Q-symmetry generators and exact solutions for nonlinear heat conduction

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We investigate conditional invariance by considering Q-symmetry generators of the nonlinear heat equation $\partial u/\partial x_0 - \lambda\partial^2 u/\partial x_1^2 = f(u)$, where λ is a real constant and f an arbitrary differentiable function. With the obtained Q-generators we construct exact solutions by the use of similarity ansatze and reductions to ordinary differential equations. A generalization to m-space dimensions is performed.

1. Introduction

Most nonlinear partial differential equations are not integrable and cannot be treated via the inverse scattering transform, nor its generalization. Such equations are mostly treated by numerical methods. Interesting qualitative and quantitative features are however often missed in this manner and it is of great value to be able to obtain exact analytic solutions of nonintegrable equations. The application of Lie transformation groups, whereby a transformation is obtained that leaves the differential equation invariant, is useful in finding exact solutions (see [1–8]). If an equation is invariant under some Lie transformation group, the equation is said to have a symmetry. It is known that the integrability and the existence of symmetries is connected. This was studied in connection with the Painlevé test (see [1–3]). Many important nonintegrable partial differential equations have no significant symmetries. In this article we consider conditional symmetries of partial differential equations as introduced in [9–13]. We make use of these conditional symmetries to obtain exact solutions. The following equation is studied:

$$\frac{\partial u}{\partial x_0} - \lambda\frac{\partial^2 u}{\partial x_1^2} = f(u), \tag{1}$$

where x_0 indicates time, λ is a real constant, and f an arbitrary differentiable function. For nonlinear functions f this equation plays an important role in nonlinear heat transfer processes.

Before we consider conditional symmetries of (1), let us briefly describe the classical Lie approach and introduce our notation [1]. We are concerned with a partial differential equation of order r with $m+1$ independent variables $(x_0, x_1, \ldots, x_m)$ and one field variable u, i.e. an equation of the form

$$F\left(x_0, \ldots, x_m, u, \frac{\partial_u}{\partial x_0}, \ldots, \frac{\partial^r u}{\partial x_{j_1}\cdots\partial x_{j_r}}\right) = 0, \tag{2}$$

where $0 \le j_1 \le j_2 \le \cdots \le j_r \le m$, $j = 0, \ldots, m$. The submanifold $\mathbb{R}^r$ of the r-jet bundle $J^r(M,1)$ is determined by the constrained equation

$$F(x_0, \ldots, x_m, u, u_0, \ldots, u_{j_1\cdots j_r}) = 0, \tag{3}$$

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where the dimension of the differential manifold M is m. A Lie transformation group that leaves (3) invariant is generated by a Lie (point) symmetry generator Z, defined by

$$Z = \sum_{j=0}^{m} \xi_j(x_0, \ldots, x_m, u)\frac{\partial}{\partial x_j} + \eta(x_0, \ldots, x_m, u)\frac{\partial}{\partial u}. \tag{4}$$

$Z_{\rm v}$ is the associated vertical form of (4) on $J^1(M,1)$, defined by

$$Z_{\rm v} = \left(\eta - \sum_{j=0}^{m} \xi_j u_j\right)\frac{\partial}{\partial u}, \tag{5}$$

where $Z_{\rm v}\rfloor\theta = Z\rfloor\theta$. Here θ is a differential 1-form, called the contact form on $J^1(M,1)$, defined by

$$\theta = du - \sum_{j=0}^{m} u_j dx_j$$

with $js^*\theta = 0$. Here js^* denotes the pull-back map. Equation (3) is called invariant under the prolonged Lie symmetry generator $\bar{Z}_{\rm v}$ if

$$L_{\bar{Z}_{\rm v}} F \,\hat{=}\, 0, \tag{6}$$

where $\hat{=}$ indicates the restriction to solutions of (3) and its prolongations. L denotes the Lie derivative. $\bar{Z}_{\rm v}$ is found by prolonging the vertical generator $Z_{\rm v}$, i.e.,

$$\bar{Z}_{\rm v} = U\frac{\partial}{\partial u} = \sum_{j=0}^{m} D_j(U)\frac{\partial}{\partial u_j} + \cdots + \sum_{j_1,\ldots,j_r=0}^{m} D_{j_1\cdots j_r}(U)\frac{\partial}{\partial u_{j_1\cdots j_r}} + \cdots,$$

where

$$U = \eta - \sum_{j=0}^{m} \xi_j u_j \equiv Z_{\rm v}\rfloor\theta$$

and D_j is the total derivative operator. A similarity ansatz for (2) is obtained by solving the linear partial differential equation

$$js^*(Z_{\rm v}\rfloor\theta) = 0, \tag{7}$$

with $Z_{\rm v}$ an associated vertical Lie symmetry generator for the equation. This ansatz will reduce the dimension of (2) by one. The solution of the reduced equation is known as a similarity solution of (2). Thus, the existence of a symmetry provides us with a similarity ansatz and a possible exact solution can be calculated. The converse is however not and true, i.e., any exact solution of a partial differential equation is not associated with a symmetry of the equation. For such solutions one can introduce conditional symmetries, i.e. symmetries that leave the equation invariant under some additional condition.

2. *Q*-symmetry generators

Following [9–13] we give the definition for conditional invariance of (2).

Definition. *Equation (3) is called Q-conditionally invariant if*

$$L_{\bar{Q}_{\mathrm{v}}} F \,\hat{\hat{=}}\, 0 \tag{8}$$

under the condition

$$Q_{\mathrm{v}} \rfloor \theta = 0. \tag{9}$$

Q is called the Q-symmetry generator and $\bar{Q}_{\mathrm{v}}$ the prolonged vertical Q-symmetry generator.

Here $\hat{\hat{=}}$ indicates the restriction to solution of $F = 0$ and (9) together with their prolongations. Q is considered in the form of a Lie symmetry generator.

Let us now study (1) by the used of the above definition. We are interested only in nonlinear functions f. From the definition it follows that the Lie derivative (8), for the equation

$$F \equiv u_0 - \lambda u_{11} - f(u) = 0 \tag{10}$$

under the condition

$$Q_{\mathrm{v}} \rfloor \theta \equiv \eta - \xi_0 u_0 - \xi_1 u_1 = 0, \tag{11}$$

has to be studied. Let us consider the Q-symmetry generator in the form

$$Q = c\frac{\partial}{\partial x_0} + \xi_1(u)\frac{\partial}{\partial x_1} + \eta(u)\frac{\partial}{\partial u}, \tag{12}$$

where c is an arbitrary real constant. We can state the following

Theorem 1. *The generator*

$$Q = k_1\frac{\partial}{\partial x_1} + \eta(u)\frac{\partial}{\partial u} \tag{13}$$

is a Q-symmetry generator for (1) if an only if

$$f(u) = \eta(u)\left(-\frac{\lambda}{k_1^2}\frac{d\eta}{du} + c_1\right), \tag{14}$$

where η is an arbitrary differentiate function of u and k_1, c_1 are arbitrary real constants.

Proof. By applying the Lie derivative (8) and condition (9), with generator (13), we obtain the following determining equations using computer algebra [15, 16]:

$$c^3\lambda\frac{d^2\xi_1}{du^2} = 0, \quad -3c\lambda\frac{d^2\xi_1}{du^2}\eta^2 + (3cf + 2\eta)\frac{d\xi_1}{du}\xi_1^2 + 2c\lambda\frac{d^2\eta}{du^2}\xi_1\eta = 0,$$
$$c\left(3c\lambda\frac{d^2\xi_1}{du^2}\eta - 2\frac{d\xi_1}{du}\xi_1^2 - c\lambda\frac{d^2\eta}{du^2}\xi_1\right) = 0$$

and

$$\lambda\frac{d^2\xi_1}{du^2}\eta^3 - 3f\frac{d\xi_1}{du}\xi_1^2\eta + f\frac{d\eta}{du}\xi_1^3 - \lambda\frac{d^2\eta}{du^2}\xi_1\eta^2 - \frac{df}{du}\xi_1^3\eta = 0.$$

For $c \neq 0$ the general solution of the above four equations gives only a linear function $f(u) = a_1 u + a_2$, where a_1 and a_2 are arbitrary real constants. For $c = 0$ the general solution

$$\xi_1(u) = k_1, \quad f(u) = \eta(u)\left(-\frac{\lambda}{k_1^2}\frac{d\eta}{du} + c_1\right),$$

follows. ■

Consider the Q-symmetry generator in the form

$$Q = \xi_0(u)\frac{\partial}{\partial x_0} + c\frac{\partial}{\partial x_1} + \eta(u)\frac{\partial}{\partial u}, \tag{15}$$

where c is an arbitrary constant. We can state the following

Theorem 2. *The generator*

$$Q = \frac{k_2}{u+k_1}\frac{\partial}{\partial x_0} + c\frac{\partial}{\partial x_1} + \left[-\frac{2}{3}\frac{c^2}{\lambda k_2}\left(\frac{1}{2}u^2 + k_1 u\right) - \frac{k_3}{u+k_1} + k_4\right]\frac{\partial}{\partial u} \tag{16}$$

is a Q-symmetry generator for (1) if an only if

$$f(u) = \frac{2}{3}\frac{\eta}{\xi_0}, \tag{17}$$

where

$$\xi_0 = \frac{k_2}{u+k_1}$$

and

$$\eta = -\frac{2}{3}\frac{c^2}{\lambda k_2}\left(\frac{1}{2}u^2 + k_1 u\right) - \frac{k_3}{u+k_1} + k_4.$$

Here $k_1, \ldots, k_4$ are arbitrary real constants.

Proof. Applying the Lie derivative (8) and condition (9), with generator (15), the determining equations are given by

$$c(3f\xi_0 - 2\eta) = 0,$$
$$\lambda\frac{d^2\xi_0}{du^2}\xi_0\eta - 2\lambda\left(\frac{d\xi_0}{du}\right)^2\eta + 2\lambda\frac{d\xi_0}{du}\frac{d\eta}{du}\xi_0 + 2c^2\frac{d\xi_0}{du} - \lambda\frac{d^2\eta}{du^2}\xi_0^2 = 0,$$
$$c\left[-\frac{d^2\xi_0}{du^2}\xi_0 + 2\left(\frac{d\xi_0}{du}\right)^2\right] = 0$$

and

$$-f\frac{d\xi_0}{du}\eta + f\frac{d\eta}{du}\xi_0 - \frac{df}{du}\xi_0\eta = 0.$$

For $c = 0$ only linear functions f are obtained for the general solution of the above system. If $c \neq 0$, f follows from the first determining equation and the condition on η is in the form of a linear second order equation, namely

$$\frac{d^2\eta}{du^2} + \frac{2}{u+k_1}\frac{d\eta}{du} + \frac{2c^2}{\lambda k^2} = 0.$$

The general solution, given in theorem 2, follows. ■

For the nonlinear equation

$$\frac{\partial u}{\partial x_0} + \frac{\partial^2 u}{\partial x_1^2} = au^k, \tag{18}$$

with a and k arbitrary real constants, and $k \neq 1$, we can state the following

Theorem 3. *The generator*

$$Q = \frac{\partial}{\partial x_0} + \xi_1(x_0, x_1)\frac{\partial}{\partial x_1} + \alpha(x_0, x_1)u\frac{\partial}{\partial u}, \tag{19}$$

is a Q-symmetry generator for (18) if and only if the following conditions on ξ_1 and α are satisfied:

$$\frac{\partial \alpha}{\partial x_0} + \frac{\partial^2 \alpha}{\partial x_1^2} = (k-1)\alpha^2, \tag{20}$$

$$\frac{\partial \xi_1}{\partial x_0} - (k-1)\alpha\xi_1 = \frac{k+3}{2}\frac{\partial \alpha}{\partial x_1} \tag{21}$$

and

$$\frac{\partial \xi_1}{\partial x_1} = \frac{1-k}{2}\alpha. \tag{22}$$

The proof follows directly from the invariance condition (8) together with (9). Note that the above condition on ξ_1 reduces to the following third order ordinary differential equation:

$$\frac{2}{k-1}\frac{d^3\xi_1}{dx_1^3} + \xi_1\frac{d^2\xi_1}{dx_1^2} = 0,$$

which can be transformed to the Abel equation of the second kind

$$xy\frac{dy}{dx} + y^2 + \left(7x + \frac{k-1}{2}\right)y + 6x^2 + (k-1)x = 0$$

where

$$\frac{\xi_1}{x_1} = P(\xi_1), \quad P(\xi_1) = \xi_1^2 x(z), \quad z = \ln(\xi_1), \quad \frac{dx}{dz} = y(x).$$

With other special ansätze for ξ_0, ξ_1 and η we obtain the following results for (1) with $\lambda = -1$.

Theorem 4. 1. *The generator*

$$Q = 2\sqrt{x_0}\frac{\partial}{\partial x_1} + f(u)\frac{\partial}{\partial u} \tag{23}$$

is a Q-symmetry generator for (1) ($\lambda = -1$), if and only if f satisfies the equation

$$f\frac{d^2 f}{du^2} = 2. \tag{24}$$

The general solution of (24) is given by

$$\pm\int \frac{df}{\sqrt{4\ln f + \tilde{k}_1}} = u + \tilde{k}_2,$$

where $\tilde{k}_1$ and $\tilde{k}_2$ are integrating constants.

2. *The generator*

$$Q = x_1\frac{\partial}{\partial x_1} + f(u)\frac{\partial}{\partial u} \tag{25}$$

is a Q-symmetry generator for (1) $(\lambda = -1)$, if and only if f satisfies the equation

$$f\frac{d^2 f}{du^2} = 2\left(\frac{df}{du} - 1\right). \tag{26}$$

The general solution of (26) is given by

$$f(u) = \pm\sqrt{(w-1)\exp(w)/\tilde{k}_1},$$

where w is obtained from

$$\pm\int \sqrt{\tilde{k}_1^{-1}(w-1)^{-1}\exp(w)}dw = u + \tilde{k}_2.$$

Here $\tilde{k}_1$ and $\tilde{k}_2$ are integrating constants.

The proof follows by applying the invariance condition (8) together with (9).

Let us make some remarks on Q-symmetries. The determining equations for Q-generators are nonlinear over-determined systems of differential equations. This is in contrast to Lie symmetry generators where the determining equations are linear differential equations. It is obvious that every Lie symmetry of an equation is also a Q-symmetry but that the converse is not true, so that the above Q-symmetries do not generate Lie transformation groups that leave the equation invariant. If we multiply a Q-symmetry (or Lie symmetry) of a particular equation by an arbitrary function, we again find a Q-symmetry for that equation.

3. Q-similarity solutions

Let us now make use of theorems 1 to 4 to construct exact solutions of (1). The similarity ansatz is obtained by solving the linear partial differential equation

$$js^*(Q\rfloor\theta) \equiv \xi_0\frac{\partial u}{\partial x_0} + \xi_1\frac{\partial u}{\partial x_1} - \eta = 0. \tag{27}$$

We seek the general solution of (27) in the form

$$\psi(x_0, x_1, u) = \phi\{\omega[x_0, x_1, u(x_0, x_1)]\},$$

where ψ is an arbitrary function of its arguments and ϕ is an arbitrary function of the similarity variable ω. We call solutions, obtained by Q-symmetries, the Q-similarity solutions.

Let us consider the following cases:

Case 1(a): Consider the equation

$$\frac{\partial u}{\partial x_0} - \lambda \frac{\partial^2 u}{\partial x_1^2} = -\frac{\lambda}{k_1^2} \exp(2u) + k_2 \exp(u). \tag{28}$$

This corresponds to $\eta = \exp(u)$ for the Q-symmetry given in theorem 1. By solving (27) for the Q-symmetry

$$Q = k_1 \frac{\partial}{\partial x_1} + \exp(u) \frac{\partial}{\partial u}$$

we obtain the similarity ansatz

$$u = -\ln \left(\phi(\omega) - \frac{x_1}{k_1} \right) \quad \text{and} \quad \omega = x_0.$$

On insertion into (28) we obtain the reduced equation

$$\frac{d\phi}{d\omega} - k_2 = 0. \tag{29}$$

An exact solution of (28) is thus given by

$$u(x_0, x_1) = -\ln \left(-\frac{x_1}{k_1} + k_2 x_0 + k_3 \right),$$

where k_1, k_2, k_3 are arbitrary real constants.

Case 1(b): Consider the equation

$$\frac{\partial u}{\partial x_0} - \lambda \frac{\partial^2 u}{\partial x_1^2} = -(b_5 u^5 + b_3 u^3 + b_1 u). \tag{30}$$

This corresponds to $\eta = a_3 u^3 + a_1 u$ for the Q-symmetry given in theorem 1 with $k_1 = 1$ and

$$a_1 = \frac{1}{6\lambda} \left(b_3 \sqrt{\frac{3\lambda}{b_5}} + \sqrt{\frac{3\lambda b_3^2}{b_5} - 12 b_1 \lambda} \right), \quad a_3 = \sqrt{\frac{b_5}{3\lambda}},$$

$$c_1 = -\frac{1}{3} \sqrt{\frac{3\lambda b_3^2}{b_5}} + \frac{2}{3} \left(\sqrt{\frac{3 b_3^2 \lambda}{b_5} - 12 b_1 \lambda} \right).$$

By solving (27) for the Q-symmetry

$$Q = k_1 \frac{\partial}{\partial x_1} + (a_3 u^3 + a_1 u) \frac{\partial}{\partial u},$$

we obtain the similarity ansatz

$$u = \frac{\sqrt{a_1}}{\phi(\omega)} \exp(a_1 x_1 / c) \sqrt{1 - a_3 \phi^2(\omega) \exp(2 a_1 x_1 / c)} \quad \text{and} \quad \omega = x_0.$$

The reduced equation is given by

$$\frac{d\phi}{d\omega} - a_1 c_1 \phi = 0, \tag{31}$$

with the general solution

$$\phi(\omega) = \tilde{c}\exp(a_1c_1\omega).$$

Here $\tilde{c}$ is an arbitrary real constant. An exact solution of (30) is thus given by

$$u = \frac{\tilde{c}\sqrt{a_1}\exp[a_1(x_1 + c_1x_0)]}{\sqrt{1 - a_3\tilde{c}^2\exp[2a_1(x_1 + c_1x_0)]}}.$$

Case 2: Consider the equation

$$\frac{\partial u}{\partial x_0} - \lambda\frac{\partial^2 u}{\partial x_1^2} = -b_3u^3 - b_2u^2 + b_1u + b_0. \tag{32}$$

This corresponds to $\eta(u) = q_3u^2 + q_2u + q_1$ for the Q-symmetry in theorem 2. Here

$$q_3 = -c\sqrt{\frac{b_3}{2\lambda}}, \quad q_2 = -\frac{c}{3}\sqrt{\frac{2}{\lambda b_3}}b_2, \quad q_1 = \frac{c}{\sqrt{\lambda b_3}}\left(\frac{b_1}{\sqrt{2}} + \frac{\sqrt{2}b_2^2}{9b_3}\right).$$

The real constants b_1, b_2, b_3 are related to the constants k_1, k_2, k_3 and k_4 in the Q-symmetry given in theorem 2 by the relations

$$k_1 = \frac{b_2}{3b_3}, \quad k_2 = \frac{c}{3}\sqrt{\frac{2}{\lambda b_3}}, \quad k_3 = 0, \quad k_4 = \frac{b_1c}{\sqrt{2\lambda b_3}} + \frac{2cb_2^2}{9\sqrt{2\lambda b_3}b_3},$$

where

$$b_0 = \frac{b_1b_2}{3b_3} + \frac{2b_2^3}{27b_3^2}.$$

In terms of b_2 and b_3, ξ_0 is given by

$$\xi_0(u) = \frac{c\sqrt{2b_3}}{\sqrt{\lambda}(3b_3u + b_2)}.$$

In order to solve (27) for the above given ξ_0 and η we must solve the equation

$$\frac{d^2y}{d\varepsilon^2} - q_2\frac{dy}{d\varepsilon} + q_3q_1y = 0,$$

where

$$u = -\frac{1}{q_3}\frac{d}{d\varepsilon}(\ln y).$$

ε is the group parameter. Thus, there are three cases to be studied:

$$q_2^2 - 4q_1q_3 = 0, \quad q_2^2 - 4q_1q_3 < 0, \quad q_2^2 - 4q_1q_3 > 0.$$

Case 2(a): Consider $q_2^2 - 4q_1q_3 = 0$, i.e.,

$$b_1 = -\frac{1}{3}\frac{b_2^2}{b_3}.$$

The similarity ansatz is given by

$$u = \frac{b_2[x_1 - \phi(\omega)] - 3\sqrt{2b_3\lambda}}{3b_3[\phi(\omega) - x_1]} \quad \text{and} \quad \omega = x_0 - \frac{3b_3}{(3ub_3 + b_2)^2}.$$

The reduced equation then takes the form

$$\frac{d^2\phi}{d\omega^2} - \frac{1}{3\lambda}\left(\frac{d\phi}{d\omega}\right)^3 = 0, \tag{33}$$

which has the general solution

$$\phi(\omega) = -3\lambda\sqrt{-\frac{2}{3\lambda}\omega + \tilde{c}_1} + \tilde{c}_2.$$

Here $\tilde{c}_1$, $\tilde{c}_2$ are arbitrary real constants. Solving for u, an exact solution of (32) takes the form

$$u = \frac{6\sqrt{2b_3\lambda}(x_1 - \tilde{c}_2) - b_2(x_1^2 - 2\tilde{c}_2 x_1 - 9\lambda^2\tilde{c}_1 + 6\lambda x_0 + \tilde{c}_2^2)}{3b_3(x_1^2 - 2\tilde{c}_2 x_1 - 9\lambda^2\tilde{c}_1 + 6\lambda x_0 + \tilde{c}_2^2)}.$$

Case 2(b): Consider $q_2^2 - 4q_1q_3 < 0$, i.e.

$$b_1 + \frac{b_2^2}{3b_3} < 0.$$

The similarity ansatz is then given by

$$u = -\frac{\beta}{q_3}\frac{1}{\tan\{\beta[x_1 - \phi(\omega)]/c\}} - \frac{\alpha}{q_3},$$
$$\omega = x_0 + \frac{c^2}{3\lambda\beta^2}\ln\left\{\left[1 + \frac{\beta^2}{(q_3u + \alpha)^2}\right]^{-1/2}\right\},$$

where

$$\alpha = \frac{q_2}{2}, \quad \beta = \frac{1}{2}\sqrt{4q_1q_3 - q_2^2}.$$

The reduced equation takes the form

$$A\frac{d^2\phi}{d\omega^2} + B\frac{d\phi}{d\omega} + C\left(\frac{d\phi}{d\omega}\right)^3 = 0, \tag{34}$$

where

$$A = 6b_3(81b_1^4b_3^4 + 108b_1^3b_2^2b_3^3 + 54b_1^2b_2^4b_3^2 + 12b_1b_2^6b_3 + b_2^8),$$
$$B = 9b_1b_3(81b_1^4b_3^4 + 135b_1^3b_2^2b_3^3 + 90b_1^2b_2^4b_3^2 + 30b_1b_2^6b_3 + 5b_2^8) + 3b_1^{10},$$

and

$$C = -\frac{1}{3}A.$$

The general solution of (34) is

$$\phi(\omega) = \frac{A}{B}\sqrt{\frac{B}{C}}\arctan\sqrt{\exp(2B\omega/A) - C\tilde{c}_1/A} + \tilde{c}_2,$$

where $\tilde{c}_1$, $\tilde{c}_2$ are arbitrary real constants. An implicit solution of (32) can then be given in the form

$$\frac{A}{B}\sqrt{\frac{B}{C}}\arctan\sqrt{\exp\left(\frac{2Bx_0}{A}\right)\left[1+\frac{\beta^2}{(q_3u+\alpha)^2}\right]^{-c^2B/(2A\beta^2\lambda)}-\frac{C\tilde{c}_1}{A}}+$$
$$+\frac{c}{\beta}\arctan\left(-\frac{\beta}{q_3u}\right)=x_1-\tilde{c}_2.$$

Case 2(c): Consider $q_2^2-4q_1q_1>0$, i.e.

$$b_1+\frac{b_2^2}{3b_3}>0.$$

The similarity ansatz is then given by

$$u=\frac{\phi(\omega)\exp(A_1+A_2)(q_2+\sqrt{\Delta})-4q_3\sqrt{\Delta}b_2}{12q_3\sqrt{\Delta}b_3-2q_3\phi(\omega)\exp(A_1+A_2)}$$

and

$$\omega=-\frac{2uq_3+q_2-\sqrt{\Delta}}{2uq_3+q_2+\sqrt{\Delta}}\exp(-x_1\sqrt{\Delta}/c),$$

where

$$A_1=\frac{x_0\sqrt{\Delta}}{cq_3\sqrt{2b_3}}\left(-b_2q_2q_3+3b_3q_1q_3+\frac{b_2^2q_3^2}{3b_3}\right),$$
$$A_2=\frac{x_1}{c}\left(\frac{q_2}{2}+\frac{\sqrt{\Delta}}{2}-\frac{b_2q_3}{3b_3}\right)$$

and

$$\Delta=\frac{2c^2}{\lambda}\left(\frac{b_2^2}{3b_3}+b_1\right).$$

The reduced equation is given by

$$\frac{d^2\phi}{d\omega^2}=0 \tag{35}$$

so that the two nontrivial exact solution of (32) take the form

$$\begin{aligned}u={}&[\pm 2\{6S_1S_2\sqrt{2S_2}c\tilde{c}_1\exp[(3\sqrt{2S_2}x_1b_3+S_1S_2x_0)/(2S_1b_3)]+\\&+3S_2\lambda\tilde{c}_1^2\exp[(\sqrt{2S_2}x_1b_3+S_1S_2x_0)/(S_1b_3)]+\\&+18S_2^2b_3c^2\exp[(2\sqrt{2S_2}x_1/S_1]\}^{1/2}+\\&+2\sqrt{\lambda}\tilde{c}_2(b_2-\sqrt{3S_2})\exp[(3\sqrt{2S_2}x_1b_3+S_1S_2x_0)/(2S_1b_3)]-\\&-2\sqrt{6S_2b_3}c(2b_2-\sqrt{3S_2})\exp[\sqrt{2S_2}x_1/S_1]-\\&-2\sqrt{\lambda}b_2\tilde{c}_1\exp[(\sqrt{2S_2}x_1b_3+S_1S_2x_0)/(2S_1b_3)]\times\\&\times\frac{1}{6b_3}[2\sqrt{6S_2b_3}c\exp[\sqrt{2S_2}x_1/S_1]+\sqrt{\lambda}\tilde{c}_1\times\\&\times\exp[(\sqrt{2S_2}x_1b_3+S_1S_2x_0)/(2S_1b_3)]-\\&-\sqrt{\lambda}\tilde{c}_1\exp[(3\sqrt{2S_2}x_1b_3+S_1S_2x_0)/(2S_1b_3)]^{-1},\end{aligned}$$

where

$$S_1 = \sqrt{3\lambda b_3}, \quad S_2 = 3b_1b_3 + b_2^2.$$

Note. The case $\lambda = -1$ and $f(u) = \tilde{b}_3u^3 + \tilde{b}_1u + \tilde{b}_0$, i.e. the equation

$$\frac{\partial u}{\partial x_0} + \frac{\partial^2 u}{\partial x_1^2} = \tilde{b}_3u^3 + \tilde{b}_1u + \tilde{b}_0 \tag{36}$$

has been studied by Fushchych et al. [14]. This case can be obtained from theorem 2 by considering

$$k_1 = 0, \quad k_2 = \frac{c}{3}\sqrt{\frac{2}{\tilde{b}_3}}, \quad k_3 = -\frac{c}{2}\tilde{b}_0\sqrt{\frac{2}{\tilde{b}_3}}, \quad k_4 = \frac{c}{2}\tilde{b}_1\sqrt{\frac{2}{\tilde{b}_3}},$$

i.e.

$$Q = \frac{\partial}{\partial x_0} + \frac{3}{2}\sqrt{2\tilde{b}_3}u\frac{\partial}{\partial x_1} + \frac{3}{2}(\tilde{b}_3u^3 + \tilde{b}_1u + \tilde{b}_0)\frac{\partial}{\partial u}.$$

Case 3: Consider the equation

$$\frac{\partial}{\partial x_0} + \frac{\partial^2 u}{\partial x_1^2} = au^3. \tag{37}$$

From theorem 3, with $\alpha = x_1^{-2}$, it follows that

$$Q = x_1^2\frac{\partial}{\partial x_0} + 3x_1\frac{\partial}{\partial x_1} + 3u\frac{\partial}{\partial u}.$$

The similarity ansatz is then given by

$$u = x_1\phi(\omega), \quad \omega = x_0 - \frac{1}{6}x_1^2$$

so that the reduced equation takes the form

$$\frac{d^2\phi}{d\omega^2} = 9a\phi^3. \tag{38}$$

The general solution, in terms of an elliptic integral, is given by

$$\int_0^\phi \frac{d\tau}{\sqrt{c_1 + \tau^4}} = \frac{3}{2}\sqrt{3a}(\omega + c_2)$$

so that a solution of (37) can be given in the form

$$\int_0^{u/x_1} \frac{d\tau}{\sqrt{c_1 + \tau^4}} = \frac{3}{2}\sqrt{2a}\left(x_0 - \frac{1}{6}x_1^2 + c_2\right).$$

Case 4(a): From theorem 4.1 we obtain the implicit ansatz

$$\frac{df}{du} = \phi(x_0) + \frac{x_1}{\sqrt{x_0}}$$

for (1) where f satisfies (26) and $\lambda = -1$. The reduced equation takes the following form

$$\frac{d\phi}{dx_0} + \frac{1}{2x_0}\phi - 2 = 0 \tag{39}$$

so that an exact solution of the nonlinear partial differential equation is

$$\frac{df}{du} = \frac{x_1}{\sqrt{x_0}} + \frac{4}{3}x_0.$$

Case 4(b): From theorem 4.2 we obtain the implicit ansatz

$$\frac{df}{du} = x_1^2\phi(x_0) + 1$$

for (1) where f satisfies (26) and $\lambda = -1$. The reduced equation takes the following form

$$\frac{d\phi}{dx_0} - 2\phi + 2\phi^2 = 0 \tag{40}$$

so that an exact solution of the nonlinear partial differential ansatz equation is given by

$$\frac{df}{du} = \frac{x_1^2}{1 + c_1\exp(-2x_0)} + 1$$

4. Generalization to $m + 1$ dimensions

For a generalization for m space dimensions we consider the equations

$$\frac{\partial u}{\partial x_0} + \frac{1}{2n}\Delta u = au^3, \tag{41}$$

$$\frac{\partial u}{\partial x_0} + \frac{1}{2n}\Delta u = f(u), \tag{42}$$

where a and n are real constants, f satisfies

$$f\frac{d^2f}{du^2} = 2, \quad \text{and} \quad \Delta \equiv \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{2x_m^2}.$$

An exact solution for (41) is found to be

$$u = \frac{2\boldsymbol{\beta}\cdot\boldsymbol{x}}{3ax_0 - (\boldsymbol{\beta}\cdot\boldsymbol{x})^2},$$

where $\boldsymbol{\beta}\cdot\boldsymbol{x} = \sum\limits_{j=1}^{m}\beta_j x_j$, etc., with β_j arbitrary real constants.

This solution is obtained from the Q-symmetries

$$Q_j = 2\rho_j\frac{\partial}{\partial x_0} + 3au\frac{\partial}{\partial x_j} + 3a\rho_j u^3\frac{\partial}{\partial u},$$

where $\boldsymbol{\rho}^2 = an$ and $j = 1, \ldots, m$. This leads to the ansatz

$$u = \frac{2}{\phi(\omega) - 2\boldsymbol{\beta}\cdot\boldsymbol{x}}, \quad \omega = -\frac{1}{u^2} - 2ax_0$$

from which the reduced equation

$$2\frac{d^2\phi}{d\omega^2} = \left(\frac{d\phi}{d\omega}\right)^3$$

follows. From the Q-symmetry

$$Q_j = \alpha_j(\boldsymbol{\alpha}\cdot\boldsymbol{x})^2\frac{\partial}{\partial x_0} + 3\boldsymbol{\alpha}\cdot\boldsymbol{x}\frac{\partial}{\partial x_j} + 3\alpha_j u\frac{\partial}{\partial u}$$

with $\boldsymbol{\alpha}^2 = 1$ and $j = 1, \ldots, m$, the ansatz

$$u = \boldsymbol{\alpha}\cdot\boldsymbol{x}\phi(\omega), \quad \omega = x_0 - \frac{1}{6}(\boldsymbol{\alpha}\cdot\boldsymbol{x})^2$$

reduced (41) to the equation

$$\frac{d^2\phi}{d\omega^2} = 9a\phi^3. \tag{43}$$

An exact solution for (41) is then given by

$$\int_0^{u/(\boldsymbol{\alpha}\cdot\boldsymbol{x})}\frac{d\tau}{\sqrt{c_1+\tau^4}} = \frac{3}{2}\sqrt{2a}\left[x_0 - \frac{1}{6}(\boldsymbol{\alpha}\cdot\boldsymbol{x})^2 + c_2\right].$$

For (42) we obtain the Q-symmetries

$$Q_j = 2\sqrt{x_0}\frac{\partial}{\partial x_0} + \gamma_j f(u)\frac{\partial}{\partial u},$$

where $\boldsymbol{\gamma}^2 = 2n$ and $j = 1, \ldots, m$. This leads to the implicit ansatz

$$\frac{df}{du} = \frac{\boldsymbol{\gamma}\cdot\boldsymbol{x}}{\sqrt{x_0}} + \phi(x_0),$$

and the reduces equation

$$\frac{d\phi}{dx_0} + \frac{\phi}{2x_0} - 2 = 0. \tag{44}$$

An exact solution of (42) is then given by

$$\frac{df}{du} = \frac{\boldsymbol{\gamma}\cdot\boldsymbol{x} + c_1}{\sqrt{x_0}} + \frac{4}{3}(x_0).$$

5. Concluding remarks

From the above results it is clear that the study of Q-symmetries provides a useful method for obtaining exact solutions for nonlinear partial differential equations. Note that all the reduced equations: (31), (33)–(35), (38)–(40), (43), (44) that were obtained by Q-symmetry reductions are integrable and we were able to solve these reduced equations in general.

Generalized Q-symmetries, in the form of Q-Bäcklund symmetries, defined by

$$Q_{\mathrm{B}} = g(x_0, \ldots, x_m, u, u_0, \ldots, u_{j_1\cdots j_q})\frac{\partial}{\partial u}, \tag{45}$$

can be considered for eq. (2). Here $q > r$. This will extend the number of exact solutions for (2). We could find no Q-Bäcklund symmetry for eq. (1) with nonlinear f. In [1, 17] an example is given to demonstrate the method by which one can obtain exact solutions with Lie–Bäcklund generators. Note that, in the case of Lie–Bäcklund or Q-Bäcklund symmetries for (2), one can, in general, not find the general solution of the equation

$$js^*(Q_B \rfloor \theta) = 0. \tag{46}$$

This is due to the fact that (46) is usually more complicated, in that it has a higher order of derivatives and of nonlinearity, than (2). By, however, considering linear combinations of symmetries in the contraction (46), one can combine (2) and (46) to eliminate some derivatives or non-linearities (see [1, 17]).

In the study of conditional symmetries one can consider additional differential equations as conditions for (2), and then study the symmetry properties of the combined two equations. However, one then has to consider the compatibility problem between (2) and the additional equation. This approach was studied, and exact solutions were obtained, for the multi-dimensional d'Alembert equation [18, 19] and some nonlinear equations of acoustics [20].

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