nonhomogeneous heating of boudaries [25]. In this case w^i are the coordinates of the flow velocity vector, w^3 is the flow temperature, s is the pressure, the Grasshoff number λ is equal to $-\alpha_1$, and the Prandtl number σ is equal to 1. Some similarity solutions of these equations were constructed in [22]. The particular case of system (6.1) for $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = 0$ and $\alpha_3 = 1$ was considered in [31].

In this section we study symmetry properties of system (6.1) and construct large sets of its exact solutions.

Theorem 6.1 *The MIA of (6.1) is the algebra*

1. $E_1 = \langle \partial_1,\ \partial_2,\ \partial_s\rangle$ *if* $\alpha_1 \neq 0,\ \alpha_4 \neq 0.$
2. $E_2 = \langle \partial_1,\ \partial_2,\ \partial_s,\ \partial_{w^3} - \alpha_1 z_2\partial_s\rangle$ *if* $\alpha_1 \neq 0,\ \alpha_4 = 0,\ (\alpha_1, \alpha_2, \alpha_5) \neq (0,0,0).$
3. $E_3 = \langle \partial_1,\ \partial_2,\ \partial_s,\ \partial_{w^3} - \alpha_1 z_2\partial_s,\ \tilde{D} - 3w^3\partial_{w^3}\rangle$ *if* $\alpha_1 \neq 0,\ \alpha_k = 0,\ k = \overline{2,5}.$
4. $E_4 = \langle \partial_1,\ \partial_2,\ \partial_s,\ J,\ (w^3 + \alpha_5/\alpha_4)\partial_{w^3}\rangle$ *if* $\alpha_1 = 0,\ \alpha_4 \neq 0.$
5. $E_5 = \langle \partial_1,\ \partial_2,\ \partial_s,\ J,\ \partial_{w^3}\rangle$ *if* $\alpha_1 = \alpha_4 = 0,\ (\alpha_2, \alpha_3) \neq (0,0),\ \alpha_5 \neq 0.$
6. $E_6 = \langle \partial_1,\ \partial_2,\ \partial_s,\ J,\ \partial_{w^3},\ w^3\partial_{w^3}\rangle$ *if* $\alpha_1 = \alpha_4 = \alpha_5 = 0,\ (\alpha_2, \alpha_3) \neq (0,0).$
7. $E_7 = \langle \partial_1,\ \partial_2,\ \partial_s,\ J,\ \partial_{w^3},\ \tilde{D} + 2w^3\partial_{w^3}\rangle$ *if* $\alpha_5 \neq 0,\ \alpha_l = 0,\ l = \overline{1,4}.$
8. $E_8 = \langle \partial_1,\ \partial_2,\ \partial_s,\ J,\ \partial_{w^3},\ \tilde{D}, w^3\partial_{w^3}\rangle$ *if* $\alpha_n = 0,\ n = \overline{1,5}.$

Here $\tilde{D} = z_i\partial_i - w^i\partial_{w^i} - 2s\partial_s$, $J = z_1\partial_2 - z_2\partial_1 + w^1\partial_{w^2} - w^2\partial_{w^1}$, $\partial_i = \partial_{z_i}$.

Note 6.1 The bases of the algebras E_6 and E_8 contain the operator $w^3\partial_{w^3}$ that is not induced by elements of $A(NS)$.

Note 6.2 If $\alpha_4 \neq 0$, the constant α_5 can be made to vanish by means of local transformation

$$\tilde{w}^3 = w^3 + \alpha_5/\alpha_4, \quad \tilde{s} = s - \alpha_1\alpha_5\alpha_4^{-1} z_2, \tag{6.2}$$

where the independent variables and the functions w^i are not transformed. Therefore, we consider below that $\alpha_5 = 0$ if $\alpha_4 \neq 0$.

Note 6.3 Making the non-local transformation

$$\tilde{s} = s + \alpha_2\Psi, \tag{6.3}$$

where $\Psi_1 = w^2$, $\Psi_2 = -w^1$ (such a function Ψ exists in view of the last equation of (6.1)), in system (6.1) with $\alpha_3 = 0$, we obtain a system of form (6.1) with $\tilde{\alpha}_3 = \tilde{\alpha}_2 = 0$. In some cases ($\alpha_1 \neq 0$, $\alpha_3 = \alpha_4 = \alpha_5 = 0$, $\alpha_2 \neq 0$; $\alpha_1 = \alpha_3 = \alpha_4 = 0$, $\alpha_2 \neq 0$) transformation (6.3) allows the symmetry of (6.1) to be extended and non-Lie solutions to be constructed. Moreover, it means that in the cases listed above system (6.1) is invariant under the non-local transformation

$$\hat{z}_i = e^{\varepsilon} z_i, \quad \hat{w}^i = e^{-\varepsilon} w^i, \quad \hat{w}^3 = e^{\delta\varepsilon} w^3, \quad \hat{s} = e^{-2\varepsilon} s + \alpha_2(e^{-2\varepsilon} - 1)\Psi,$$

where

$\delta = -3$ if $\alpha_3 = \alpha_4 = \alpha_5 = 0$, $\alpha_1, \alpha_2 \neq 0$;

$\delta = 2$ if $\alpha_1 = \alpha_3 = \alpha_4 = 0$, $\alpha_2, \alpha_5 \neq 0$;

$\delta = 0$ if $\alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = 0$, $\alpha_2 \neq 0$.

Let us consider an ansatz of the form:

$$
\begin{aligned}
&w^1 = a_1\varphi^1 - a_2\varphi^3 + b_1\omega_2,\\
&w^2 = a_2\varphi^1 + a_1\varphi^3 + b_2\omega_2,\\
&w^3 = \varphi^2 + b_3\omega_2,\\
&s = h + d_1\omega_2 + d_2\omega_1\omega_2 + \tfrac{1}{2}d_3\omega_2^2,
\end{aligned}
\tag{6.4}
$$

where $a_1^2 + a_2^2 = 1$, $\omega = \omega_1 = a_1z_2 - a_2z_1$, $\omega_2 = a_1z_1 + a_2z_2$, $B, b_a, d_a = \text{const}$,

$$
\begin{aligned}
&b_i = Ba_i, \quad b_3(B + \alpha_4) = 0,\\
&d_2 = \alpha_2B - \alpha_1b_3a_1, \quad d_3 = -B^2 - \alpha_1b_3a_2,
\end{aligned}
\tag{6.5}
$$

Here and below $\varphi^a = \varphi^a(\omega)$ and $h = h(\omega)$. Indeed, formulas (6.4) and (6.5) determine a whole set of ansatzes for system (6.1). This set contains both Lie ansatzes, constructed by means of subalgebras of the form

$$
\langle a_1\partial_1 + a_2\partial_2 + a_3(\partial_{w^3} - \alpha_1z_2\partial_s) + a_4\partial_s\rangle,
\tag{6.6}
$$

and non-Lie ansatzes. Equation (6.5) is the necessary and sufficient condition to reduce (6.1) by means of an ansatz of form (6.3). As a result of reduction we obtain the following system of ODEs:

$$
\begin{aligned}
&\varphi^3\varphi^1_\omega - \varphi^1_{\omega\omega} + \mu_{1j}\varphi^j + d_1 + d_2\omega + \alpha_2\varphi^3 = 0,\\
&\varphi^3\varphi^2_\omega - \varphi^2_{\omega\omega} + \mu_{2j}\varphi^j + \alpha_5 = 0,\\
&\varphi^3\varphi^3_\omega - \varphi^3_{\omega\omega} + h_\omega - \alpha_2\varphi^1 + \alpha_1a_1\varphi^2 = 0,\\
&\varphi^3_\omega = \sigma,
\end{aligned}
\tag{6.7}
$$

where $\mu_{11} = -B$, $\mu_{12} = -\alpha_1a_2$, $\mu_{21} = -b_3$, $\mu_{22} = -\alpha_4$, $\sigma = \alpha_3 - B$. If $\sigma = 0$, system (6.7) implies that

$$
\begin{aligned}
&\varphi^3 = C_0 = \text{const},\\
&h = \alpha_2\textstyle\int\varphi^1(\omega)d\omega - \alpha_1a_1\int\varphi^2(\omega)d\omega,
\end{aligned}
$$

and the functions φ^i satisfy system (4.23), where $\nu_{11} = d_1 + \alpha_2C_0$, $\nu_{21} = d_2$, $\nu_{12} = \alpha_5$, $\nu_{22} = 0$. If $\sigma \neq 0$, then $\varphi^3 = \sigma\omega$ (translating ω, the integration constant can be made to vanish),

$$
h = -\tfrac{1}{2}\sigma^2\omega^2 + \alpha_2\textstyle\int\varphi^1(\omega)d\omega - \alpha_1a_1\int\varphi^2(\omega)d\omega,
$$

and the functions satisfy system (4.29), where $\nu_{11} = d_1$, $\nu_{21} = d_2 + \alpha_2\sigma$, $\nu_{12} = \alpha_5$, $\nu_{22} = 0$.

Note 6.4 Step-by-step reduction of the NSEs (1.1) by means of ansatzes (3.4)–(3.7) and (6.4) is equivalent to a particular case of immediate reduction of the NSEs (1.1) to ODEs by means of ansatzes 5 and 6 from Subsection 4.1.

Table 1. Complete sets of inequivalent one-dimensional subalgebras of the algebras $E_1 - E_8$ (a and a_l ($l = \overline{1,4}$) are real constants)

Algebra	Subalgebras	Values of parameters
E_1	$\langle a_1\partial_1 + a_2\partial_2 + a_3\partial_s\rangle,\ \langle\partial_s\rangle$	$a_1^2 + a_2^2 = 1$
E_2	$\langle a_1\partial_1 + a_2\partial_2 + a_3(\partial_{w^3} - \alpha_1 z_2\partial_s)\rangle,$ $\langle\partial_1 + a_4\partial_s\rangle,\ \langle\partial_{w^3} - \alpha_1 z_2\partial_s\rangle,\ \langle\partial_s\rangle$	$a_1^2 + a_2^2 = 1,$ $a_4 \neq 0$
E_3	$\langle a_1\partial_1 + a_2\partial_2 + a_3(\partial_{w^3} - \alpha_1 z_2\partial_s)\rangle,\ \langle\partial_1 + a_4\partial_s\rangle,$ $\langle\tilde{D} - 3w^3\partial_{w^3}\rangle,\ \langle\partial_{w^3} - \alpha_1 z_2\partial_s\rangle,\ \langle\partial_s\rangle$	$a_1^2 + a_2^2 = 1,$ $a_3 \in \{-1; 0; 1\},$ $a_4 \in \{-1; 1\}$
E_4	$\langle J + a_1\partial_s + a_2 w^3\partial_{w^3}\rangle,\ \langle\partial_2 + a_1\partial_s + a_2 w^3\partial_{w^3}\rangle,$ $\langle w^3\partial_{w^3} + a_1\partial_s\rangle,\ \langle\partial_s\rangle$	
E_5	$\langle J + a_1\partial_s + a_2\partial_{w^3}\rangle,\ \langle\partial_2 + a_1\partial_s + a_2\partial_{w^3}\rangle,$ $\langle\partial_{w^3} + a_1\partial_s >, < \partial_s\rangle$	
E_6	$\langle J + a_1\partial_s + a_2 w^3\partial_{w^3}\rangle,\ \langle\partial_2 + a_1\partial_s + a_2 w^3\partial_{w^3}\rangle,$ $\langle J + a_1\partial_s + a_3\partial_{w^3}\rangle,\ \langle\partial_2 + a_1\partial_s + a_3\partial_{w^3}\rangle,$ $\langle w^3\partial_{w^3} + a_1\partial_s\rangle,\ \langle\partial_{w^3} + a_1\partial_s\rangle,\ \langle\partial_s\rangle$	$a_2 \neq 0,$ $a_3 \in \{-1; 0; 1\}$
E_7	$\langle\tilde{D} + aJ + 2w^3\partial_{w^3}\rangle,\ \langle J + a_1\partial_s + a_2\partial_{w^3}\rangle,$ $\langle\partial_2 + a_1\partial_s + a_2\partial_{w^3}\rangle, \langle\partial_{w^3} + a_2\partial_s\rangle,\ \langle\partial_s\rangle$	$a_2 \in \{-1; 0; 1\},$ $a_1 \in \{-1; 0; 1\}$ if $a_2 = 0$
E_8	$\langle\tilde{D} + aJ + a_3 w^3\partial_{w^3}\rangle,\ \langle\tilde{D} + aJ + a_3\partial_{w^3}\rangle,$ $\langle J + a_1\partial_s + a_4 w^3\partial_{w^3}\rangle,\ \langle\partial_2 + a_1\partial_s + a_4 w^3\partial_{w^3}\rangle,$ $\langle J + a_1\partial_s + a_2\partial_{w^3}\rangle,\ \langle\partial_2 + a_1\partial_s + a_2\partial_{w^3}\rangle,$ $\langle w^3\partial_{w^3} + a_1\partial_s\rangle,\ \langle\partial_{w^3} + a_1\partial_s\rangle,\ \langle\partial_s\rangle$	$a_i \in \{-1; 0; 1\},$ $a_4 \neq 0$

Now let us choose such algebras, among the algebras from Table 1, that can be used to reduce system (6.1) and do not belong to the set of algebras (6.6). By means of the chosen algebras we construct ansatzes that are tabulated in the form of Table 2.

Table 2. Ansatzes reducing system (6.1) $(r=(z_1^2+z_2^2)^{1/2})$

N	Values of α_n	Algebra	Invariant variable	Ansatz
1	$\alpha_1\neq 0$, $\alpha_k=0$, $k=\overline{2,5}$	$\langle\tilde{D}-3w^3\partial_{w^3}\rangle$	$\omega=\arctan\frac{z_2}{z_1}$	$w^1=r^{-2}(z_1\varphi^1-z_2\varphi^2)$, $w^2=r^{-2}(z_2\varphi^1+z_1\varphi^2)$, $w^3=r^{-3}\varphi^3$, $s=r^{-2}h$
2	$\alpha_1=0$, $\alpha_5=0$	$\langle\partial_2+a_1\partial_s+a_2w^3\partial_{w^3}\rangle$, $a_2\neq 0$	$\omega=z_1$	$w^1=\varphi^1$, $w^2=\varphi^2$, $w^3=\varphi^3e^{a_2z_2}$, $s=h+a_1z_2$
3	$\alpha_1=0$, $\alpha_4=0$	$\langle J+a_1\partial_s+a_2\partial_{w^3}\rangle$	$\omega=r$	$w^1=z_1\varphi^1-z_2r^{-2}\varphi^2$, $w^2=z_2\varphi^1+z_1r^{-2}\varphi^2$, $w^3=\varphi^3+a_2\arctan\frac{z_2}{z_1}$, $s=h+a_1\arctan\frac{z_2}{z_1}$
4	$\alpha_1=0$, $\alpha_5=0$	$\langle J+a_1\partial_s+a_2w_3\partial_{w^3}\rangle$ $a_2\neq 0$ if $\alpha_4=0$	$\omega=r$	$w^1=z_1\varphi^1-z_2r^{-2}\varphi^2$, $w^2=z_2\varphi^1+z_1r^{-2}\varphi^2$, $w^3=\varphi^3e^{a_2\arctan\frac{z_2}{z_1}}$, $s=h+a_1\arctan\frac{z_2}{z_1}$
5	$\alpha_5\neq 0$, $\alpha_l=0$, $l=\overline{1,4}$	$\langle\tilde{D}+aJ+2w^3\partial_{w^3}\rangle$	$\omega=\arctan\frac{z_2}{z_1}-$ $-a\ln r$	$w^1=r^{-2}(z_1\varphi^1-z_2\varphi^2)$, $w^2=r^{-2}(z_2\varphi^1+z_1\varphi^2)$, $w^3=r^2\varphi^3$, $s=r^{-2}h$
6	$\alpha_n=0$, $n=\overline{1,5}$	$\langle\tilde{D}+aJ+a_1\partial_{w^3}\rangle$	$\omega=\arctan\frac{z_2}{z_1}-$ $-a\ln r$	$w^1=r^{-2}(z_1\varphi^1-z_2\varphi^2)$, $w^2=r^{-2}(z_2\varphi^1+z_1\varphi^2)$, $w^3=\varphi^3+a_1\ln r$, $s=r^{-2}h$
7	$\alpha_n=0$, $n=\overline{1,5}$	$\langle\tilde{D}+aJ+a_1w^3\partial_{w^3}\rangle$, $a_1\neq 0$	$\omega=\arctan\frac{z_2}{z_1}-$ $-a\ln r$	$w^1=r^{-2}(z_1\varphi^1-z_2\varphi^2)$, $w^2=r^{-2}(z_2\varphi^1+z_1\varphi^2)$, $w^3=r^{a_1}\varphi^3$, $s=r^{-2}h$

Substituting the ansatzes from Table 2 into system (6.1), we obtain the reduced systems of ODEs in the functions φ^a and h:

$$
\begin{aligned}
1.\quad & \varphi^2\varphi^1_\omega-\varphi^1_{\omega\omega}-\varphi^1\varphi^1-\varphi^2\varphi^2-2h+\alpha_1\varphi^3\sin\omega+2\varphi^2_\omega=0,\\
& \varphi^2\varphi^2_\omega-\varphi^2_{\omega\omega}+h_\omega-2\varphi^1_\omega+\alpha_1\varphi^3\cos\omega=0,\\
& \varphi^2\varphi^3_\omega-\varphi^3_{\omega\omega}-3\varphi^1\varphi^3-9\varphi^3=0,\\
& \varphi^2_\omega=0.
\end{aligned}
\tag{6.8}
$$

$$
\begin{aligned}
2.\quad & \varphi^1\varphi^1_\omega-\varphi^1_{\omega\omega}+\alpha_2\varphi^2+h_\omega=0,\\
& \varphi^1\varphi^2_\omega-\varphi^2_{\omega\omega}-\alpha_2\varphi^1+a_1=0,\\
& \varphi^1\varphi^3_\omega-\varphi^3_{\omega\omega}+(a_2\varphi^2+\alpha_4-a_2^2)\varphi^3=0,\\
& \varphi^1_\omega=\alpha_3.
\end{aligned}
\tag{6.9}
$$

$$
\begin{aligned}
3.\quad & \omega\varphi^1\varphi^1_\omega-\varphi^1_{\omega\omega}+\varphi^1\varphi^1-\omega^{-4}\varphi^2\varphi^2-3\omega^{-1}\varphi^1_\omega+\alpha_2\omega^{-2}\varphi^2+\omega^{-1}h_\omega=0,\\
& \omega\varphi^1\varphi^2_\omega-\varphi^2_{\omega\omega}+\omega^{-1}\varphi^2_\omega-\alpha_2\omega^2\varphi^1+a_1=0,\\
& \omega\varphi^1\varphi^3_\omega-\varphi^3_{\omega\omega}+a_2\omega^{-2}\varphi^2-\omega^{-1}\varphi^3_\omega+\alpha_5=0,\\
& 2\varphi^1+\omega\varphi^1_\omega=\alpha_3.
\end{aligned}
\tag{6.10}
$$

$$
\begin{aligned}
4.\quad & \omega\varphi^1\varphi^1_\omega-\varphi^1_{\omega\omega}+\varphi^1\varphi^1-\omega^{-4}\varphi^2\varphi^2-3\omega^{-1}\varphi^1_\omega+\alpha_2\omega^{-2}\varphi^2+\omega^{-1}h_\omega=0,\\
& \omega\varphi^1\varphi^2_\omega-\varphi^2_{\omega\omega}+\omega^{-1}\varphi^2_\omega-\alpha_2\omega^2\varphi^1+a_1=0,\\
& \omega\varphi^1\varphi^3_\omega-\varphi^3_{\omega\omega}+a_2\omega^{-2}\varphi^2\varphi^3-\omega^{-1}\varphi^3_\omega+(\alpha_4-a_2^2\omega^{-2})\varphi^3=0,\\
& 2\varphi^1+\omega\varphi^1_\omega=\alpha_3.
\end{aligned}
\tag{6.11}
$$

$$
\begin{aligned}
5.\quad & (\varphi^2-a\varphi^1)\varphi^1_\omega-(1+a^2)\varphi^1_{\omega\omega}-\varphi^1\varphi^1-\varphi^2\varphi^2-ah_\omega-2h=0,\\
& (\varphi^2-a\varphi^1)\varphi^2_\omega-(1+a^2)\varphi^2_{\omega\omega}-2(a\varphi^2_\omega+\varphi^1_\omega)+h_\omega=0,\\
& (\varphi^2-a\varphi^1)\varphi^3_\omega-(1+a^2)\varphi^3_{\omega\omega}+2\varphi^1\varphi^3-4\varphi^3+4a\varphi^3_\omega+\alpha_5=0,\\
& \varphi^2_\omega-a\varphi^1_\omega=0.
\end{aligned}
\tag{6.12}
$$

$$
\begin{aligned}
6.\quad & (\varphi^2-a\varphi^1)\varphi^1_\omega-(1+a^2)\varphi^1_{\omega\omega}-\varphi^1\varphi^1-\varphi^2\varphi^2-ah_\omega-2h=0,\\
& (\varphi^2-a\varphi^1)\varphi^2_\omega-(1+a^2)\varphi^2_{\omega\omega}-2(a\varphi^2_\omega+\varphi^1_\omega)+h_\omega=0,\\
& (\varphi^2-a\varphi^1)\varphi^3_\omega-(1+a^2)\varphi^3_{\omega\omega}+a_1\varphi^1=0,\\
& \varphi^2_\omega-a\varphi^1_\omega=0.
\end{aligned}
\tag{6.13}
$$

$$
\begin{aligned}
7.\quad & (\varphi^2-a\varphi^1)\varphi^1_\omega-(1+a^2)\varphi^1_{\omega\omega}-\varphi^1\varphi^1-\varphi^2\varphi^2-ah_\omega-2h=0,\\
& (\varphi^2-a\varphi^1)\varphi^2_\omega-(1+a^2)\varphi^2_{\omega\omega}-2(a\varphi^2_\omega+\varphi^1_\omega)+h_\omega=0,\\
& (\varphi^2-a\varphi^1)\varphi^3_\omega-(1+a^2)\varphi^3_{\omega\omega}+a_1\varphi^1\varphi^3-a_1^2\varphi^3+2aa_1\varphi^3_\omega=0,\\
& \varphi^2_\omega-a\varphi^1_\omega=0.
\end{aligned}
\tag{6.14}
$$

Numeration of reduced systems (6.8)–(6.14) corresponds to that of the ansatzes in Table 2. Let us integrate systems (6.8)–(6.14) in such cases when it is possible. Below, in this section, $C_k=\mathrm{const}$ $(k=\overline{1,6})$.

1. We failed to integrate system (6.8) in the general case, but we managed to find the following particular solutions:

a) $\varphi^1 = -6\wp(\omega + C_3, \frac{1}{3}(4 - 2C_1), C_2) - 2,$
$\varphi^2 = \varphi^3 = 0, \quad h = 2\varphi^1 + C_1;$

b) $\varphi^1 = -6C_1^2 e^{2C_1\omega}\wp(e^{C_1\omega} + C_3, 0, C_2) + 3C_1^2 - 2,$
$\varphi^2 = 5C_1, \quad \varphi^3 = 0,$
$h = -12C_1^2 e^{2C_1\omega}\wp(e^{C_1\omega} + C_3, 0, C_2) - 2 - \frac{13}{2}C_1^2 - \frac{9}{4}C_1^4;$

c) $\varphi^1 = C_1, \quad \varphi^2 = C_2, \quad \varphi^3 = 0, \quad h = -\frac{1}{2}(C_1^2 + C_2^2).$

Here $\wp(\tau, \varkappa_1, \varkappa_2)$ is the Weierstrass function that satisfies the equation (see [19]):

$$(\wp_\tau)^2 = 4\wp^3 - \varkappa_1\wp - \varkappa_2. \tag{6.15}$$

2. If $\alpha_3 = 0$, the last equation of (6.9) implies that $\varphi^1 = C_1$. It follows from the other equations of (6.9) that

$$\varphi^2 = C_3 + C_2 e^{C_1\omega} - (a_1 C_1^{-1} - \alpha_2)\omega,$$
$$h = C_6 - \alpha_2 C_3\omega - \alpha_2 C_2 C_1^{-1} e^{C_1\omega} + \tfrac{1}{2}\alpha_2(a_1 C_1^{-1} - \alpha_2)\omega^2$$

if $C_1 \neq 0$, and

$$\varphi^2 = C_3 + C_2\omega + \tfrac{1}{2}a_1\omega^2,$$
$$h = C_6 - \alpha_2 C_3\omega - \tfrac{1}{2}\alpha_2 C_2\omega^2 - \tfrac{1}{6}\alpha_2 a_1\omega^3$$

if $C_1 = 0$. The function φ^3 satisfies the equation

$$\varphi^3_{\omega\omega} - C_1\varphi^3_\omega + (a_2^2 - \alpha_4 - a_2\varphi^2)\varphi^3 = 0. \tag{6.16}$$

We solve equation (6.16) for the following cases:

A. $C_2 = a_1 - \alpha_2 C_1 = 0$:

$$\varphi^3 = \begin{cases} e^{\frac{1}{2}C_1\omega}\big(C_4 e^{\mu^{1/2}\omega} + C_5 e^{-\mu^{1/2}\omega}\big), & \mu > 0, \\ e^{\frac{1}{2}C_1\omega}\big(C_4 + C_5\omega\big), & \mu = 0, \\ e^{\frac{1}{2}C_1\omega}\big(C_4\cos((-\mu)^{1/2}\omega) + C_5\sin((-\mu)^{1/2}\omega)\big), & \mu < 0, \end{cases}$$

where $\mu = \frac{1}{4}C_1^2 - a_2^2 + \alpha_4 + a_2 C_3$.

B. $C_1 = a_1 = 0$, $C_2 \neq 0$ ([19]):

$$\varphi^3 = \xi^{1/2} Z_{1/3}\big(\tfrac{2}{3}(-a_2 C_2)^{1/2}\xi^{3/2}\big),$$

where $\xi = \omega + (C_3 a_2 - a_2^2 - \alpha_4)/(a_2 C_2)$. Here $Z_\nu(\tau)$ is the general solution of the Bessel equation (4.22).

C. $C_1 = 0$, $a_1 \neq 0$ ([19]):

$$\varphi^3 = (\omega + C_2 a_1^{-1})^{-1/2} W\big(\nu, \tfrac{1}{4}, (\tfrac{1}{2}a_1 a_2)^{-1/2}(\omega + C_2 a_1^{-1})^2\big),$$

where $\nu = \frac{1}{4}(\frac{1}{2}a_1 a_2)^{-1/2}\big(a_2^2 - \alpha_4 - a_2 C_3 + \frac{1}{2}a_2 C_3^2 a_1^{-1}\big)$. Here $W(\varkappa, \mu, \tau)$ is the general solution of the Whittaker equation (4.21).

D. $C_1\neq 0$, $C_2\neq 0$, $a_1-\alpha_2C_1=0$ ([19]):

$$\varphi^3=e^{\frac12 C_1\omega}Z_\nu\big(2C_1^{-1}(-a_2C_2)^{1/2}e^{\frac12 C_1\omega}\big),$$

where $\nu=C_1^{-1}\big(C_1^2+4(\alpha_4+a_2C_3-a_2^2)\big)^{1/2}$. Here $Z_\nu(\tau)$ is the general solution of the Bessel equation (4.22).

E. $C_1\neq 0$, $a_1-\alpha_2C_1\neq 0$, $C_2=0$ ([19]):

$$\varphi^3=e^{\frac12 C_1\omega}\xi^{1/2}Z_{1/3}\Big(\tfrac23\big(a_2(a_1C_1^{-1}-\alpha_2)\big)^{1/2}\xi^{3/2}\Big),$$

where $\xi=\omega+\big(a_2^2-\frac14 C_1^2-C_3a_2-\alpha_4\big)/\big(a_2(a_1C_1^{-1}-\alpha_2)\big)$. Here $Z_\nu(\tau)$ is the general solution of the Bessel equation (4.22).

If $\alpha_3\neq 0$, then $\varphi^1=\alpha_3\omega$ (translating ω, the integration constant can be made to vanish),

$$\varphi^2=C_1+C_2\int e^{\frac12\alpha_3\omega^2}d\omega+a_1\int e^{\frac12\alpha_3\omega^2}\Big(\int e^{-\frac12\alpha_3\omega^2}d\omega\Big)d\omega+\alpha_2\omega,$$
$$h=C_3-\tfrac12(\alpha_2^2+\alpha_3^2)\omega^2-\alpha_2C_1\omega-\alpha_2C_2\Big(\omega\int e^{\frac12\alpha_3\omega^2}d\omega-\alpha_3^{-1}e^{\frac12\alpha_3\omega^2}\Big)-$$
$$-\alpha_2a_1\Big(\omega\int e^{\frac12\alpha_3\omega^2}\big(\int e^{-\frac12\alpha_3\omega^2}d\omega\big)d\omega-\alpha_3^{-1}e^{\frac12\alpha_3\omega^2}\int e^{-\frac12\alpha_3\omega^2}d\omega+\alpha_3^{-1}\omega\Big),$$

and the function φ^3 satisfies the equation

$$\varphi^3_{\omega\omega}-\alpha_3\omega\varphi^3_\omega+(a_2^2-\alpha_4-a_2\varphi^2)\varphi^3=0. \tag{6.17}$$

We managed to find a solution of (6.17) only for the case $a_1=C_2=0$, i.e.,

$$\varphi^3=e^{\frac14\alpha_3\omega^2}V\big(\alpha_3^{1/2}(\omega+2a_2\alpha_2\alpha_3^{-2}),\nu\big),$$

where $\nu=4\alpha_3^{-1}\big(\alpha_4+a_2C_1-a_2^2(\alpha_2^2\alpha_3^{-2}+1)\big)$. Here $V(\tau,\nu)$ is the general solution of the Weber equation

$$4V_{\tau\tau}=(\tau^2+\nu)V. \tag{6.18}$$

3. The general solution of system (6.10) has the form:

$$\varphi^1=C_1\omega^{-2}+\tfrac12\alpha_3, \tag{6.19}$$

$$\begin{aligned}\varphi^2&=C_2+C_3\int\omega^{C_1+1}e^{\frac14\alpha_3\omega^2}d\omega-\tfrac12\alpha_2\omega^2+\\&+a_1\int\omega^{C_1+1}e^{\frac14\alpha_3\omega^2}\Big(\int\omega^{-C_1-1}e^{-\frac14\alpha_3\omega^2}d\omega\Big)d\omega,\end{aligned} \tag{6.20}$$

$$\begin{aligned}\varphi^3&=C_4+C_5\int\omega^{C_1-1}e^{\frac14\alpha_3\omega^2}d\omega+\\&+\int\omega^{C_1-1}e^{\frac14\alpha_3\omega^2}\Big(\int\omega^{1-C_1}e^{-\frac14\alpha_3\omega^2}(\alpha_5+a_2\omega^{-2}\varphi^2)d\omega\Big)d\omega,\end{aligned}$$

$$h=C_6-\tfrac18\alpha_3^2\omega^2-\tfrac12C_1^2\omega^{-2}+\int(\varphi^2(\omega))^2\omega^{-3}d\omega-\alpha_2\int\omega^{-1}\varphi^2(\omega)d\omega. \tag{6.21}$$

4. System (6.11) implies that the functions φ^i and h are determined by (6.19)–(6.21), and the function φ^3 satisfies the equation

$$\varphi^3_{\omega\omega}-\big((C_1-1)\omega^{-1}+\tfrac12\alpha_3\omega\big)\varphi^3_\omega+\big(a_2\omega^{-2}(a_2-\varphi^2)-\alpha_4\big)\varphi^3=0. \tag{6.22}$$

We managed to solve equation (6.22) in following cases:

A. $C_3 = a_1 = 0$, $\alpha_3 \neq 0$:

$$\varphi^3 = \omega^{\frac{1}{2}C_1 - 1} e^{\frac{1}{8}\alpha_3\omega^2} W(\varkappa, \mu, \tfrac{1}{4}\alpha_3\omega^2),$$

where $\varkappa = \frac{1}{4}\big(2 - C_1 - (4\alpha_4 + 2\alpha_2 a_2)\alpha_3^{-1}\big)$, $\mu = \frac{1}{4}(C_1^2 - 4a_2^2 + 4a_2C_2)^{1/2}$. Here $W(\varkappa, \mu, \tau)$ is the general solution of the Whittaker equation (4.21).

Let $\alpha_3 = 0$, then

$$\varphi^2 = \begin{cases} C_2 + C_3 \ln\omega + \frac{1}{4}(a_1 + 2\alpha_2)\omega^2, & C_1 = -2, \\ C_2 + \frac{1}{2}C_3\omega^2 + \frac{1}{2}a_1\omega^2(\ln\omega - \frac{1}{2}), & C_1 = 0, \\ C_2 + C_3(C_1+2)^{-1}\omega^{C_1+2} - \frac{1}{2}C_1^{-1}(a_1 - \alpha_2 C_1)\omega^2, & C_1 \neq 0, -2. \end{cases}$$

B. $C_3 = a_1 - \alpha_2 C_1 = 0$:

$$\varphi^3 = \begin{cases} \omega^{\frac{1}{2}C_1} Z_\nu(\mu^{1/2}\omega), & \mu \neq 0, \\ \omega^{\frac{1}{2}C_1}(C_5\omega^\nu + C_6\omega^{-\nu}), & \mu = 0, \nu \neq 0, \\ \omega^{\frac{1}{2}C_1}(C_5 + C_6\ln\omega), & \mu = 0, \nu = 0, \end{cases} \tag{6.23}$$

where $\mu = -\alpha_4$, $\nu = \frac{1}{2}(C_1^2 - 4a_2^2 + 4a_2C_2)^{1/2}$. Here and below $Z_\nu(\tau)$ is the general solution of the Bessel equation (4.22).

C. $C_3 = 0$, $C_1 \neq 0$: φ^3 is determined by (6.23), where

$$\mu = \tfrac{1}{2}a_2C_1^{-1}(a_1 - \alpha_2C_1) - \alpha_4, \quad \nu = \tfrac{1}{2}(C_1^2 - 4a_2^2 + 4a_2C_2)^{1/2}.$$

D. $C_1 = a_1 = 0$: φ^3 is determined by (6.23), where

$$\mu = -\tfrac{1}{2}a_2C_3 - \alpha_4, \quad \nu = (-a_2^2 + a_2C_2)^{1/2}.$$

E. $C_3 \neq 0$, $C_1 \notin \{0; -2\}$, $a_2(a_1 - \alpha_2C_1) - 2\alpha_4C_1 = 0$:

$$\varphi^3 = \omega^{\frac{1}{2}C_1} Z_\nu(\mu\omega^{1+\frac{1}{2}C_1}),$$

where $\mu = 2C_3^{1/2}(C_1+2)^{-3/2}$, $\nu = (C_1+2)^{-1}(C_1^2 - 4a_2^2 + 4a_2C_2)^{1/2}$.

F. $C_1 = -2$, $C_3 \neq 0$, $a_2(a_1 + 2\alpha_2) + 4\alpha_4 = 0$ ([19]):

$$\varphi^3 = \omega^{-1}\xi^{1/2} Z_{1/3}(\tfrac{2}{3}C_3^{1/2}\xi^{3/2}),$$

where $\xi = \ln\omega + C_3^{-1}(a_2^2 - a_2C_2 - 1)$.

G. $C_1 = 2$, $C_3 < 0$, $1 - a_2^2 + a_2C_2 \geq 0$:

$$\varphi^3 = W(\varkappa, \mu, \tfrac{1}{2}(-C_3)^{1/2}\omega^2),$$

where $\varkappa = \frac{1}{8}(-C_3)^{-1/2}(-4\alpha_4 + a_2^2 - 2\alpha_2a_2)$, $\mu = \frac{1}{2}(1 - a_2^2 + a_2C_2)^{1/2}$. Here $W(\varkappa, \mu, \tau)$ is the general solution of the Whittaker equation (4.21).

5–7. Identical corollaries of system (6.12), (6.13), and (6.14) are the equations

$$\varphi^2 = a\varphi^1 + C_1, \tag{6.24}$$

$$h = a(1 + a^2)\varphi^1_\omega + (2 + 2a^2 - aC_1)\varphi^1 + C_2, \tag{6.25}$$

$$(1+a^2)\varphi^1_{\omega\omega}+(4a-C_1)\varphi^1_\omega+\varphi^1\varphi^1+4\varphi^1+(1+a^2)^{-1}(C_1^2+2C_2)=0. \quad (6.26)$$

We found the following solutions of (6.26):

A. If $(1+a^2)^{-1}(C_1^2+2C_2)<4$:

$$\varphi^1=\big(4-(1+a^2)^{-1}(C_1^2+2C_2)\big)^{1/2}-2. \quad (6.27)$$

B. If $C_1=4a$:

$$\varphi^1=-6\wp\left(\frac{\omega}{(1+a^2)^{1/2}}+C_4,\ \frac{4}{3}-\frac{(C_1^2+2C_2)}{3(1+a^2)},\ C_3\right)-2. \quad (6.28)$$

Here and below $\wp(\tau,\varkappa_1,\varkappa_2)$ is the Weierstrass function satisfying equation (6.15). If $C_2=2-6a^2$ and $C_3=0$, a particular case of (6.28) is the function

$$\varphi^1=-6(1+a^2)\omega^2-2 \quad (6.29)$$

(the constant C_4 is considered to vanish).

C. If $_1\neq 4a$, $(1+a^2)^{-1}(C_1^2+2C_2)-4=-9\mu^4$:

$$\varphi^1=-6\mu^2e^{-2\xi}\wp(e^{-\xi}+C_4,0,C_3)+3\mu^2-2, \quad (6.30)$$

where $\xi=(1+a^2)^{-1/2}\mu\omega$, $\mu=\frac{1}{5}(4a-C_1)(1+a^2)^{-1/2}$. If $C_3=0$, a paticular case of (6.30) is the function

$$\varphi^1=-6\mu^2e^{-2\xi}(e^{-\xi}+C_4)^{-2}+3\mu^2-2, \quad (6.31)$$

where the constant C_4 is considered not to vanish.

The function φ^3 has to be found for systems (6.12), (6.13), and (6.14) individually.

5. The function φ^3 satisfy the equation

$$(1+a^2)\varphi^3_{\omega\omega}-(C_1+4a)\varphi^3_\omega-(2\varphi^1-4)\varphi^3-\alpha_5=0.$$

If φ^1 is determined by (6.27), we obtain

$$\begin{aligned}\varphi^3=\exp\big(\tfrac{1}{2}(1+a^2)^{-1}(C_1+4a)\omega\big)\times&\\ \times\left\{\begin{array}{ll} C_5\exp(\nu^{1/2}\omega)+C_6\exp(-\nu^{1/2}\omega), & \nu>0\\ C_5\cos((-\nu)^{1/2}\omega)+C_6\sin((-\nu)^{1/2}\omega), & \nu<0\\ C_5+C_6\omega, & \nu=0\end{array}\right\}&+\\ +\left\{\begin{array}{lll} -\alpha_5(2\varphi^1-4)^{-1}, & 2\varphi^1-4\neq0 & \\ -\alpha_5(4a+C_1)^{-1}\omega, & 2\varphi^1-4=0, & C_1+4a\neq0\\ \tfrac{1}{2}\alpha_5(1+a^2)^{-1}\omega^2, & 2\varphi^1-4=0, & C_1+4a=0\end{array}\right\}&,\end{aligned}$$

where $\nu=\frac{1}{4}(1+a^2)^{-2}(C_1+4a)^2-(1+a^2)^{-1}(4-2\varphi^1)$.

6. In this case φ^3 satisfy the equation

$$(1+a^2)\varphi^3_{\omega\omega}-C_1\varphi^3_\omega=a_1\varphi^1.$$

Therefore,

$$\varphi^3 = C_5 + C_6 \exp\big((1+a^2)^{-1}C_1\omega\big) + a_1 C_1^{-1}\Big(\int \varphi^1(\omega)d\omega + \\ + \exp\big((1+a^2)^{-1}C_1\omega\big)\int \exp\big(-(1+a^2)^{-1}C_1\omega\big)\varphi^1(\omega)d\omega\Big)$$

for $C_1 \neq 0$, and

$$\varphi^3 = C_5 + C_6\omega + a_1(1+a^2)^{-1}\big(\omega\int\varphi^1(\omega)d\omega - \int\omega\varphi^1(\omega)d\omega\big)$$

for $C_1 = 0$.

7. The function φ^3 satisfy the equation

$$(1+a^2)\varphi^3_{\omega\omega} - (C_1 + 2a_1 a)\varphi^3_\omega + (a_1^2 - a_1\varphi^1)\varphi^3 = 0. \tag{6.32}$$

A. If φ^1 is determined by (6.27), it follows that

$$\varphi^3 = \exp\big(\tfrac{1}{2}(1+a^2)^{-1}(C_1+2a_1a)\omega\big) \times \\ \times \left\{\begin{array}{ll} C_5\exp(\nu^{1/2}\omega) + C_6\exp(-\nu^{1/2}\omega), & \nu > 0 \\ C_5\cos((-\nu)^{1/2}\omega) + C_6\sin((-\nu)^{1/2}\omega), & \nu < 0 \\ C_5 + C_6\omega, & \nu = 0 \end{array}\right\},$$

where $\nu = \frac{1}{4}(1+a^2)^{-2}(C_1+2a_1a)^2 - (1+a^2)^{-1}(a_1^2 - a_1\varphi^1)$.

B. If $C_1 = 4a$, that is, φ^1 is determined by (6.27), we obtain

$$\varphi^3 = \exp\big(a(a_1+2)(1+a^2)^{-1}\omega\big)\theta(\tau),$$

where $\tau = (1+a^2)^{-1/2}\omega + C_4$. Here the function $\theta = \theta(\tau)$ is the general solution of of the following Lame equation ([19]):

$$\theta_{\tau\tau} + \big(6a_1\wp(\tau) + a_1^2 + 2a_1 - a^2(2+a_1)^2(1+a^2)^{-1}\big)\theta = 0$$

with the Weierstrass function

$$\wp(\tau) = \wp\Big(\tau, \tfrac{1}{3}\big(4 - (1+a^2)^{-1}(C_1^2 + 2C_2)\big), C_3\Big).$$

Consider the particular case when $C_2 = 2 - 6a^2$ and $C_3 = 0$ additionally, i.e., φ^1 can be given in form (6.29). Depending on the values of a and a_1, we obtain the following expression for φ^3:

Case 1. $a_1 \neq -2$, $a_1 \neq 2a^2$:

$$\varphi^3 = |\omega|^{1/2} exp\left(\frac{a(2+a_1)}{1+a^2}\omega\right) Z_\nu\left(\frac{\big((2+a_1)(a_1-2a^2)\big)^{1/2}}{1+a^2}\omega\right),$$

where $\nu = (\frac{1}{4} - 6a_1)^{1/2}$.

Case 2. $a_1 = -2$: $\varphi^3 = C_5\omega^4 + C_6\omega^{-3}$.

Case 3. $a_1 = 2a^2$:

Case 3.1. $48a^2 < 1$: $\varphi^3 = |\omega|^{1/2}e^{2a\omega}\big(C_5\omega^\sigma + C_6\omega^{-\sigma}\big)$, where $\sigma = \frac{1}{2}\sqrt{1-48a^2}$.

Case 3.2. $48a^2 = 1$, that is, $a = \pm\frac{1}{12}\sqrt{3}$: $\varphi^3 = |\omega|^{1/2}(C_5 + C_6\ln\omega)$.

Case 3.3. $48a^2 > 1$: $\varphi^3 = |\omega|^{1/2}e^{2a\omega}\bigl(C_5\cos(\gamma\ln\omega) + C_6\sin(\gamma\ln\omega)\bigr)$, where $\gamma = \frac{1}{2}\sqrt{48a^2-1}$.

C. Let the conditions

$$C_1 \neq 4a, \quad (1+a^2)^{-1}(C_1^2 + 2C_2) - 4 = -9\mu^4$$

be satisfied, that is, let φ^1 be determined by (6.30). Transforming the variables in equation (6.32) by the formulas:

$$\varphi^3 = \tau^{-1/2}\exp\Bigl(\tfrac{1}{2}(C_1 + 2aa_1)(1+a^2)^{-1}\omega\Bigr)\theta(\tau),$$
$$\tau = \exp\bigl(-\mu(1+a^2)^{-1/2}\omega\bigr),$$

we obtain the following equation in the function $\theta = \theta(\tau)$:

$$\tau^2\theta_{\tau\tau} + \bigl(6a_1\tau^2\wp(\tau + C_4, 0, C_3) + \sigma\bigr)\theta = 0, \qquad (6.33)$$

where $\sigma = \mu^{-2}\bigl(a_1^2 + 2a_1 - \frac{1}{4}(1+a^2)^{-1}(C_1^2 + 2aa_1)^2\bigr) - 3a_1 + \frac{1}{4}$. If $\sigma = 0$, equation (6.33) is a Lame equation.

In the particular case when φ^1 is determined by (6.31), equation (6.33) has the form:

$$\tau^2(\tau + C_4)^2\theta_{\tau\tau} + \bigl(6a_1\tau^2 + \sigma(\tau + C_4)^2\bigr)\theta = 0. \qquad (6.34)$$

By means of the following transformation of variables:

$$\theta = |\xi|^{\nu_1}|\xi - 1|^{\nu_2}\psi(\xi), \quad \xi = -C_4^{-1}\tau,$$

where $\nu_1(\nu_1 - 1) + \sigma = 0$ and $\nu_2(\nu_2 - 1) + 6a_1 = 0$, equation (6.34) is reduced to a hypergeometric equation of the form (see [19]):

$$\xi(\xi - 1)\psi_{\xi\xi} + \bigl(2(\nu_1 + \nu_2)\xi - 2\nu_1\bigr)\psi_\xi + 2\nu_1\nu_2\psi = 0.$$

If $\sigma = 0$, equation (6.34) implies that

$$(\tau + C_4)^2\theta_{\tau\tau} + 6a_1\theta = 0.$$

Therefore,

$$\theta = C_5|\tau + C_4|^{1/2-\nu} + C_6|\tau + C_4|^{1/2+\nu}$$

if $a_1 < \frac{1}{24}$, where $\nu = (\frac{1}{4} - 6a_1)^{1/2}$,

$$\theta = |\tau + C_4|^{1/2}\bigl(C_5 + C_6\ln|\tau + C_4|\bigr)$$

if $a_1 = \frac{1}{24}$, and

$$\theta = |\tau + C_4|^{1/2}\bigl(C_5\cos(\nu\ln|\tau + C_4|) + C_6\sin(\nu\ln|\tau + C_4|)\bigr)$$

if $a_1 > \frac{1}{24}$, where $\nu = (6a_1 - \frac{1}{4})^{1/2}$.

7 Exact solutions of system (2.9)

Among the reduced systems from Section 2, only particular cases of system (2.9) have Lie symmetry operators that are not induced by elements from $A(NS)$. Therefore, Lie reductions of the other systems from Section 2 give only solutions that can be obtained by means of reducing the NSEs with two- and three-dimensional subalgebras of $A(NS)$.

Here we consider system (2.9) with ρ^i vanishing. As mentioned in Note 2.5, in this case the vector-function $\vec{m}$ has the form $\vec{m} = \eta(t)\vec{e}$, where $\vec{e} = \text{const}$, $|\vec{e}| = 1$, and $\eta = \eta(t) = |\vec{m}(t)| \neq 0$. Without loss of generality we can assume that $\vec{e} = (0,0,1)$, i.e.,

$$\vec{m} = (0, 0, \eta(t)).$$

For such vector $\vec{m}$, conditions (2.5) are satisfied by the following vector $\vec{n}^i$:

$$\vec{n}^1 = (1,0,0), \quad \vec{n}^2 = (0,1,0).$$

Therefore, ansatz (2.4) and system (2.9) can be written, respectively, in the forms:

$$\begin{aligned} &u^1 = v^1, \quad u^2 = v^2, \quad u^3 = \big(\eta(t)\big)^{-1}\big(v^3 + \eta_t(t)x_3\big), \\ &p = q - \tfrac{1}{2}\eta_{tt}(t)\big(\eta(t)\big)^{-1}x_3^2, \end{aligned} \tag{7.1}$$

where $v = v(y_1, y_2, y_3)$, $q = q(y_1, y_2, y_3)$, $y_i = x_i$, $y_3 = t$, and

$$\begin{aligned} &v^i_t + v^j v^i_j - v^i_{jj} + q_i = 0, \\ &v^3_t + v^j v^3_j - v^3_{jj} = 0, \\ &v^i_i + \rho^3 = 0, \end{aligned} \tag{7.2}$$

where $\rho^3 = \rho^3(t) = \eta_t/\eta$.

It was shown in Note 2.8 that there exists a local transformation which make ρ^3 vanish. Therefore, we can consider system (7.2) only with ρ^3 vanishing and extend the obtained results in the case $\rho^3 \neq 0$ by means of transformation (2.12). However it will be sometimes convenient to investigate, at once, system (7.2) with an arbitrary function ρ^3.

The MIA of (7.2) with $\rho^3 = 0$ is given by the algebra

$$B = \langle R_3(\bar{\psi}),\ Z^1(\lambda),\ D^1_3,\ \partial_t,\ J^1_{12},\ \partial_{v^3},\ v^3\partial_{v^3}\rangle$$

(see notations in Subsection 2.1). We construct complete sets of inequivalent one-dimensional subalgebras of B and choose such algebras, among these subalgebras, that can be used to reduce system (7.2) and do not lie in the linear span of the operators $R_3(\bar{\psi})$, $Z^1(\lambda)$, J^1_{12}, i.e., the operators which are induced by operators from $A(NS)$ for arbitrary ρ^3. As a result we obtain the following algebras (more exactly, the following classes of algebras):

The one-dimentional subalgebras:

1. $B^1_1 = \langle D^1_3 + 2\varkappa J^1_{12} + 2\gamma v^3\partial_{v^3} + 2\beta\partial_{v^3}\rangle$, where $\gamma\beta = 0$.
2. $B^1_2 = \langle \partial_t + \varkappa J^1_{12} + \gamma v^3\partial_{v^3} + \beta\partial_{v^3}\rangle$, where $\gamma\beta = 0$, $\varkappa \in \{0; 1\}$.
3. $B^1_3 = \langle J^1_{12} + \gamma v^3\partial_{v^3} + Z^1(\lambda(t))\rangle$, where $\gamma \neq 0$, $\lambda \in C^\infty((t_0, t_1), \mathbb{R})$.
4. $B^1_4 = \langle R_3(\bar{\psi}(t)) + \gamma v^3\partial_{v^3}\rangle$, where $\gamma \neq 0$,
   $\bar{\psi}(t) = (\psi^1(t), \psi^2(t)) \neq (0,0)\ \forall\, t \in (t_0, t_1)$, $\psi^i \in C^\infty((t_0, t_1), \mathbb{R})$.

The two-dimentional subalgebras:

1. $B^2_1 = \langle \partial_t + \beta_2\partial_{v^3},\ D^1_3 + \varkappa J^1_{12} + \gamma v^3\partial_{v^3} + \beta_1\partial_{v^3}\rangle$,
   where $\gamma\beta_1 = 0, (\gamma-2)\beta_2 = 0$.
2. $B^2_2 = \langle D^1_3 + 2\gamma_1 v^3\partial_{v^3} + 2\beta_1\partial_{v^3},\ J^1_{12} + \gamma_2 v^3\partial_{v^3} + \beta_2\partial_{v^3} + Z^1(\varepsilon|t|^{-1})\rangle$,
   where $\gamma_1\beta_1 = 0,\ \gamma_2\beta_2 = 0,\ \gamma_1\beta_2 - \gamma_2\beta_1 = 0$.
3. $B^2_3 = \langle D^1_3 + 2\varkappa J^1_{12} + 2\gamma_1 v^3\partial_{v^3} + 2\beta_1\partial_{v^3},\ R_3(|t|^{\sigma+1/2}\cos\tau, |t|^{\sigma+1/2}\sin\tau) +$
   $+ \gamma_2 v^3\partial_{v^3} + \beta_2\partial_{v^3} + Z^1(\varepsilon|t|^{\sigma-1})\rangle$, where $\tau = \varkappa\ln|t|$,
   $(\gamma_1+\sigma)\beta_1 - \gamma_2\beta_1 = 0,\ \sigma\gamma_2 = 0,\ \varepsilon\sigma = 0$.
4. $B^2_4 = \langle \partial_t + \gamma_1 v^3\partial_{v^3} + \beta_1\partial_{v^3},\ J^1_{12} + \gamma_2 v^3\partial_{v^3} + \beta_2\partial_{v^3} + Z^1(\varepsilon)\rangle$,
   where $\gamma_1\beta_1 = 0,\ \gamma_2\beta_2 = 0,\ \gamma_1\beta_2 - \gamma_2\beta_1 = 0$.
5. $B^2_5 = \langle \partial_t + \varkappa J^1_{12} + \gamma_1 v^3\partial_{v^3} + \beta_1\partial_{v^3},\ R_3(e^{\sigma t}\cos\varkappa t, e^{\sigma t}\sin\varkappa t) +$
   $+ Z^1(\varepsilon e^{\sigma t}) + \gamma_2 v^3\partial_{v^3} + \beta_2\partial_{v^3}\rangle$, where $(\gamma_1+\sigma)\beta_1 - \gamma_2\beta_1 = 0$,
   $\sigma\gamma_2 = 0,\ \varepsilon\sigma = 0$.
6. $B^2_6 = \langle R_3(\bar\psi^1) + \gamma v^3\partial_{v^3},\ R_3(\bar\psi^2)\rangle$, where $\bar\psi^i = (\psi^{i1}(t), \psi^{i2}(t)) \neq (0,0)$
   $\forall\, t\in(t_0,t_1),\ \psi^{ij}\in C^\infty((t_0,t_1),\mathbb{R}),\ \bar\psi^1_{tt}\cdot\bar\psi^2 - \bar\psi^1\cdot\bar\psi^2_{tt} = 0,\ \gamma\neq 0$.
   Hereafter $\bar\psi^1\cdot\bar\psi^2 := \psi^{1i}\psi^{2i}$.

Let us reduce system (7.2) to systems of PDEs in two independent variables. With the algebras B^1_1–B^1_4 we can construct the following complete set of Lie ansatzes of codimension 1 for system (7.2) with $\rho^3 = 0$:

1.
$$\begin{aligned} &v^1 = |t|^{-1/2}(w^1\cos\tau - w^2\sin\tau) + \tfrac12 y_1 t^{-1} - \varkappa y_2 t^{-1},\\ &v^2 = |t|^{-1/2}(w^1\sin\tau + w^2\cos\tau) + \tfrac12 y_2 t^{-1} + \varkappa y_1 t^{-1},\\ &v^3 = |t|^\gamma w^3 + \beta\ln|t|,\\ &q = |t|^{-1}s + \tfrac12(\varkappa^2 + \tfrac14)t^{-2}r^2, \end{aligned} \tag{7.3}$$

where $\tau = \varkappa\ln|t|$, $\gamma\beta = 0$,

$$z_1 = |t|^{-1/2}(y_1\cos\tau + y_2\sin\tau), \quad z_2 = |t|^{-1/2}(-y_1\sin\tau + y_2\cos\tau).$$

Here and below $w^a = w^a(z_1,z_2)$, $s = s(z_1,z_2)$, $r = (y_1^2+y_2^2)^{1/2}$.

2.
$$\begin{aligned} &v^1 = w^1\cos\varkappa t - w^2\sin\varkappa t - \varkappa y_2,\\ &v^2 = w^1\sin\varkappa t + w^2\cos\varkappa t + \varkappa y_1,\\ &v^3 = w^3 e^{\gamma t} + \beta t,\\ &q = s + \tfrac12\varkappa^2 r^2, \end{aligned} \tag{7.4}$$

where $\varkappa\in\{0;1\}$, $\gamma\beta = 0$,

$$z_1 = y_1\cos\varkappa t + y_2\sin\varkappa t, \quad z_2 = -y_1\sin\varkappa t + y_2\cos\varkappa t.$$

3.
$$\begin{aligned} &v^1 = y_1 r^{-1}w^3 - y_2 r^{-2}w^1 - \gamma y_2 r^{-2},\\ &v^2 = y_2 r^{-1}w^3 + y_1 r^{-2}w^1 + \gamma y_1 r^{-2},\\ &v^3 = w^2 e^{\gamma\arctan y_2/y_1},\\ &q = s + \lambda(t)\arctan y_2/y_1, \end{aligned} \tag{7.5}$$

where $_1 = t$, $z_2 = r$, $\gamma \neq 0$, $\lambda \in C^\infty((t_0, t_1), \mathbb{R})$.

$$\begin{aligned} 4.\quad & \bar v = (\bar\psi\cdot\bar\psi)^{-1}\Big((w^1+\gamma)\bar\psi + w^3\bar\theta + (\bar\psi\cdot\bar y)\bar\psi_t - z_2\bar\theta_t\Big) \\ & v^3 = w^2\exp\big(\gamma(\bar\psi\cdot\bar\psi)^{-1}(\bar\psi\cdot\bar y)\big) \\ & q = s - (\bar\psi\cdot\bar\psi)^{-1}(\bar\psi_{tt}\cdot\bar y)(\bar\psi\cdot\bar y) + \tfrac12(\bar\psi\cdot\bar\psi)^{-2}(\bar\psi_{tt}\cdot\bar\psi)(\bar\psi\cdot\bar y)^2, \end{aligned} \tag{7.6}$$

where $z_1 = t$, $z_2 = (\bar\theta\cdot\bar y)$, $\gamma\neq 0$, $\bar v = (v^1, v^2)$, $\bar y = (y_1, y_2)$, $\psi^i \in C^\infty((t_0,t_1),\mathbb{R})$, $\bar\theta = (-\psi^2, \psi^1)$.

Substituting ansatzes (7.3) and (7.4) into system (7.2) with $\rho^3 = 0$, we obtain a reduced system of the form (6.1), where

$$\begin{aligned} &\alpha_1 = 0,\quad \alpha_2 = -1,\quad \alpha_3 = -2\varkappa,\quad \alpha_4 = \gamma,\quad \alpha_5 = \beta \quad \text{if}\quad t>0 \quad \text{and} \\ &\alpha_1 = 0,\quad \alpha_2 = 1,\quad \alpha_3 = 2\varkappa,\quad \alpha_4 = -\gamma,\quad \alpha_5 = -\beta \quad \text{if}\quad t<0 \end{aligned}$$

for ansatz (7.3) and

$$\alpha_1 = 0,\quad \alpha_2 = 0,\quad \alpha_3 = -2\varkappa,\quad \alpha_4 = \gamma,\quad \alpha_5 = \beta$$

for ansatz (7.4). System (6.1) is investigated in Section 6 in detail.

Because the form of ansatzes (7.3) is not changed after transformation (2.12), it is convinient to substitute their into a system of form (7.2) with an arbitrary function ρ^3. As a result of substituting, we obtain the following reduced systems:

$$\begin{aligned} 3.\quad & w^3_1 + w^3w^3_2 - z_2^{-3}(w^1+\gamma)^2 - (w^3_{22} + z_2^{-1}w^3_2 - z_2^{-2}w^3) + s_2 = 0, \\ & w^1_1 + w^3w^1_2 - w^1_{22} + z_2^{-1}w^1_2 + \lambda = 0, \\ & w^2_1 + w^3w^2_2 - w^2_{22} - z_2^{-1}w^2_2 + \gamma z_2^{-2}w^1w^2 = 0, \\ & w^3_2 + z_2^{-1}w^3 = -\eta_1/\eta. \end{aligned} \tag{7.7}$$

$$\begin{aligned} 4.\quad & w^1_1 + w^3w^1_2 - (\bar\psi\cdot\bar\psi)w^1_{22} = 0, \\ & w^3_1 + w^3w^3_2 - (\bar\psi\cdot\bar\psi)w^3_{22} + (\bar\psi\cdot\bar\psi)s_2 + 2(w^1+\gamma)(\bar\psi\cdot\bar\theta)(\bar\psi\cdot\bar\psi)^{-1} - \\ & \quad - 2(\bar\psi_t\cdot\bar\psi)(\bar\psi\cdot\bar\psi)^{-1}w^3 + (2\bar\psi_t\cdot\bar\psi_t - \bar\psi_{tt}\cdot\bar\psi)(\bar\psi\cdot\bar\psi)^{-1}z_2 = 0, \\ & w^2_1 + w^3w^2_2 - (\bar\psi\cdot\bar\psi)w^2_{22} + \gamma(\bar\psi\cdot\bar\psi)^{-1}\big(w^1 + (\bar\psi_t\cdot\bar\theta)(\bar\psi\cdot\bar\psi)^{-1}z_2\big)w^2 = 0, \\ & w^3_2 + \eta_t/\eta = 0. \end{aligned} \tag{7.8}$$

Unlike systems 8 and 9 from Subsection 3.2, systems (7.7) and (7.8) are not reduced to linear systems of PDEs.

Let us investigate system (7.7). The last equation of (7.7) immediately gives

$$\begin{aligned} & (w^3_2 + z_2^{-1}w^3)_2 = w^3_{22} + z_2^{-1}w^3_2 - z_2^{-2}w^3 = 0, \\ & w^3 = (\chi-1)z_2^{-1} - \tfrac12\eta_t\eta^{-1}z_2, \end{aligned} \tag{7.9}$$

where $\chi = \chi(t)$ is an arbitrary differentiable function of $t = z_2$. Then it follows from the first equation of (7.7) that

$$s = \int z_2^{-3}(w^1+\gamma)^2dz_2 - \tfrac12(\chi-1)^2z_2^{-2} + \tfrac14 z_2^2\Big((\eta_t/\eta)_t - \tfrac12(\eta_t/\eta)^2\Big) - \chi_t\ln|z_2|.$$

Substituting (7.9) into the remaining equations of (7.7), we get

$$\begin{aligned} &w^1_1 - w^1_{22} + \big(\chi z_2^{-1} - \tfrac{1}{2}\eta_t\eta^{-1}z_2\big)w^1_2 + \lambda = 0,\\ &w^2_1 - w^2_{22} + \big((\chi-2)z_2^{-1} - \tfrac{1}{2}\eta_t\eta^{-1}z_2\big)w^2_2 + \gamma z_2^{-2}w^1w^2 = 0. \end{aligned} \tag{7.10}$$

By means of changing the independent variables

$$\tau = \textstyle\int |\eta(t)|dt, \quad z = |\eta(t)|^{1/2}z_2, \tag{7.11}$$

system (7.10) can be transformed to a system of a simpler form:

$$\begin{aligned} &w^1_\tau - w^1_{zz} + \hat\chi z^{-1}w^2_z + \hat\lambda|\hat\eta|^{-1} = 0,\\ &w^2_\tau - w^2_{zz} + (\hat\chi - 2)z^{-1}w^2_z + \gamma z^{-2}w^1w^2 = 0, \end{aligned} \tag{7.12}$$

where $\hat\eta(\tau) = \eta(t)$, $\hat\chi(\tau) = \chi(t)$, and $\hat\lambda(\tau) = \lambda(t)$.

If $\lambda(t) = -2C\eta(t)(\chi(t)-1)$ for some fixed constant C, particular solutions of (7.10) are functions

$$w^1 = C\eta(t)z_2^2, \quad w^2 = f(z_1,z_2)\exp\big(\gamma C\textstyle\int\eta(t)dt\big),$$

where f is an arbitrary solution of the following equation

$$f_1 - f_{22} + \big((\chi-2)z_2^{-1} - \tfrac{1}{2}\eta_t\eta^{-1}z_2\big)f_2 = 0. \tag{7.13}$$

In the variables from (7.11), equation (7.13) has form (5.22) with $\tilde\eta(\tau) = \chi(t) - 2$.

In the case $\lambda(t) = 8C(\chi(t)-1)\eta(t)\int\eta(t)(\chi(t)-3)dt$ ($C = \mathrm{const}$), particular solutions of (7.10) are functions

$$\begin{aligned} &w^1 = C\Big((\eta(t))^2z_2^4 - 4z_2^2\eta(t)\textstyle\int\eta(t)(\chi(t)-3)dt\Big),\\ &w^2 = f(z_1,z_2)\exp\big(\tfrac{1}{2}(\gamma C)^{1/2}\eta(t)z_2^2 + \xi(t)\big), \end{aligned}$$

where $\xi(t) = -(\gamma C)^{1/2}\int\eta(t)(\chi(t)-3)dt + 4\gamma C\int\eta(t)\big(\int\eta(t)(\chi(t)-3)dt\big)dt$ and f is an arbitrary solution of the following equation

$$f_1 - f_{22} + \big((\chi-2)z_2^{-1} - (\tfrac{1}{2}\eta_t\eta^{-1} + 2(\gamma C)^{1/2})z_2\big)f_2 = 0. \tag{7.14}$$

After the change of the independent variables

$$\tau = \textstyle\int|\eta(t)|\exp\big(4(\gamma C)^{1/2}\int\eta(t)dt\big)dt, \quad z = |\eta(t)|^{1/2}\exp\big(2(\gamma C)^{1/2}\int\eta(t)dt\big)z_2$$

in (7.14), we obtain equation (5.22) with $\tilde\eta(\tau) = \chi(t) - 2$ again.

Let us continue to system (7.8). The last equation of (7.8) integrates with respect to z_2 to the following expression: $w^3 = -\eta_t\eta^{-1}z_2 + \chi$. Here $\chi = \chi(t)$ is an differentiable function of $z_1 = y_3 = t$. Let us make the transformation from the symmetry group of (7.2):

$$\tilde{\bar v}(t,\bar y) = \bar v(t,\bar y - \bar\xi(t)) + \bar\xi_t(t), \quad \tilde v^3 = v^3, \quad \tilde q(t,\bar y) = q(t,\bar y - \bar\xi(t)) - \bar\xi_{tt}(t)\cdot\bar y,$$

where $\bar\xi_{tt}\cdot\bar\psi - \bar\xi\cdot\bar\psi_{tt} = 0$ and

$$\bar\xi_t\cdot\bar\theta + \chi + \eta_t\eta^{-1}(\bar\xi\cdot\bar\theta) - |\bar\psi|^{-2}(\bar\xi\cdot\bar\psi)(\bar\psi_t\cdot\bar\theta) + |\bar\psi|^{-2}(\bar\xi\cdot\bar\theta)(\bar\theta_t\cdot\bar\theta) = 0.$$

Hereafter $|\bar\psi|^2=\bar\psi\cdot\bar\psi$. This transformation does not modify ansatz (7.6), but it makes the function χ vanish, i.e., $\tilde w^3=-\eta_t\eta^{-1}z_2$. Therefore, without loss of generality we may assume, at once, that $w^3=-\eta_t\eta^{-1}z_2$.

Substituting the expression for w^3 in the other equations of (7.8), we obtain that

$$s=z_2^2|\bar\psi|^{-2}\Big(\big(\tfrac12\bar\psi_{tt}\cdot\bar\psi-\bar\psi_t\cdot\bar\psi_t-(\bar\psi_t\cdot\bar\psi)\eta_t\eta^{-1}\big)|\bar\psi|^{-2}+\tfrac12\eta_{tt}\eta^{-1}-(\eta_t)^2\eta^{-2}\Big)-\\ -2(\bar\psi_t\cdot\bar\theta)|\bar\psi|^{-2}\int w^1(z_1,z_2)dz_2,$$

$$\begin{aligned}&w^1_1-\eta_1\eta^{-1}z_2w^1_2-|\bar\psi|^2w^1_{22}=0,\\&w^2_1-\eta_1\eta^{-1}z_2w^2_2-|\bar\psi|^2w^2_{22}+\gamma|\bar\psi|^{-2}\big(2(\bar\psi_t\cdot\bar\theta)|\bar\psi|^{-2}z_2+w^1\big)w^2=0.\end{aligned}\tag{7.15}$$

The change of the independent variables

$$\tau=\int(\eta(t)|\bar\psi|)^2dt,\quad z=\eta(t)z_2$$

reduces system (7.15) to the following form:

$$\begin{aligned}&w^1_\tau-w^1_{zz}=0,\\&w^2_\tau-w^2_{zz}+\gamma|\bar{\hat\psi}|^{-4}\hat\eta^{-2}\big(2(\bar{\hat\psi}_t\cdot\bar{\hat\theta})\hat\eta z+w^1\big)w^2=0,\end{aligned}\tag{7.16}$$

where $\bar{\hat\psi}(\tau)=\bar\psi(t)$, $\bar{\hat\theta}(\tau)=\bar\theta(t)$, $\hat\eta(\tau)=\eta(t)$.

Particular solutions of (7.15) are the functions

$$\begin{aligned}&w^1=C_1+C_2\eta(t)z_2+C_3\big(\tfrac12(\eta(t)z_2)^2+\int(\eta(t)|\bar\psi|)^2dt\big),\\&w^2=f(t,z_2)\exp\big(\xi^2(t)z_2^2+\xi^1(t)z_2+\xi^0(t)\big),\end{aligned}$$

where $(\xi^2(t),\xi^1(t),\xi^0(t))$ is a particular solution of the system of ODEs:

$$\begin{aligned}&\xi^2_t-2\eta_t\eta^{-1}\xi^2-4|\bar\psi|^2(\xi^2)^2+\tfrac12C_3\gamma\eta^2|\bar\psi|^{-2}=0,\\&\xi^1_t-\eta_t\eta^{-1}\xi^1-4|\bar\psi|^2\xi^2\xi^1+2\gamma(\bar\psi_t\cdot\bar\theta)|\bar\psi|^{-4}+C_2\gamma\eta|\bar\psi|^{-2}=0,\\&\xi^0_t-2|\bar\psi|^2\xi^2-|\bar\psi|^2(\xi^1)^2+\gamma\big(C_1+C_3\int(\eta(t)|\bar\psi|)^2dt\big)|\bar\psi|^{-2}=0,\end{aligned}$$

and f is an arbitrary solution of the following equation

$$f_1-|\bar\psi|^2f_{22}+\big((\eta_t\eta^{-1}+4|\bar\psi|^2\xi^2)z_2+2|\bar\psi|^2\xi^1\big)f_2=0.\tag{7.17}$$

Equation (7.17) is reduced by means of a local transformation of the independent variables to the heat equation.

Consider the Lie reductions of system (7.2) to systems of ODEs. The second basis operator of the each algebra B^2_k, $k=\overline{1,5}$ induces, for the reduced system obtained from system (7.2) by means of the first basis operator, either a Lie symmetry operator from Table 2 or a operator giving a ansatz of form (6.4). Therefore, the Lie reduction of system (7.2) with the algebras $B^2_1-B^2_5$ gives only solutions that can be constructed for system (7.2) by means of reducing with the algebras B^1_1 and B^1_2 to system (6.1).

With the algebra B^2_6 we obtain an ansatz and a reduced system of the following forms:

$$\begin{aligned}&\bar v=\bar\phi+\lambda^{-1}(\bar\theta^i\cdot\bar y)\bar\psi^i_t,\quad v^3=\phi^3\exp\big(\gamma\lambda(\bar\theta^1\cdot\bar y)\big),\\&s=h-\tfrac12\lambda^{-1}(\bar\psi^i_{tt}\cdot\bar y)(\bar\theta^i\cdot\bar y),\end{aligned}\tag{7.18}$$

where $\phi^a = \phi^a(\omega)$, $h = h(\omega)$, $\omega = t$, $\lambda = \psi^{11}\psi^{22} - \psi^{12}\psi^{21} = \bar\psi^1 \cdot \bar\theta^1 = \bar\psi^2 \cdot \bar\theta^2$, $\bar\theta^1 = (\psi^{22}, -\psi^{21})$, $\bar\theta^2 = (-\psi^{12}, \psi^{11})$, and

$$\begin{aligned} &\bar\phi_t + \lambda^{-1}(\bar\theta^i \cdot \bar\phi)\bar\psi^i_t = 0, \quad \phi^3_t + \big(\gamma\lambda^{-1}(\bar\theta^1 \cdot \bar\phi) - \gamma^2\lambda^{-2}(\bar\theta^1 \cdot \bar\theta^1)\big)\phi^3 = 0, \\ &\lambda^{-1}(\bar\theta^i \cdot \bar\psi^i_t) + \eta_t\eta^{-1} = 0. \end{aligned} \tag{7.19}$$

Let us make the transformation from the symmetry group of system (7.2):

$$\tilde{\bar v}(t,\bar y) = \bar v(t, \bar y - \bar\xi) + \bar\xi_t, \quad \tilde v^3(t,\bar y) = v^3(t,\bar y - \bar\xi), \quad \tilde s(t,\bar y) = s(t,\bar y - \bar\xi) - \bar\xi_{tt}\cdot\bar y,$$

where

$$\bar\xi_t + \lambda^{-1}(\bar\theta^i \cdot \bar\xi)\bar\psi^i_t + \bar\phi = 0. \tag{7.20}$$

It follows from (7.20) that $\bar\xi_{tt} = \lambda^{-1}(\bar\theta^i \cdot \bar\xi)\bar\psi^i_{tt}$, i.e., $\bar\theta^i_{tt}\cdot\bar\xi - \bar\theta^i\cdot\bar\xi_{tt} = 0$. Therefore, this trasformation does not modify ansatz (7.18), but it makes the functions ϕ^i vanish. And without loss of generality we may assume, at once, that $\phi^i \equiv 0$. Then

$$\phi^3 = C\exp\Big(\int \big(\gamma\lambda^{-1}|\theta|\big)^2 dt\Big), \quad C = \text{const}.$$

The last equation of system (7.19) is the compatibility condition of system (7.2) and ansatz (7.18).

8 Conclusion

In this article we reduced the NSEs to systems of PDEs in three and two independent variables and systems of ODEs by means of the Lie method. Then, we investigated symmetry properties of the reduced systems of PDEs and made Lie reductions of systems which admitted non-trivial symmetry operators, i.e., operators that are not induced by operators from $A(NS)$. Some of the systems in two independent variables were reduced to linear systems of either two one-dimensional heat equations or two translational equations. We also managed to find exact solutions for most of the reduced systems of ODEs.

Now, let us give some remaining problems. Firstly, we failed, for the present, to describe the non-Lie ansatzes of form (1.6) that reduce the NSEs. (These ansatzes include, for example, the well-known ansatzes for the Karman swirling flows (see bibliography in [16]). One can also consider non-local ansatzes for the Navier–Stokes field, i.e., ansatzes containing derivatives of new unknown functions.

Second problem is to study non-Lie (i.e., non-local, conditional, and Q-conditional) symmetries of the NSEs [13].

And finally, it would be interesting to investigate compatability and to construct exact solutions of overdetermined systems that are obtained from the NSEs by means of different additional conditions. Usually one uses the condition where the nonlinearity has a simple form, for example, the potential form (see review [36]), i.e., $\mathrm{rot}((\vec u\cdot\vec\nabla)\vec u) = \vec 0$ (the NS fields satisfying this condition is called the generalized Beltrami flows). We managed to describe the general solution of the NSEs with the additional condition where the convective terms vanish [29, 30]. But one can give other conditions, for example,

$$\triangle\vec u = \vec 0, \quad \vec u_t + (\vec u\cdot\vec\nabla)\vec u = \vec 0,$$

and so on.

We will consider the problems above elsewhere.

Appendix

A Inequivalent one-, two-, and three-dimensional subalgebras of $A(NS)$

To find complete sets of inequivalent subalgebras of $A(NS)$, we use the method given, for example, in [27, 28]. Let us describe it briefly.

1. We find the commutation relations between the basis elements of $A(NS)$.

2. For arbitrary basis elements V, W^0 of $A(NS)$ and each $\varepsilon \in \mathbb{R}$ we calculate the adjoint action

$$W(\varepsilon) = \mathrm{Ad}(\varepsilon V)W^0 = \mathrm{Ad}(\exp(\varepsilon V))W^0$$

of the element $\exp(\varepsilon V)$ from the one-parameter group generated by the operator V on W^0. This calculation can be made in two ways: either by means of summing the Lie series

$$W(\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\{V^n, W^0\} = W^0 + \frac{\varepsilon}{1!}[V, W^0] + \frac{\varepsilon^2}{2!}[V,[V,W^0]] + \cdots, \qquad \text{(A.1)}$$

where $\{V^0, W^0\} = W^0$, $\{V^n, W^0\} = [V, \{V^{n-1}, W^0\}]$, or directly by means of solving the initial value problem

$$\frac{dW(\varepsilon)}{d\varepsilon} = [V, W(\varepsilon)], \quad W(0) = W^0. \qquad \text{(A.2)}$$

3. We take a subalgebra of a general form with a fixed dimension. Taking into account that the subalgebra is closed under the Lie bracket, we try to simplify it by means of adjoint actions as much as possible.

A.1 The commutation relations and the adjoint representation of the algebra $A(NS)$

Basis elements (1.2) of $A(NS)$ satisfy the following commutation relations:

$$\begin{aligned}
&[J_{12}, J_{23}] = -J_{31}, \quad [J_{23}, J_{31}] = -J_{12}, \quad [J_{31}, J_{12}] = -J_{23},\\
&[\partial_t, J_{ab}] = [D, J_{ab}] = 0, \quad [\partial_t, D] = 2\partial_t,\\
&[\partial_t, R(\vec{m})] = R(\vec{m}_t), \quad [D, R(\vec{m})] = R(2t\vec{m}_t - \vec{m}),\\
&[\partial_t, Z(\chi)] = Z(\chi_t), \quad [D, Z(\chi)] = Z(2t\chi_t + 2\chi),\\
&[R(\vec{m}), R(\vec{n})] = Z(\vec{m}_{tt}\cdot\vec{n} - \vec{m}\cdot\vec{n}_{tt}), \quad [J_{ab}, R(\vec{m})] = R(\vec{\tilde{m}}),\\
&[J_{ab}, Z(\chi)] = [Z(\chi), R(\vec{m})] = [Z(\chi), Z(\eta)] = 0,
\end{aligned} \qquad \text{(A.3)}$$

where $\tilde{m}^a = m^b$, $\tilde{m}^b = -m^a$, $\tilde{m}^c = 0$, $a \neq b \neq c \neq a$.

Note A.1 Relations (A.3) imply that the set of operators (1.2) generates an algebra when, for example, the parameter-functions m^a and χ belong to $C^\infty((t_0,t_1),\mathbb{R})$ ($C_0^\infty((t_0,t_1),\mathbb{R})$, $A((t_0,t_1),\mathbb{R})$), i.e., the set of infinite-differentiable (infinite-differentiable finite, real analytic) functions from (t_0,t_1) in $\mathbb{R}$, where $-\infty \leq t_0 < t_1 \leq +\infty$.

But the NSEs (1.1) admit operators (1.3) and (1.4) with parameter-functions of a less degree of smoothness. Moreover, the minimal degree of their smoothness depends on the smoothness that is demanded for the solutions of the NSEs (1.1). Thus, if $u^a \in C^2((t_0,t_1)\times\Omega,\mathbb{R})$ and $p \in C^1((t_0,t_1)\times\Omega,\mathbb{R})$, where Ω is a domain in $\mathbb{R}^3$, then it is sufficient that $m^a \in C^3((t_0,t_1),\mathbb{R})$ and $\chi \in C^1((t_0,t_1),\mathbb{R})$. Therefore, one can consider the "pseudoalgebra" generated by operators (1.2). The prefix "pseudo-" means that in this set of operators the commutation operation is not determined for all pairs of its elements, and the algebra axioms are satisfied only by elements, where they are defined. It is better to indicate the functional classes that are sets of values for the parameters m^a and χ in the notation of the algebra $A(NS)$. But below, for simplicity, we fix these classes, taking $m^a,\chi \in C^\infty((t_0,t_1),\mathbb{R})$, and keep the notation of the algebra generated by operators (1.2) in the form $A(NS)$. However, all calculations will be made in such a way that they can be translated for the case of a less degree of smoothness.

Most of the adjoint actions are calculated simply as sums of their Lie series. Thus,

$$
\begin{aligned}
&\mathrm{Ad}(\varepsilon\partial_t)D = D + 2\varepsilon\partial_t, \quad \mathrm{Ad}(\varepsilon D)\partial_t = e^{-2\varepsilon}\partial_t,\\
&\mathrm{Ad}(\varepsilon Z(\chi))\partial_t = \partial_t - \varepsilon Z(\chi_t), \quad \mathrm{Ad}(\varepsilon Z(\chi))D = D - \varepsilon Z(2t\chi_t + 2\chi),\\
&\mathrm{Ad}(\varepsilon R(\vec m))\partial_t = \partial_t - \varepsilon R(\vec m_t) - \tfrac12\varepsilon^2 Z(\vec m_t\cdot\vec m_{tt} - \vec m\cdot\vec m_{ttt}),\\
&\mathrm{Ad}(\varepsilon R(\vec m))D = D - \varepsilon R(2t\vec m_t - \vec m) -\\
&\qquad\qquad - \tfrac12\varepsilon^2 Z(2t\vec m_t\cdot\vec m_{tt} - 2t\vec m\cdot\vec m_{ttt} - 4\vec m\cdot\vec m_{tt}),\\
&\mathrm{Ad}(\varepsilon R(\vec m))J_{ab} = J_{ab} - \varepsilon R(\vec{\tilde m}) + \varepsilon^2 Z(m^a m^b_{tt} - m^a_{tt}m^b),\\
&\mathrm{Ad}(\varepsilon R(\vec m))R(\vec n) = R(\vec n) + \varepsilon Z(\vec m_{tt}\cdot\vec n - \vec m\cdot\vec n_{tt}), \quad \mathrm{Ad}(\varepsilon J_{ab})R(\vec m) = R(\vec{\hat m}),\\
&\mathrm{Ad}(\varepsilon J_{ab})J_{cd} = J_{cd}\cos\varepsilon + [J_{ab},J_{cd}]\sin\varepsilon \quad \big((a,b)\neq(c,d)\neq(b,a)\big),
\end{aligned}
\tag{A.4}
$$

where

$$
\begin{aligned}
&\tilde m^a = m^b, \quad \tilde m^b = -m^a, \quad \tilde m^c = 0, \quad a\neq b\neq c\neq a,\\
&\hat m^d = m^d\cos\varepsilon + \tilde m^d\sin\varepsilon, \quad \hat m^c = m^c, \quad a\neq b\neq c\neq a, \quad d\in\{a;b\}.
\end{aligned}
$$

Four adjoint actions are better found by means of integrating a system of form (A.2). As a result we obtain that

$$
\begin{aligned}
&\mathrm{Ad}(\varepsilon\partial_t)Z(\chi(t)) = Z(\chi(t+\varepsilon)), \quad \mathrm{Ad}(\varepsilon D)Z(\chi(t)) = Z(e^{2\varepsilon}\chi(te^{2\varepsilon})),\\
&\mathrm{Ad}(\varepsilon\partial_t)R(\vec m(t)) = R(\vec m(t+\varepsilon)), \quad \mathrm{Ad}(\varepsilon D)R(\vec m(t)) = R(e^{-\varepsilon}\vec m(te^{2\varepsilon})).
\end{aligned}
\tag{A.5}
$$

Cases where adjoint actions coincide with the identical mapping are omitted.

Note A.2 If $Z(\chi(t)) \in A(NS)[C^\infty((t_0,t_1),\mathbb{R})]$ with $-\infty < t_0$ or $t_1 < +\infty$, the adjoint representation $\mathrm{Ad}(\varepsilon\partial_t)$ ($\mathrm{Ad}(\varepsilon D)$) gives an equivalence relation between the operators $Z(\chi(t))$ and $Z(\chi(t+\varepsilon))$ ($Z(\chi(t))$ and $Z(e^{2\varepsilon}\chi(te^{2\varepsilon}))$) that belong to the different algebras

$$
\begin{aligned}
&A(NS)[C^\infty((t_0,t_1),\mathbb{R})] \quad\text{and}\quad A(NS)[C^\infty((t_0-\varepsilon,t_1-\varepsilon),\mathbb{R})]\\
&(A(NS)[C^\infty((t_0,t_1),\mathbb{R})] \quad\text{and}\quad A(NS)[C^\infty((t_0e^{-2\varepsilon},t_1e^{-2\varepsilon}),\mathbb{R})])
\end{aligned}
$$

respectively. An analogous statement is true for the operator $R(\vec m)$. Equivalence of subalgebras in Theorems A.1 and A.2 is also meant in this sense.

Note A.3 Besides the adjoint representations of operators (1.2) we make use of discrete transformation (1.6) for classifying the subalgebras of $A(NS)$,

To prove the theorem of this section, the following obvious lemma is used.

Lemma A.1 *Let* $N\in\mathbb{N}$.

A. *If* $\chi\in C^N((t_0,t_1),\mathbb{R})$, *then* $\exists\,\eta\in C^N((t_0,t_1),\mathbb{R}):\ 2t\eta_t+2\eta=\chi$.

B. *If* $\chi\in C^N((t_0,t_1),\mathbb{R})$, *then* $\exists\,\eta\in C^N((t_0,t_1),\mathbb{R}):\ 2t\eta_t-\eta=\chi$.

C. *If* $m^i\in C^N((t_0,t_1),\mathbb{R})$ *and* $a\in\mathbb{R}$, *then* $\exists\, l^i\in C^N((t_0,t_1),\mathbb{R}):$ $2tl^1_t-l^1+al^2=m^1,\quad 2tl^2_t-l^2-al^1=m^2.$

A.2 One-dimensional subalgebras

Theorem A.1 *A complete set of $A(NS)$-inequivalent one-dimensional subalgebras of $A(NS)$ is exhausted by the following algebras:*

1. $A^1_1(\varkappa)=\langle D+2\varkappa J_{12}\rangle$, *where* $\varkappa\ge 0$.

2. $A^1_2(\varkappa)=\langle\partial_t+\varkappa J_{12}\rangle$, *where* $\varkappa\in\{0;1\}$.

3. $A^1_3(\eta,\chi)=\langle J_{12}+R(0,0,\eta(t))+Z(\chi(t))\rangle$ *with smooth functions* η *and* χ. *Algebras* $A^1_3(\eta,\chi)$ *and* $A^1_3(\tilde\eta,\tilde\chi)$ *are equivalent if* $\exists\,\varepsilon,\delta\in\mathbb{R}$, $\exists\,\lambda\in C^\infty((t_0,t_1),\mathbb{R})$:

$$\tilde\eta(\tilde t)=e^{-\varepsilon}\eta(t),\quad \tilde\chi(\tilde t)=e^{2\varepsilon}(\chi(t)+\lambda_{tt}(t)\eta(t)-\lambda(t)\eta_{tt}(t)),\tag{A.6}$$

where $\tilde t=te^{-2\varepsilon}+\delta$.

4. $A^1_4(\vec m,\chi)=\langle R(\vec m(t))+Z(\chi(t))\rangle$ *with smooth functions* $\vec m$ *and* χ: $(\vec m,\chi)\not\equiv(\vec 0,0)$. *Algebras* $A^1_4(\vec m,\chi)$ *and* $A^1_4(\tilde{\vec m},\tilde\chi)$ *are equivalent if* $\exists\,\varepsilon,\delta\in\mathbb{R}$, $\exists\, C\neq 0$, $\exists\, B\in O(3)$, $\exists\,\vec l\in C^\infty((t_0,t_1),\mathbb{R}^3)$:

$$\tilde{\vec m}(\tilde t)=Ce^{-\varepsilon}B\vec m(t),\quad \tilde\chi(\tilde t)=Ce^{2\varepsilon}\big(\chi(t)+\vec l_{tt}(t)\cdot\vec m(t)-\vec m_{tt}(t)\cdot\vec l(t)\big),\tag{A.7}$$

where $\tilde t=te^{-2\varepsilon}+\delta$.

Proof. Consider an arbitrary one-dimensional subalgebra generated by

$$V=a_1D+a_2\partial_t+a_3J_{12}+a_4J_{23}+a_5J_{31}+R(\vec m)+Z(\chi).$$

The coefficients a_4 and a_5 are omitted below since they always can be made to vanish by means of the adjoint representations $\mathrm{Ad}(\varepsilon_1J_{12})$ and $\mathrm{Ad}(\varepsilon_2J_{31})$.

If $a_1\neq 0$ we get $\tilde a_1=1$ by means of a change of basis. Next, step-by-step we make a_2, $\vec m$, and χ vanish by means of the adjoint representations $\mathrm{Ad}(-\frac12 a_2a_1^{-1}\partial_t)$, $\mathrm{Ad}(R(\vec l))$, and $\mathrm{Ad}(Z(\chi))$, where

$$\vec l\in C^\infty((t_0+\tfrac12 a_2a_1^{-1},t_1+\tfrac12 a_2a_1^{-1}),\mathbb{R}^3),$$
$$\eta\in C^\infty((t_0+\tfrac12 a_2a_1^{-1},t_1+\tfrac12 a_2a_1^{-1}),\mathbb{R}),$$

and $\vec l$, η are solutions of the equations

$$2t\vec l_t-\vec l+a_3a_1^{-1}(l^2,-l^1,0)^T=\vec m,\quad 2t\eta_t+2\eta=\hat\chi+\frac12(\vec l_{tt}\cdot\vec m-\vec l\cdot\vec m_{tt})$$

with $\hat{\vec{m}}(t) = a_1^{-1}\vec{m}(t - \frac{1}{2}a_2a_1^{-1})$ and $\hat{\chi}(t) = a_1^{-1}\chi(t - \frac{1}{2}a_2a_1^{-1})$. Such $\vec{l}$ and η exist in virtue of Lemma A.1. As a result we obtain the algebra $A_1^1(\varkappa)$, where $2\varkappa = a_3a_1^{-1}$. In case $\varkappa < 0$ additionally one has to apply transformation (1.6) with $b = 1$.

If $a_1 = 0$ and $a_2 \neq 0$, we make $\tilde{a}_2 = 1$ by means of a change of basis. Next, step-by-step we make $\vec{m}$ and χ vanish by means of the adjoint representations $\mathrm{Ad}(R(\vec{l}))$ and $\mathrm{Ad}(Z(\chi))$, where $\vec{l} \in C^\infty((t_0,t_1),\mathbb{R}^3)$, $\eta \in C^\infty((t_0,t_1),\mathbb{R})$, and

$$a_2\vec{l}_t + a_3(l^2, -l^1, 0)^T = \vec{m}, \quad a_2\eta_t = \chi + \frac{1}{2}(\vec{l}_{tt}\cdot\vec{m} - \vec{l}\cdot\vec{m}_{tt}).$$

If $a_3 = 0$ we obtain the algebra $A_2^1(0)$ at once. If $a_3 \neq 0$, using the adjoint representation $\mathrm{Ad}(\varepsilon D)$ and transformation (1.6) (in case of need), we obtain the algebra $A_2^1(1)$.

If $a_1 = a_2 = 0$ and $a_3 \neq 0$, after a change of basis and applying the adjoint representation $\mathrm{Ad}(R(-a_3^{-1}m^2, a_3^{-1}m^1, 0))$ we get the algebra $A_3^1(\eta,\tilde{\chi})$, where $\eta = a_3^{-1}m^3$ and $\tilde{\chi} = a_3^{-1}\chi + a_3^{-2}(m^1_{tt}m^2 - m^1m^2_{tt})$. Equivalence relation (A.6) is generated by the adjoint representations $\mathrm{Ad}(\varepsilon D)$, $\mathrm{Ad}(\delta\partial_t)$, and $\mathrm{Ad}(R(0,0,\lambda))$.

If $a_1 = a_2 = a_3 = 0$, at once we get the algebra $A_4^1(\vec{m},\chi)$. Equivalence relation (A.7) is generated by the adjoint representations $\mathrm{Ad}(\varepsilon D)$, $\mathrm{Ad}(\delta\partial_t)$, $\mathrm{Ad}(R(\vec{l}))$, and $\mathrm{Ad}(\varepsilon_{ab}J_{ab})$.

A.3 Two-dimensional subalgebras

Theorem A.2 *A complete set of $A(NS)$-inequivalent two-dimensional subalgebras of $A(NS)$ is exhausted by the following algebras:*

1. $A_1^2(\varkappa) = \langle\partial_t,\ D + \varkappa J_{12}\rangle$, *where* $\varkappa \geq 0$.

2. $A_2^2(\varkappa,\varepsilon) = \langle D,\ J_{12} + R(0,0,\varkappa|t|^{1/2}) + Z(\varepsilon t^{-1})\rangle$, *where* $\varkappa \geq 0$, $\varepsilon \geq 0$.

3. $A_3^2(\varkappa,\varepsilon) = \langle\partial_t,\ J_{12} + R(0,0,\varkappa) + Z(\varepsilon)\rangle$, *where* $\varkappa \in \{0;1\}$, $\varepsilon \geq 0$ *if* $\varkappa = 1$ *and* $\varepsilon \in \{0;1\}$ *if* $\varkappa = 0$.

4. $A_4^2(\sigma,\varkappa,\mu,\nu,\varepsilon) = \langle D + 2\varkappa J_{12},\ R\big(|t|^{\sigma+1/2}(\nu\cos\tau,\ \nu\sin\tau,\ \mu)\big) + Z(\varepsilon|t|^{\sigma-1})\rangle$, *where* $\tau = \varkappa\ln|t|$, $\varkappa > 0$, $\mu \geq 0$, $\nu \geq 0$, $\mu^2 + \nu^2 = 1$, $\varepsilon\sigma = 0$, *and* $\varepsilon \geq 0$.

5. $A_5^2(\sigma,\varepsilon) = \langle D,\ R(0,0,|t|^{\sigma+1/2}) + Z(\varepsilon|t|^{\sigma-1})\rangle$, *where* $\varepsilon\sigma = 0$ *and* $\varepsilon \geq 0$.

6. $A_6^2(\sigma,\mu,\nu,\varepsilon) = \langle\partial_t + J_{12},\ R(\nu e^{\sigma t}\cos t, \nu e^{\sigma t}\sin t, \mu e^{\sigma t}) + Z(\varepsilon e^{\sigma t})\rangle$, *where* $\mu \geq 0$, $\nu \geq 0$, $\mu^2 + \nu^2 = 1$, $\varepsilon\sigma = 0$, *and* $\varepsilon \geq 0$.

7. $A_7^2(\sigma,\varepsilon) = \langle\partial_t,\ R(0,0,e^{\sigma t}) + Z(\varepsilon e^{\sigma t})\rangle$, *where* $\sigma \in \{-1;0;1\}$, $\varepsilon\sigma = 0$, *and* $\varepsilon \geq 0$.

8. $A_8^2(\lambda,\psi^1,\rho,\psi^2) = \langle J_{12} + R(0,0,\lambda) + Z(\psi^1),\ R(0,0,\rho) + Z(\psi^2)\rangle$ *with smooth functions (of t)* λ, ρ, *and* ψ^i: $(\rho,\psi^2) \not\equiv (0,0)$ *and* $\lambda_{tt}\rho - \lambda\rho_{tt} \equiv 0$. *Algebras* $A_8^2(\lambda,\psi^1,\ \rho,\psi^2)$ *and* $A_8^2(\tilde{\lambda},\tilde{\psi}^1,\tilde{\rho},\tilde{\psi}^2)$ *are equivalent if* $\exists C_1 \neq 0$, $\exists\varepsilon,\delta,C_2 \in \mathbb{R}$, $\exists\theta \in C^\infty((t_0,t_1),\mathbb{R})$:

$$\begin{aligned}
&\tilde{\lambda}(\tilde{t}) = e^{\varepsilon}(\lambda(t) + C_2\rho(t)), \quad \tilde{\rho}(\tilde{t}) = C_1e^{-\varepsilon}\rho(t),\\
&\tilde{\psi}^1(\tilde{t}) = e^{2\varepsilon}\big(\psi^1(t) + \theta_{tt}(t)\lambda(t) - \theta(t)\lambda_{tt}(t) +\\
&\qquad + C_2(\psi^2(t) + \theta_{tt}(t)\rho(t) - \theta(t)\rho_{tt}(t))\big),\\
&\tilde{\psi}^2(\tilde{t}) = C_1e^{2\varepsilon}(\psi^2(t) + \theta_{tt}(t)\rho(t) - \theta(t)\rho_{tt}(t)),
\end{aligned} \tag{A.8}$$

where $\tilde{t} = te^{-2\varepsilon} + \delta$.

9. $A^2_9(\vec m^1,\chi^1,\vec m^2,\chi^2)=\langle R(\vec m^1(t))+Z(\chi^1(t)),\ R(\vec m^2(t))+Z(\chi^2(t))\rangle$ *with smooth functions* $\vec m^i$ *and* χ^i*:*

$$\vec m^1_{tt}\cdot\vec m^2-\vec m^1\cdot\vec m^2_{tt}=0,\quad \operatorname{rank}\big((\vec m^1,\chi^1),(\vec m^2,\chi^2)\big)=2.$$

Algebras $A^2_9(\vec m^1,\chi^1,\vec m^2,\chi^2)$ *and* $A^2_9(\tilde{\vec m}^1,\tilde\chi^1,\tilde{\vec m}^2,\tilde\chi^2)$ *are equivalent if* $\exists\,\varepsilon,\delta\in\mathbb{R}$, $\exists\,\{a_{ij}\}_{i,j=1,2}:\det\{a_{ij}\}\ne0$, $\exists\,B\in O(3)$, $\exists\,\vec l\in C^\infty((t_0,t_1),\mathbb{R}^3)$*:*

$$\begin{aligned}&\tilde{\vec m}^i(\tilde t)=e^{-\varepsilon}a_{ij}B\vec m^j(t),\\&\tilde\chi^i(\tilde t)=e^{2\varepsilon}a_{ij}\big(\chi^j(t)+\vec l_{tt}(t)\cdot\vec m^j(t)-\vec l(t)\cdot\vec m^j_{tt}(t)\big),\end{aligned}\tag{A.9}$$

where $\tilde t=te^{-2\varepsilon}+\delta$*.*

10. $A^2_{10}(\varkappa,\sigma)=\langle D+\varkappa J_{12},\ Z(|t|^\sigma)\rangle$*, where* $\varkappa\ge0$, $\sigma\in\mathbb{R}$*.*

11. $A^2_{11}(\sigma)=\langle\partial_t+J_{12},\ Z(e^{\sigma t})\rangle$*, where* $\sigma\in\mathbb{R}$*.*

12. $A^2_{12}(\sigma)=\langle\partial_t,\ Z(e^{\sigma t})\rangle$*, where* $\sigma\in\{-1;0;1\}$*.*

The proof of Theorem A.2 is analogous to that of Theorem A.1. Let us take an arbitrary two-dimensional subalgebra generated by two linearly independent operators of the form

$$V^i=a^i_1D+a^i_2\partial_t+a^i_3J_{12}+a^i_4J_{23}+a^i_5J_{31}+R(\vec m^i)+Z(\chi^i),$$

where $a^i_n=\mathrm{const}$ $(n=\overline{1,5})$ and $[V^1,V^2]\in\langle V^1,V^2\rangle$. Considering the different possible cases we try to simplify V^i by means of adjoint representation as much as possible. Here we do not present the proof of Theorem A.2 as it is too cumbersome.

A.4 Three-dimensional subalgebras

We also constructed a complete set of $A(NS)$-inequivalent three-dimensional subalgebras. It contains 52 classes of algebras. By means of 22 classes from this set one can obtain ansatzes of codimension three for the Navier–Stokes field. Here we only give 8 superclasses that arise from unification of some of these classes:

1. $A^3_1=\langle D,\ \partial_t,\ J_{12}\rangle$.

2. $A^3_2=\langle D+\varkappa J_{12},\ \partial_t,\ R(0,0,1)\rangle$, where $\varkappa\ge0$. Here and below $\varkappa$, σ, ε_1, ε_2, μ, ν, and a_{ij} are real constants.

3. $A^3_3(\sigma,\nu,\varepsilon_1,\varepsilon_2)=\langle D,\ J_{12}+\nu\big(R(0,0,|t|^{1/2}\ln|t|)+Z(\varepsilon_2|t|^{-1}\ln|t|)\big)+Z(\varepsilon_1|t|^{-1}),$ $R(0,0,|t|^{\sigma+1/2})+Z(\varepsilon_2|t|^{\sigma-1})\rangle$, where $\nu\sigma=0$, $\varepsilon_1\ge0$, $\nu\ge0$, and $\sigma\varepsilon_2=0$.

4. $A^3_4(\sigma,\nu,\varepsilon_1,\varepsilon_2)=\langle\partial_t,\ J_{12}+Z(\varepsilon_1)+\nu\big(R(0,0,t)+Z(\varepsilon_2t)\big),\ R(0,0,e^{\sigma t})+Z(\varepsilon_2e^{\sigma t})\rangle$, where $\nu\sigma=0$, $\sigma\varepsilon_2=0$, and, if $\sigma=0$, the constants ν, ε_1, and ε_2 satisfy one of the following conditions:

$$\nu=1,\ \varepsilon_1\ge0;\quad \nu=0,\ \varepsilon_1=1,\ \varepsilon_2\ge0;\quad \nu=\varepsilon_1=0,\ \varepsilon_2\in\{0;1\}.$$

5. $A^3_5(\varkappa,\vec m^1,\vec m^2,\chi^1,\chi^2)=\langle D+2\varkappa J_{12},\ R(\vec m^1)+Z(\chi^1),\ R(\vec m^2)+Z(\chi^2)\rangle$, where $\varkappa\ge0$, $\operatorname{rank}(\vec m^1,\vec m^2)=2$,

$$\begin{aligned}&t\vec m^i_t-\tfrac12\vec m^i+\varkappa(m^{i2},-m^{i1},0)^T=a_{ij}\vec m^j,\\&t\chi^i_t+\chi^i=a_{ij}\chi^j,\quad a_{ij}=\mathrm{const},\end{aligned}$$

$$(a_{11}+a_{22})\big(a_{21}\vec m^1\cdot\vec m^1+(a_{22}-a_{11})\vec m^1\cdot\vec m^2-a_{12}\vec m^2\cdot\vec m^2+ \\ +2\varkappa(m^{12}m^{21}-m^{11}m^{22})\big)=0. \tag{A.10}$$

This superclass contains eight inequivalent classes of subalgebras that can be obtained from it by means of a change of basis and the adjoint actions

$$\mathrm{Ad}(\delta_1 D),\quad \mathrm{Ad}(\delta_2 J_{12}),\quad \mathrm{Ad}(R(\vec n)+Z(\eta)),$$
$$\big(\mathrm{Ad}(\delta D),\quad \mathrm{Ad}(\varepsilon_{ab}J_{ab}),\quad \mathrm{Ad}(R(\vec n)+Z(\eta))\big)$$

if $\varkappa>0$ ($\varkappa=0$) respectively. Here the functions $\vec n$ and η satisfy the following equations:

$$t\vec n_t-\tfrac12\vec n+\varkappa(n^2,-n^1,0)^T=b_i\vec m^i,$$
$$t\eta_t+\eta=b_i\chi_i+\tfrac12 t(\vec n_{ttt}\cdot\vec n-\vec n_{tt}\cdot\vec n_t)+\vec n_{tt}\cdot\vec n+\varkappa(n^1n^2_{tt}-n^1_{tt}n^2).$$

6. $A^3_6(\varkappa,\vec m^1,\vec m^2,\chi^1,\chi^2)=\langle\partial_t+\varkappa J_{12},\ R(\vec m^1)+Z(\chi^1),\ R(\vec m^2)+Z(\chi^2)\rangle$, where $\varkappa\in\{0;1\}$, $\mathrm{rank}(\vec m^1,\vec m^2)=2$,

$$\vec m^i_t-\varkappa(m^{i2},-m^{i1},0)^T=a_{ij}\vec m^j,\quad t\chi^i_t=a_{ij}\chi^j,$$

and a_{ij} are constants satisfying (A.10). This superclass contains eight inequivalent classes of subalgebras that can be obtained from it by means of a change of basis and the adjoint actions

$$\mathrm{Ad}(\delta_1\partial_t),\quad \mathrm{Ad}(\delta_2 J_{12}),\quad \mathrm{Ad}(R(\vec n)+Z(\eta)),$$
$$\big(\mathrm{Ad}(\delta_1\partial_t),\quad \mathrm{Ad}(\delta_2 D),\quad \mathrm{Ad}(\varepsilon_{ab}J_{ab}),\quad \mathrm{Ad}(R(\vec n)+Z(\eta))\big)$$

if $\varkappa=1$ ($\varkappa=0$) respectively. Here the functions $\vec n$ and η satisfy the following equations:

$$\vec n_t+\varkappa(n^2,-n^1,0)^T=b_i\vec m^i,$$
$$\eta_t=b_i\chi_i+\tfrac12(\vec n_{ttt}\cdot\vec n-\vec n_{tt}\cdot\vec n_t)+\varkappa(n^1n^2_{tt}-n^1_{tt}n^2).$$

7. $A^3_7(\eta^1,\eta^2,\eta^3,\chi)=\langle J_{12}+R(0,0,\eta^3),\ R(\eta^1,\eta^2,0),\ R(-\eta^2,\eta^1,0)\rangle$, where

$$\eta^a\in C^\infty((t_0,t_1),\mathbb R),\quad \eta^1_{tt}\eta^2-\eta^1\eta^2_{tt}\equiv0,\quad \eta^i\eta^i\not\equiv0,\quad \eta^3\neq0.$$

Algebras $A^3_7(\eta^1,\eta^2,\eta^3)$ and $A^3_7(\tilde\eta^1,\tilde\eta^2,\tilde\eta^3)$ are equivalent if $\exists\,\delta_a\in\mathbb R$, $\exists\,\delta_4\neq0$:

$$\begin{aligned}\tilde\eta^1(\tilde t)&=\delta_4(\eta^1(t)\cos\delta_3-\eta^2(t)\sin\delta_3),\\ \tilde\eta^2(\tilde t)&=\delta_4(\eta^1(t)\sin\delta_3+\eta^2(t)\cos\delta_3),\\ \tilde\eta^3(\tilde t)&=e^{-\delta_1}\eta^3(t),\end{aligned}\tag{A.11}$$

where $\tilde t=te^{-2\delta_1}+\delta_2$.

8. $A^3_8(\vec m^1,\vec m^2,\vec m^3)=\langle R(\vec m^1),\ R(\vec m^2),\ R(\vec m^3)\rangle$, where

$$\vec m^a\in C^\infty((t_0,t_1),\mathbb R^3),\quad \mathrm{rank}(\vec m^1,\vec m^2,\vec m^3)=3,\quad \vec m^a_{tt}\cdot\vec m^b-\vec m^a\cdot\vec m^b_{tt}=0.$$

Algebras $A^3_8(\vec m^1,\vec m^2,\vec m^3)$ and $A^3_8(\tilde{\vec m}^1,\tilde{\vec m}^2,\tilde{\vec m}^3)$ are equivalent if $\exists\,\delta_i\in\mathbb R^3$, $\exists\,B\in O(3)$, $\exists\,\{d_{ab}\}:\det\{d_{ab}\}\neq0$ such that

$$\tilde{\vec m}^a(\tilde t)=d_{ab}B\vec m^b(t),\tag{A.12}$$

where $\tilde t=te^{-2\delta_1}+\delta_2$.

B On construction of ansatzes for the Navier–Stokes field by means of the Lie method

The general method for constructing a complete set of inequivalent Lie ansatzes of a system of PDEs are well known and described, for example, in [27, 28]. However, in some cases when the symmetry operators of the system have a special form, this method can be modified [9]. Thus, in the case of the NSEs, coefficients of an arbitrary operator

$$Q = \xi^0\partial_t + \xi^a\partial_a + \eta^a\partial_{u^a} + \eta^0\partial_p$$

from $A(NS)$ satisfy the following conditions:

$$\begin{aligned} &\xi^0 = \xi^0(t,\vec{x}), \quad \xi^a = \xi^a(t,\vec{x}), \quad \eta^a = \eta^{ab}(t,\vec{x})u^b + \eta^{a0}(t,\vec{x}), \\ &\eta^0 = \eta^{01}(t,\vec{x})p + \eta^{00}(t,\vec{x}). \end{aligned} \tag{B.1}$$

(The coefficients ξ^a, ξ^0, η^a, and η^0 also satisfy stronger conditions than (B.1). For example if $Q \in A(NS)$, then $\xi^0 = \xi^0(t)$, $\eta^{ab} = \mathrm{const}$, and so on. But conditions (B.1) are sufficient to simplify the general method.) Therefore, ansatzes for the Navier–Stokes field can be constructing in the following way:

1. We fix a M-dimensional subalgebra of $A(NS)$ with the basis elements

$$Q^m = \xi^{m0}\partial_t + \xi^{ma}\partial_a + (\eta^{mab}u^b + \eta^{ma0})\partial_{u^a} + (\eta^{m01}p + \eta^{m00})\partial_p, \tag{B.2}$$

where $M \in \{1;2;3\}$, $m = \overline{1,M}$, and

$$\mathrm{rank}\{(\xi^{m0}, \xi^{m1}, \xi^{m2}, \xi^{m3}),\ m = \overline{1,M}\} = M. \tag{B.3}$$

To construct a complete set of inequivalent Lie ansatzes of codimension M for the Navier–Stokes field, we have to use the set of M-dimensional subalgebras from Section A. Condition (B.3) is neeeded for the existance of ansatzes connected with this subalgebra.

2. We find the invariant independent variables $\omega_n = \omega_n(t,\vec{x})$, $n = \overline{1,N}$, where $N = 4 - M$, as a set of functionally independent solutions of the following system:

$$L^m\omega = Q^m\omega = \xi^{m0}\partial_t\omega + \xi^{ma}\partial_a\omega = 0, \quad m = \overline{1,M}, \tag{B.4}$$

where $L^m := \xi^{m0}\partial_t + \xi^{ma}\partial_a$.

3. We present the Navier–Stokes field in the form:

$$u^a = f^{ab}(t,\vec{x})v^b(\bar{\omega}) + g^a(t,\vec{x}), \quad p = f^0(t,\vec{x})q(\bar{\omega}) + g^0(t,\vec{x}), \tag{B.5}$$

where v^a and q are new unknown functions of $\bar{\omega} = \{\omega_n,\ n = \overline{1,N}\}$. Acting on representation (B.5) with the operators Q^m, we obtain the following equations on functions f^{ab}, g^a, f^0, and g^0:

$$\begin{aligned} &L^m f^{ab} = \eta^{mac}f^{cb}, \quad L^m g^a = \eta^{mab}g^b + \eta^{ma0}, \quad c = \overline{1,3}, \\ &L^m f^0 = \eta^{m01}f^0, \quad L^m g^0 = \eta^{m01}g^0 + \eta^{m00}. \end{aligned} \tag{B.6}$$

If the set of functions f^{ab}, f^0, g^a, and g^0 is a particular solution of (B.6) and satisfies the conditions $\mathrm{rank}\{(f^{1b}, f^{2,b}, f^{3b}),\ b = \overline{1,3}\} = 3$ and $f^0 \neq 0$, formulas (B.5) give an ansatz for the Navier–Stokes field.

The ansatz connected with the fixed subalgebra is not determined in an unique manner. Thus, if

$$\begin{aligned}
&\tilde\omega_l=\tilde\omega_l(\bar\omega), \quad \det\left\{\frac{\partial\tilde\omega_l}{\partial\omega_n}\right\}_{l,n=\overline{1,N}}\neq 0,\\
&\tilde f^{ab}(t,\vec x)=f^{ac}(t,\vec x)F^{cb}(\bar\omega), \quad \tilde g^a(t,\vec x)=g^a(t,\vec x)+f^{ac}(t,\vec x)G^c(\bar\omega),\\
&\tilde f^0(t,\vec x)=f^0(t,\vec x)F^0(\bar\omega), \quad \tilde g^0(t,\vec x)=g^0(t,\vec x)+f^0(t,\vec x)G^0(\bar\omega),
\end{aligned} \tag{B.7}$$

the formulas

$$u^a=\tilde f^{ab}(t,\vec x)\tilde v^b(\bar{\tilde\omega})+\tilde g^a(t,\vec x), \quad p=\tilde f^0(t,\vec x)q(\bar{\tilde\omega})+\tilde g^0(t,\vec x) \tag{B.8}$$

give an ansatz which is equivalent to ansatz (B.5). The reduced system of PDEs on the functions $\tilde v^a$ and $\tilde q$ is obtained from the system on v^a and q by means of a local transformation. Our problem is to find or "to guess", at once, such an ansatz that the corresponding reduced system has a simple and convenient form for our investigation. Otherwise, we can obtain a very complicated reduced system which will be not convenient for investigation and we can not simplify it.

Consider a simple example.

Let $M=1$ and let us give the algebra $\langle\partial_t+\varkappa J_{12}\rangle$, where $\varkappa\in\{0;1\}$. For this algebra, the invariant independent variables $y_a=y_a(t,\vec x)$ are functionally independent solutions of the equation $Ly=0$ (see (B.4)), where

$$L:=\partial_t+\varkappa(x_1\partial_{x_2}-x_2\partial_{x_1}). \tag{B.9}$$

There exists an infinite set of choices for the variables y_a. For example, we can give the following expressions for y_a:

$$y_1=\arctan\frac{x_1}{x_2}-\varkappa t, \quad y_2=(x_1^2+x_2^2)^{1/2}, \quad y_3=x_3.$$

However choosing y_a in such a way, for $\varkappa\neq 0$ we obtain a reduced system which strongly differs from the "natural" reduced system for $\varkappa=0$ (the NSEs for steady flows of a viscous fluid in Cartesian coordinates). It is better to choose the following variables y_a:

$$y_1=x_1\cos\varkappa t+x_2\sin\varkappa t, \quad y_2=-x_1\sin\varkappa t+x_2\cos\varkappa t, \quad y_3=x_3.$$

The vector-functions $\vec f^{\,b}=(f^{1b},f^{2b},f^{3b})$, $b=\overline{1,3}$, should be linearly independent solutions of the system

$$Lf^1=-\varkappa f^2, \quad Lf^2=\varkappa f^1, \quad Lf^3=0$$

and the function f^0 should satisfy the equation $Lf^0=0$ and the condition $f^0\neq 0$. Here the operator L is defined by (B.9). We give the following values of these functions:

$$\vec f^{\,1}=(\cos\varkappa t,\sin\varkappa t,0), \quad \vec f^{\,2}=(-\sin\varkappa t,\cos\varkappa t,0), \quad \vec f^{\,3}=(0,0,1), \quad f^0=1.$$

The functions g^a and g^0 are solutions of the equations

$$Lg^1=-\varkappa g^2, \quad Lg^2=\varkappa g^1, \quad Lg^3=0, \quad Lg^0=0.$$

We can make, for example, g^a and g^0 vanish. Then the corresponding ansatz has the form:

$$u^1 = \tilde v^1 \cos \varkappa t - \tilde v^2 \sin \varkappa t, \quad u^2 = \tilde v^1 \sin \varkappa t + \tilde v^2 \cos \varkappa t, \quad u^3 = \tilde v^3, \quad p = \tilde q, \tag{B.10}$$

where $\tilde v^a = \tilde v^a(y_1, y_2, y_3)$ and $\tilde q = \tilde q(y_1, y_2, y_3)$ are the new unknown functions. Substituting ansatz (B.10) into the NSEs, we obtain the following reduced system:

$$\begin{aligned}
&\tilde v^a \tilde v^1_a - \tilde v^1_{aa} + \tilde q_1 + \varkappa y_2 \tilde v^1_1 - \varkappa y_1 \tilde v^1_2 - \varkappa \tilde v^2 = 0,\\
&\tilde v^a \tilde v^2_a - \tilde v^2_{aa} + \tilde q_2 + \varkappa y_2 \tilde v^2_1 - \varkappa y_1 \tilde v^2_2 + \varkappa \tilde v^1 = 0,\\
&\tilde v^a \tilde v^3_a - \tilde v^3_{aa} + \tilde q_3 + \varkappa y_2 \tilde v^3_1 - \varkappa y_1 \tilde v^3_2 = 0,\\
&\tilde v^a_a = 0.
\end{aligned} \tag{B.11}$$

Here subscripts 1,2, and 3 of functions in (B.11) denote differentiation with respect to y_1, y_2, and y_3 accordingly. System (B.11), having variable coefficients, can be simplified by means of the local transformation

$$\tilde v^1 = v^1 - \varkappa y_2, \quad \tilde v^2 = v^2 + \varkappa y_1, \quad \tilde v^3 = v^3, \quad \tilde q = q + \tfrac{1}{2}(y_1^2 + y_2^2). \tag{B.12}$$

Ansatz (B.10) and system (B.11) are transformed under (B.12) into ansatz (2.2) and system (2.7), where

$$g^1 = -\varkappa x_2, \quad g^2 = \varkappa x_1, \quad g_3 = 0, \quad g^0 = \tfrac{1}{2}\varkappa^2(x_1^2 + x_2^2), \tag{B.13}$$

$\gamma_1 = -2\varkappa$, and $\gamma_2 = 0$. Therefore, we can give the values of g^a and g^0 from (B.13) and obtain ansatz (2.2) and system (2.7) at once.

The above is a good example how a reduced system can be simplified by means of modifying (complicating) an ansatz corresponding to it. Thus, system (2.7) is simpler than system (B.11) and ansatz (2.2) is more complicated than ansatz (B.10).

Finally, let us make several short notes about constructing other ansatzes for the Navier–Stokes field.

Ansatz corresponding to the algebra $A^1_4(\vec m, \chi)$ (see Subsection A.2) can be constructed only for such t that $\vec m(t) \neq \vec 0$. For these values of t, the parameter-function χ can be made to vanish by means of equivalence transformations (A.7).

Ansatz corresponding to the algebra $A^2_8(\lambda, \psi^1, \rho, \psi^2)$ (see Subsection A.3) can be constructed only for such t that $\rho(t) \neq 0$. For these values of t, the parameter-function ψ^2 can be made to vanish by means of equivalence transformations (A.8). Moreover, it can be considered that $\lambda_t \rho - \lambda \rho_t \in \{0; 1\}$. The algebra obtained finally is denoted by $A^2_8(\lambda, \chi, \rho, 0)$.

Ansatz corresponding to the algebra $A^2_9(\vec m^1, \chi^1, \vec m^2, \chi^2)$ (see Subsection A.3) can be constructed only for such t that $\operatorname{rank}(\vec m^1, \vec m^2) = 2$. For these values of t, the parameter-functions χ^i can be made to vanish by means of equivalence transformations (A.9).

The algebras $A^2_{10}(\varkappa, \sigma)$, $A^2_{11}(\sigma)$, and $A^2_{12}(\sigma)$ can not be used to construct ansatzes by means of the Lie algorithm.

In view of equivalence transformation (A.11), the functions η^i in the algebra $A^3_7(\eta^1, \eta^2, \eta^3)$ (see Subsection A.4) can be considered to satisfy the following condition:

$$\eta^1_t \eta^2 - \eta^1 \eta^2_t \in \{0; \tfrac{1}{2}\}.$$

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