Separation of variables in the two-dimensional wave equation with potential

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The paper is devoted to solution of a problem of separation of variables in the wave equation $u_{tt} - u_{xx} + V(x)u = 0$. We give a complete classification of potentials $V(x)$ for which this equation admits a nontrivial separation of variables. Furthermore, we obtain all coordinate systems that provide separability of the equation considered.

Дана стаття присвячена розв'язанню проблеми розділення змінних для хвильового рівняння $u_{tt} - u_{xx} + V(x)u = 0$. Вказані всі потенціали $V(x)$, для яких дане рівняння допускає нетривіальне розділення змінних. Крім того, одержані всі системи координат, в яких розділюється досліджуване рівняння.

1. Introduction. In this paper, we study the two-dimensional wave equation with potential

$$(\Box + V(x))u \equiv u_{tt} - u_{xx} + V(x)u = 0, \tag{1}$$

where $u = u(t,x) \in C^2(\mathbb{R}^2, \mathbb{R}^1)$ and $V(x) \in C(\mathbb{R}^1, \mathbb{R}^1)$, by using the method of separation of variables (SV). Equations belonging to class (1) are widely used in the modern quantum physics and can be related to other linear and nonlinear equations of mathematical physics (these relations will be discussed below, at the end of the article). In particular, class (1) contains the d'Alembert equation (with $V(x) = 0$) and the Klein– Gordon–Fock equation (with $V(x) = m \equiv \mathrm{const}$).

The separation of variables in two- and three-dimensional Laplace, Helmholtz, d'Alembert, and Klein–Gordon–Fock equations had been carried out in the classical works by Bocher [1], Darboux [2], Eisenhart [3], Stepanov [4], Olevsky [5], and Kalnins and Miller (see [6] and references therein). Nevertheless, a complete solution of the problem of SV in equation (1) is not obtained yet.

When speaking about solution of equation (1) with separated variables ω_1, ω_2, we mean the ansatz

$$u(t,x) = A(t,x)\varphi_1(\omega_1(t,x))\varphi_2(\omega_2(t,x)) \tag{2}$$

reducing (1) to two ordinary differential equations for the functions $\varphi_i(\omega_i)$

$$\ddot{\varphi}_i = A_i(\omega_i,\lambda)\dot{\varphi}_i + B_i(\omega_i,\lambda)\varphi_i, \quad i = 1,2, \tag{3}$$

In formulas (2) and (3), $A, \omega_1, \omega_2 \subset C^2(\mathbb{R}^2, \mathbb{R}^1)$, $A_i, B_i \subset C^2(\mathbb{R}^1 \times \Lambda, \mathbb{R}^1)$ are some unknown functions, $\lambda \in \Lambda \subset \mathbb{R}^1$ is a real parameter (separation constant).

Definition 1. *Equation (1) admits SV in the coordinates $\omega_1(t,x)$, $\omega_2(t,x)$ if the substitution of ansatz (2) into (1) with subsequent exclusion of the second derivatives $\ddot{\varphi}_1$, $\ddot{\varphi}_2$ according to (3) yields an identity with respect to the variables $\dot{\varphi}_i$, φ_i, λ (considered as independent ones).*

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On the basis of the above definition, one can formulate the procedure of SV in equation (1). At the first step; one has to substitute expression (2) into (1) and to express the second derivatives $\ddot{\varphi}_1$, $\ddot{\varphi}_2$ via the functions $\dot{\varphi}_i$, φ_i according to equations (3). At the second step, the obtained equality is splitted with respect to the independent variables $\dot{\varphi}_i$, φ_i. As a result, one gets an overdetermined system of partial differential equations for the functions A, ω_1, ω_2 with undefined coefficients. The general solution of this system gives rise to all systems of coordinates that provide separability of equation (1).

Let us emphasize that the above approach to SV in equation (1) has much in common with the non-Lie method of reduction of nonlinear differential equations suggested in [7–9]. It is also important to note that the idea of representing solutions of linear differential equations in the "separated" form (2) goes as far as to classical works of Euler and Fourier (for a modern exposition of the problem of SV, see Miller [6] and Koornwinder [10]).

The present paper is organized as follows: In the first section, we adduce principal assertions about SV in equation (1). In the second section, the detailed proof of these assertions is given. In the last section, we briefly discuss the obtained results.

2. List of principal results. It is evident that equation (1) admits SV in the Cartesian coordinates $\omega_1 = t$, $\omega_2 = x$ for an arbitrary $V = V(x)$.

Definition 2. *Equation (1) admits a nontrivial SV if there exists at least one coordinate system $\omega_1 = (t,x)$, $\omega_2(t,x)$, different from the Cartesian system, that provides its separability.*

Next, if, in equation (1), one makes the transformations

$$t \to C_1 t, \quad x \to C_1 x, \quad t \to t, \quad x \to x + C_2, \quad C_i \in \mathbb{R}^1,$$

then the class of equations (1) transforms into itself and, moreover,

$$\begin{aligned} &V(x) \to V'(x) = C_1^2 V(C_1 x), \\ &V(x) \to V'(x) = V(x + C_2). \end{aligned} \tag{4}$$

This is why the potentials $V(x)$ and $V'(x)$ connected by one of the above relations are regarded as equivalent ones.

Theorem 1. *Equation (1) admits a nontrivial SV iff the function $V(x)$ is given up to the equivalence relations (4) by one of the result formulas:*

$$\begin{aligned}
&(1) \quad V = mx; \\
&(2) \quad V = mx^{-2}; \\
&(3) \quad V = m\sin^{-2} x; \\
&(4) \quad V = m\,\mathrm{sh}^{-2} x; \\
&(5) \quad V = m\,\mathrm{ch}^{-2} x; \\
&(6) \quad V = m\exp x; \\
&(7) \quad V = \cos^{-2} x(m_1 + m_2 \sin x); \\
&(8) \quad V = \mathrm{ch}^{-2} x(m_1 + m_2\,\mathrm{sh}\, x); \\
&(9) \quad V = \mathrm{sh}^{-2} x(m_1 + m_2\,\mathrm{ch}\, x);
\end{aligned} \tag{5}$$

(10) $V = m_1 \exp x + m_2 \exp 2x$;

(11) $V = m_1 + m_2 x^{-2}$;

(12) $V = m$.

Here, m, m_1, m_2 are arbitrary real parameters, $m_2 \neq 0$.

Note 1. Equation (1) with the potential $V(x) = m \exp x$ is transformed by the change of variables [11]

$$x' = \exp\frac{x}{2}\operatorname{ch} t, \quad t' = \exp\frac{x}{2}\operatorname{sh} t$$

into equation (1) with $V(x) = m$ (i.e., into the Klein–Gordon–Fock equation).

Note 2. Equations (1) with potentials 3, 4, 5 from (5) are transformed into equation (1) with $V(x) = mx^{-2}$ by the changes of variables [11]

$$x' = \operatorname{tg}\xi + \operatorname{tg}\eta, \quad t' = \operatorname{tg}\xi - \operatorname{tg}\eta,$$
$$x' = \operatorname{th}\xi + \operatorname{th}\eta, \quad t' = \operatorname{th}\xi - \operatorname{th}\eta,$$
$$x' = \operatorname{cth}\xi + \operatorname{th}\eta, \quad t' = \operatorname{cth}\xi - \operatorname{th}\eta.$$

Here, $\xi = (x+t)/2$, $\eta = (x-t)/2$ are cone variables.

In virtue of the above remarks, Theorem 1 implies the following assertion:

Theorem 2. *Provided that equation (1) admits a nontrivial SV, it is locally equivalent to one of the following equations:*

$$\begin{array}{ll} (1) & \Box u + mxu = 0; \\ (2) & \Box u + mx^{-2}u = 0; \\ (3) & \Box u + \cos^{-2} x(m_1 + m_2 \sin x)u = 0; \\ (4) & \Box u + \operatorname{ch}^{-2} x(m_1 + m_2 \operatorname{sh} x) = 0; \\ (5) & \Box u + \operatorname{sh}^{-2} x(m_1 + m_2 \operatorname{ch} x) = 0; \\ (6) & \Box u + e^x(m_1 + m_2 e^x)u = 0; \\ (7) & \Box u + (m_1 + m_2 x^{-2})u = 0; \\ (8) & \Box u + mu = 0. \end{array} \tag{6}$$

Thus, there exist eight inequivalent types of equations of the form (1) that admit a nontrivial SV.

It is well known that there are eleven coordinate systems that provide separability of the Klein–Gordon–Fock equation $(\Box + m)u = 0$ (see, e.g., [12]). This is why the case $V(x) = \text{const}$ is not considered here.

As is shown in Section 2, the general form of the solution of equations (6) with separated variables is as follows:

$$u(t,x) = \varphi_1(\omega_1(t,x))\varphi_2(\omega_2(t,x)); \tag{7}$$

here, $\varphi_1(\omega_1)$, $\varphi_2(\omega_2)$ are arbitrary solutions of the separated ordinary differential (6) here, equations

$$\ddot{\varphi}_i = (\lambda + g_i(\omega_i))\varphi_i, \quad i = 1, 2, \tag{8}$$

and the explicit form of the systems $\omega_i(t,x)$, $g_i(\omega_i)$ is given below.

Theorem 3. *The equation $\Box u + mxu = 0$ separated in two coordinate systems*

$$
\begin{aligned}
&(1) \quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = m\omega_2;\\
&(2) \quad \omega_1 = (x+t)^{1/2} + (x-t)^{1/2}, \quad \omega_2 = (x+t)^{1/2} - (x-t)^{1/2},\\
&\qquad g_1 = -\frac{m}{4} m\omega_1^4, \quad g_2 = -\frac{m}{4}\omega_2^4.
\end{aligned} \tag{9}
$$

Theorem 4. *The equation $\Box u + \sin^{-2} x(m_1 + m_2\cos x)u = 0$ is separated in four coordinate systems*

$$
\begin{aligned}
&(1) \quad \omega_1 = t, \quad \omega_2 = x; \quad g_1 = 0, \quad g_2 = \sin^{-2}\omega_2(m_1 + m_2\cos\omega_2);\\
&(2) \quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \operatorname{arctg}\, \operatorname{sh}(\omega_1+\omega_2) \pm \operatorname{arctg} \operatorname{sh}(\omega_1-\omega_2),\\
&\qquad g_1 = (m_1+m_2)\operatorname{sh}^{-2}\omega_1, \quad g_2 = -(m_1-m_2)\operatorname{ch}^{-2}\omega_2;\\
&(3) \quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \operatorname{arctg}\operatorname{tn}(\omega_1+\omega_2) \pm \operatorname{arctg}\ \operatorname{tn}\ (\omega_1-\omega_2)\\
&\qquad g_1 = m_1\ \operatorname{dn}^2\omega_1\operatorname{cn}^{-2}\omega_1\operatorname{sn}^{-2}\omega_1 + m_2[\operatorname{sn}^{-2}\omega_1 - \operatorname{dn}^2\omega_1\operatorname{cn}^{-2}\omega_1],\\
&\qquad g_2 = m_1k^4\ \operatorname{sn}^2\omega_2\operatorname{cn}^2\omega_2\operatorname{dn}^{-2}\omega_2 + m_2k^2[\operatorname{cn}^2\omega_2\operatorname{dn}^{-2}\omega_2 - \operatorname{sn}^2\omega_2];\\
&(4) \quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \operatorname{arctg}\left(\left(\frac{k}{k'}\right)^{1/2}\operatorname{cn}(\omega_1+\omega_2)\right) \pm \operatorname{arctg}\left(\left(\frac{k}{k'}\right)^{1/2}\operatorname{cn}(\omega_1-\omega_2)\right),\\
&\qquad g_1 = m_1[\operatorname{dn}^2\omega_1\operatorname{cn}^{-2}\omega_1 + k^2\operatorname{sn}^2\omega_1] + m_2[(k')^2\operatorname{cn}^{-2}\omega_1 + k^2\operatorname{cn}^2\omega_1],\\
&\qquad g_2 = m_1[\operatorname{dn}^2\omega_2\operatorname{cn}^{-2}\omega_2 + k^2\operatorname{sn}^2\omega_2] + m_2[(k')^2\operatorname{cn}^{-2}\omega_2 + k^2\operatorname{cn}^2\omega_2].
\end{aligned} \tag{10}
$$

In formulas (10), $k, k' = \sqrt{1-k^2}$ are the moduli of the corresponding elliptic Jacobi functions and k is an arbitrary constant satisfying the inequality $0 < k < 1$.

Theorem 5. *The equation $\Box u + \operatorname{ch}^{-2} x(m_1 + m_2\operatorname{sh} x)u = 0$ is separated in four coordinate systems*

$$
\begin{aligned}
&(1) \quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = \operatorname{ch}^{-2}\omega_2(m_1 + m_2\operatorname{sh}\omega_2);\\
&(2) \quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = -\ln\left(\left(\frac{k'}{k}\right)^{1/2}\operatorname{cn}(\omega_1+\omega_2)\right) \mp \ln\left(\left(\frac{k'}{k}\right)^{1/2}\operatorname{cn}(\omega_1-\omega_2)\right),\\
&\qquad g_1 = m_1(k')^2(\operatorname{dn}2\omega_1)^{-2} + m_2\operatorname{cn}2\omega_1(\operatorname{dn}2\omega_1)^{-2},\\
&\qquad g_2 = m_1(k')^2(\operatorname{dn}2\omega_2)^{-2} + m_2\operatorname{cn}2\omega_2(\operatorname{dn}2\omega_2)^{-2};\\
&(3) \quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = -\ln\operatorname{sh}\frac{1}{2}(\omega_1+\omega_2) \pm \ln\operatorname{ch}\frac{1}{2}(\omega_1-\omega_2),\\
&\qquad g_1 = \operatorname{ch}^{-2}\omega_1(m_1 - m_2\operatorname{sh}\omega_1), \quad g_2 = \operatorname{ch}^{-2}\omega_2(m_1 - m_2\operatorname{sh}\omega_2);\\
&(4) \quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \ln\operatorname{tn}\frac{1}{2}(\omega_1+\omega_2) \pm \ln\operatorname{dn}\frac{1}{2}(\omega_1+\omega_2).\\
&\qquad g_1 = -m_1k^2\operatorname{sn}^2\omega_1 + k^2m_2\operatorname{sn}\omega_1\operatorname{cn}\omega_1,\\
&\qquad g_2 = -m_1k^2\operatorname{sn}^2\omega_2 + k^2m_2\operatorname{sn}\omega_2\operatorname{cn}\omega_2.
\end{aligned} \tag{11}
$$

Here, $k, k' = \sqrt{1-k^2}$ are the moduli of the corresponding elliptic functions, $0 < k < 1$.

Theorem 6. *The equation* $\Box u + \mathrm{sh}^{-2}x(m_1 + m_2\mathrm{ch}x)u = 0$ *is separated in eleven coordinate systems*

$$
\begin{aligned}
&(1) && \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = \mathrm{sh}^{-2}\omega_2(m_1 + m_2\mathrm{ch}\omega_2);\\
&(2) && \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = -\ln\frac{1}{2}(\omega_1+\omega_2) \pm \ln\frac{1}{2}(\omega_1-\omega_2),\\
& && g_1 = (m_1 - m_2)\omega_1^{-2}, \quad g_2 = (m_1 + m_2)\omega_2^{-2};\\
&(3) && \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = -\ln\sin\frac{1}{2}(\omega_1+\omega_2) \pm \ln\sin\frac{1}{2}(\omega_1-\omega_2),\\
& && g_1 = (m_1 - m_2)\sin^{-2}\omega_1, \quad g_2 = (m_1 + m_2)\sin^{-2}\omega_2;\\
&(4) && \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = -\ln\mathrm{sh}\frac{1}{2}(\omega_1+\omega_2) \mp \ln\mathrm{sh}\frac{1}{2}(\omega_1-\omega_2),\\
& && g_1 = \mathrm{sh}^{-2}\omega_1(m_1 + m_2\mathrm{ch}\,\omega_1), \quad g_2 = \mathrm{sh}^{-2}\omega_2(m_1 - m_2\mathrm{ch}\,\omega_2);\\
&(5) && \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = -\ln\mathrm{ch}\frac{1}{2}(\omega_1+\omega_2) \mp \ln\mathrm{ch}\,\frac{1}{2}(\omega_1-\omega_2),\\
& && g_1 = \mathrm{sh}^{-2}\omega_1(m_1 - m_2\mathrm{ch}\,\omega_1), \quad g_2 = \mathrm{sh}^{-2}\omega_2(m_1 - m_2\mathrm{ch}\,\omega_2);\\
&(6) && \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \ln\mathrm{th}\frac{1}{2}(\omega_1+\omega_2) \pm \ln\mathrm{th}\frac{1}{2}(\omega_1-\omega_2),\\
& && g_1 = \mathrm{ch}^{-2}\omega_1(m_2 - m_1), \quad g_2 = -\mathrm{ch}^{-2}\omega_2(m_2 + m_1);\\
&(7) && \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \ln\mathrm{tg}\frac{1}{2}(\omega_1+\omega_2) \pm \ln\mathrm{tg}\,\frac{1}{2}(\omega_1-\omega_2),\\
& && g_1 = \cos^{-2}\omega_1(m_1 + m_2), \quad g_2 = \cos^{-2}\omega_2(m_1 - m_2);\\
&(8) && \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \mathrm{arth}\,\mathrm{cn}\,(\omega_1+\omega_2) \pm \mathrm{arth}\,\mathrm{cn}\,(\omega_1-\omega_2),\\
& && g_1 = (m_1 + m_2)\,\mathrm{dn}^2\omega_1\,\mathrm{cn}^{-2}\omega_1 + (m_1 - m_2)k^2\,\mathrm{sn}^2\omega_1,\\
& && g_2 = (m_1 - m_2)\,\mathrm{dn}^2\omega_2\,\mathrm{cn}^{-2}\omega_2 + (m_1 + m_2)k^2\,\mathrm{sn}^2\omega_2;\\
&(9) && \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \mathrm{arth}\,\mathrm{dn}\,(\omega_1+\omega_2) \pm \mathrm{arth}\,\mathrm{dn}\,(\omega_1-\omega_2),\\
& && g_1 = (m_1 + m_2)k^2\,\mathrm{cn}^2\omega_1\,\mathrm{dn}^{-2}\omega_1 + (m - m_2)k^2\,\mathrm{sn}^2\omega_1,\\
& && g_2 = (m_1 - m_2)k^2\,\mathrm{cn}^2\omega_2\,\mathrm{cn}^{-2}\omega_2 + (m_1 + m_2)k^2\,\mathrm{sn}^2\omega_2;\\
&(10) && \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \mathrm{arth}\,\mathrm{sn}\,(\omega_1+\omega_2) \pm \mathrm{arth}\,\mathrm{sn}\,(\omega_1-\omega_2),\\
& && g_1 = (m_1 + m_2)\,\mathrm{sn}^{-2}\omega_1 + (m_1 - m_2)k^2\,\mathrm{sn}^2\omega_1,\\
& && g_2 = (m_1 + m_2)k^2\,\mathrm{cn}^2\omega_2\,\mathrm{dn}^{-2}\omega_2 + (m_1 - m_2)\,\mathrm{dn}^2\omega_2\,\mathrm{cn}^{-2}\omega_2;\\
&(11) && \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = -\ln\,\mathrm{cn}\,(\omega_1+\omega_2) \pm \ln\,\mathrm{cn}\,(\omega_1-\omega_2),\\
& && g_1 = -m_1\,\mathrm{sn}^{-2}\omega_1 - m_2\,\mathrm{cn}\,\omega_1\,\mathrm{sn}^{-2}\omega_1,\\
& && g_2 = -m_1\,\mathrm{sn}^{-2}\omega_2 - m_2\,\mathrm{cn}\,\omega_2\,\mathrm{sn}^{-2}\omega_2.
\end{aligned}
\tag{12}
$$

Here, k *is the modulus of the corresponding elliptic functions,* $0 < k < 1$.

Theorem 7. *The equation* $\Box u + e^x(m_1 + m_2e^x)u = 0$ *is separated in six coordinate systems*

$$(1) \quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = e^{\omega_2}(m_1 + m_2e^{\omega_2});$$

$$
\begin{aligned}
&(2)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = -\ln\cos(\omega_1+\omega_2) \mp \ln\cos(\omega_1-\omega_2),\\
&\qquad g_1 = -2m_1\cos 2\omega_1 - \frac{m_2}{2}\cos 4\omega_1,\\
&\qquad g_2 = -2m_1\cos 2\omega_2 - \frac{m_2}{2}\cos 4\omega_2;\\
&(3)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \ln\mathrm{sh}\,(\omega_1+\omega_2) \pm \ln\mathrm{sh}\,(\omega_1-\omega_2),\\
&\qquad g_1 = -2m_1\mathrm{ch}2\omega_1 - \frac{m_2}{2}\mathrm{ch}\,4\omega_1,\\
&\qquad g_2 = -2m_1\mathrm{ch}\,2\omega_2 - \frac{m_2}{2}\mathrm{ch}\,4\omega_2;\\
&(4)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \ln\mathrm{ch}(\omega_1+\omega_2) \pm \ln\mathrm{ch}(\omega_1-\omega_2),\\
&\qquad g_1 = -2m_1\mathrm{ch}\,2\omega_1 - \frac{m_2}{2}\mathrm{ch}\,4\omega_1,\\
&\qquad g_2 = -2m_1\mathrm{ch}2\omega_2 - \frac{m_2}{2}\mathrm{ch}\,4\omega_2;\\
&(5)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \ln\mathrm{ch}(\omega_1+\omega_2) \pm \ln\mathrm{sh}(\omega_1-\omega_2),\\
&\qquad g_1 = -2m_1\mathrm{sh}\,2\omega_1 - \frac{m_2}{2}\mathrm{ch}\,4\omega_1,\\
&\qquad g_2 = -2m_1\mathrm{sh}\,2\omega_2 - \frac{m_2}{2}\mathrm{ch}\,4\omega_2;\\
&(6)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \ln(\omega_1+\omega_2) \pm \ln(\omega_1-\omega_2),\\
&\qquad g_1 = 2m_1 + 2m_2\omega_1^2, \quad g_2 = 2m_1 + 2m_2\omega_2^2.
\end{aligned}
\tag{13}
$$

Theorem 8. *The equation $\Box u + (m_1 + m_2 x^{-2})u = 0$ separated in six coordinate systems*

$$
\begin{aligned}
&(1)\quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = m_1 + m_2\omega_2^{-2};\\
&(2)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \exp(\omega_1+\omega_2) \pm \exp(\omega_1-\omega_2),\\
&\qquad g_1 = 4m_1\exp 2\omega_1, \quad g_2 = m_2\mathrm{ch}^{-2}\omega_2;\\
&(3)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \sin(\omega_1+\omega_2) \pm \sin(\omega_1-\omega_2),\\
&\qquad g_1 = 2m_1\cos 2\omega_1 + m_2\sin^{-2}\omega_1, \qquad g_2 = -2m_1\cos 2\omega_2 + m_2\cos^{-2}\omega_2;\\
&(4)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \mathrm{sh}(\omega_1+\omega_2) \pm \mathrm{sh}(\omega_1-\omega_2),\\
&\qquad g_1 = 2m_1\mathrm{sh}2\omega_1 + m_2\mathrm{sh}^{-2}\omega_1,\\
&\qquad g_2 = -2m_1\mathrm{sh}2\omega_2 - m_2\mathrm{sh}^{-2}\omega_2;\\
&(5)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \mathrm{ch}(\omega_1+\omega_2) \pm \mathrm{ch}(\omega_1-\omega_2),\\
&\qquad g_1 = 2m_1\mathrm{ch}2\omega_1 - m_2\mathrm{ch}^{-2}\omega_1, \quad g_2 = 2m_1\mathrm{ch}2\omega_2 - m_2\mathrm{ch}^{-2}\omega_2;\\
&(6)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = (\omega_1+\omega_2)^2 \pm (\omega_1-\omega_2)^2,\\
&\qquad g_1 = -16m_1\omega_1^2 + m_2\omega_1^{-2}, \quad g_2 = -16m_1\omega_2^2 + m_2\omega_2^{-2}.
\end{aligned}
\tag{14}
$$

It was established in [13] that the Euler–Poisson–Darboux equation

$$V_{tt} - V_{xx} - x^{-1}V_x + m^2x^{-2}V = 0$$

is separated in nine coordinate systems. Since the above equation is reduced to the equation $u_{tt} - u_{xx} + (m^2 - 1/4)x^{-2}u = 0$ by the change of dependent variable $\nu(t,x) = x^{-1/2}u(t,x)$, equation (1) with $V(x) = \lambda x^{-2}$ is also separated in nine coordinate systems.

It has been understood not long ago [6, 14] that SV is intimately connected with the symmetry properties of the equation under the study. Therefore, it is important to investigate the symmetry of equation (1).

Clearly, equation (1) with an arbitrary $V(x)$ is invariant under the two-dimensional Lie algebra that has the basis elements $Q_1 = \partial_t$, $Q_2 = u\partial_u$. Below, we adduce without a proof the assertion which gives a complete description of the potentials $V(x)$ that provide an extension of the symmetry algebra admitted by equation (1).

Theorem 9. *Equation (1) admits additional symmetry operators (i.e., operators not belonging to the algebra $\langle \partial_t, u\partial_u \rangle$) iff the potential $V(x)$ is given by one of the following formulas:*

(1) $V(x) = m\exp x$;

(2) $V(x) = mx^{-2}$;

(3) $V(x) = m\sin^{-2} x$;

(4) $V(x) = m\,\mathrm{sh}^{-2}\, x$;

(5) $V(x) = m\,\mathrm{ch}^{-2}\, x$;

(6) $V(x) = m, \quad m \in \mathbb{R}^1$,

with the additional symmetry operators having the form

(1) $Q_3 = \exp\left\{\frac{1}{2}(t-x)\right\}(\partial_x - \partial_t), \quad Q_4 = \exp\left\{-\frac{1}{2}(x+t)\right\}(\partial_x + \partial_t)$;

(2) $Q_3 = x\partial_x + t\partial_t, \quad Q_4 = (x^2+t^2)\partial_t + 2tx\partial_x$;

(3) $Q_3 = \sin t\cos x\partial_t + \sin x\cos t\partial_x, \quad Q_4 = -\cos t\cos x\partial_t + \sin x\sin t\partial_x$;

(4) $Q_3 = \mathrm{sh}\, t\,\mathrm{ch}\, x\partial_t + \mathrm{sh}\, x\,\mathrm{ch}\, t\partial_x, \quad Q_4 = \mathrm{ch}\, t\,\mathrm{ch}\, x\partial_t + \mathrm{sh}\, t\,\mathrm{sh}\, x\partial_x$;

(5) $Q_3 = \mathrm{sh}\, x\,\mathrm{ch}\, t\partial_t + \mathrm{sh}\, t\,\mathrm{ch}\, x\partial_x, \quad Q_4 = \mathrm{sh}\, t\,\mathrm{sh}\, x\partial_t + \mathrm{ch}\, t\,\mathrm{ch}\, x\partial_x$;

(6) $Q_3 = \partial_x, \quad Q_4 = t\partial_x + x\partial_t$.

This theorem is proved by the standard Lie method (see, e.g., [15, 16]).

Corollary. *If equation (1) admits additional symmetry operators, then it is locally equivalent to one of the equations $\Box u + mu = 0$ or $\Box u + mx^{-2}u = 0$.*

Thus, separability of equations 1, 3–7 from (6) is not connected with their Lie symmetry. To explain this fact one has to take into account the second-order (non-Lie) symmetry operators of equation (1). This problem will be briefly discussed in the last section.

3. Proof of Theorems 1–8. To prove the assertions listed in the previous section one has to apply the above described procedure of SV to equation (1).

By substituting ansatz (2) into equation (1), expressing the functions $\ddot{\varphi}_i$ in terms of the functions $\dot{\varphi}_i$, φ_i, with the help of equalities (3), and splitting the obtained equation with respect to independent variables $\dot{\varphi}_i$, φ_i, we get the following system of nonlinear partial differential equations:

$$1)\quad A\Box\omega_1 + 2(A_t\omega_{1t} - A_x\omega_{1x}) + AA_1(\omega_1,\lambda)(\omega_{1t}^2 - \omega_{1x}^2) = 0, \tag{15}$$

$$2)\quad A\Box\omega_2 + 2(A_t\omega_{2t} - A_x\omega_{2x}) + AA_2(\omega_2,\lambda)(\omega_{2t}^2 - \omega_{2x}^2) = 0, \tag{16}$$

$$3)\quad \Box A + A[B_1(\omega_1,\lambda)(\omega_{1t}^2 - \omega_{1x}^2) + B_2(\omega^2,\lambda)(\omega_{2t}^2 - \omega_{2x}^2) + AV(x) = 0, \tag{17}$$

$$4)\quad \omega_{1t}\omega_{2t} - \omega_{1x}\omega_{2x} = 0. \tag{18}$$

Here, $\Box \equiv \partial_t^2 - \partial_x^2$.

Thus, to separate variables in the linear differential equation (1) one has to construct a general solution of system of nonlinear partial differential equations (15)–(18). The same assertion holds true for a general linear differential equation, i.e., the problem of SV is essentially nonlinear. This is the reason why, even for the classical d'Alembert equation $\Box_4 u \equiv u_{tt} - \Delta_3 u = 0$, there is no complete description of all coordinate systems that provide its separability [6].

It is not difficult to become convinced of that from (18). Since the functions ω_1, ω_2 are real, we have

$$(\omega_{1t}^2 - \omega_{1x}^2)(\omega_{2t}^2 - \omega_{2x}^2) \neq 0. \tag{19}$$

Differentiating equations (15), (16) with respect to λ and using (19), we get $A_{1\lambda} = A_{2\lambda} = 0$.

Consequently, the relation $B_{1\lambda}B_{2\lambda} \neq 0$ holds. Differentiating with respect to λ we have

$$B_{1\lambda}(\omega_{1t}^2 - \omega_{1x}^2) + B_{2\lambda}(\omega_{2t}^2 - \omega_{2x}^2) = 0$$

or $B_{1\lambda}/B_{2\lambda} = -(\omega_{2t}^2 - \omega_{2x}^2)/(\omega_{1t}^2 - \omega_{1x}^2)$. Differentiation of the above equality with respect to λ yields $B_{1\lambda\lambda}/B_{1\lambda} = B_{2\lambda\lambda}/B_{2\lambda}$. But the functions $B_1 = B_1(\omega_1)$, $B_2 = B_2(\omega_2)$ are independent, whence it follows that there exists a function such that $B_{i\lambda\lambda} = K(\lambda)B_{i\lambda}$, $i = 1, 2$.

Integrating the above differential equation with respect to λ, we get

$$B_i(\omega_i) = \Lambda(\lambda)f_i(\omega_i) + g_i(\omega_i), \quad i = 1, 2,$$

where f_i, g_i are arbitrary smooth functions.

On redefining the parameter $\lambda \to \Lambda(\lambda)$, we have

$$B_i(\omega_i) = \Lambda f_i(\omega_i) + g_i(\omega_i), \quad i = 1, 2, \tag{20}$$

Substitution of (20) into (17) with a subsequent splitting with respect to λ yields the following equations:

$$\Box A + A[g_1(\omega_1)(\omega_{1t}^2 - \omega_{1x}^2) + g_2(\omega_2)(\omega_{2t}^2 - \omega_{2x}^2) + V(x)A = 0, \tag{21}$$

$$f_1(\omega_1)(\omega_{1t}^2 - \omega_{1x}^2) + f_2(\omega_2)(\omega_{2t}^2 - \omega_{2x}^2) = 0. \tag{22}$$

Thus, system (15)–(18) is equivalent to the system of equations (15), (16), (20)–(22). Before integrating it, we make a remark. It is evident that the structure of ansatz (2) is not changed by the transformation

$$\begin{aligned} &A \to A' = Ah_1(\omega_1)h_2(\omega_2), \\ &\omega_i \to \omega_i' = R_i(\omega_i), \quad i = 1, 2, \end{aligned} \tag{23}$$

where h_i, R_i are some smooth functions.

This is why solutions of the system under the study, connected by relations (23), are considered as equivalent ones.

By a proper choice of the functions h_i, we can put R_i, $f_1 = f_2 = 1$ and $A_1 = A_2 = 0$ in equations (15), (16), (22).

Consequently, the functions ω_1, ω_2 satisfy equations of the form

$$\omega_{1t}\omega_{2t} - \omega_{1x}\omega_{2x} = 0, \quad \omega_{1t}^2 - \omega_{1x}^2 + \omega_{2t}^2 - \omega_{2x}^2 = 0,$$

whence $(\omega_1 \pm \omega_2)_t^2 - (\omega_1 \pm \omega_2)_x^2 = 0$. Integrating the above equations, we get

$$\omega_1 = f(\xi) + g(\eta), \quad \omega_2 = f(\xi) - g(\eta), \tag{24}$$

where $f, g \subset C^2(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions, $\xi = (x+t)/2$, $\eta = (x-t)/2$.

Substitution of (24) into equations (15), (16) with $A_1 = A_2 = 0$ yields the following equations for a function $A = A(t, x)$: $(\ln A)_t = 0$, $(\ln A)_x = 0$, whence $A = 1$.

At last, substituting the obtained results into equation (21), we have

$$V(x) = [g_1(f+g) - g_2(f-g)]\frac{df}{d\xi}\frac{dg}{d\eta}. \tag{25}$$

Thus, the problem of integration of the overdetermined system of nonlinear differential equations (15)–(18) is reduced to the integration of the functional-differential equation (25).

Let us sum up the obtained results. The general form of the solution of equation (1) with separated variables is as follows:

$$u_1 = \varphi_1(f(\xi) + g(\eta))\varphi_2(f(\xi) - g(\eta)); \tag{26}$$

here, φ_i are arbitrary solutions of equations (8) and the functions $f(\xi)$, $g(\eta)$, $g_1(f+g)$, $g_2(f-g)$, $V(x)$ are determined by (25).

To integrate equation (25) we make the hodograph transformation

$$\xi = P(f), \quad \eta = R(g), \tag{27}$$

where $\dot{P} \not\equiv 0$, $\dot{R} \not\equiv 0$.

After making transformation (27), we get

$$g_1(f+g) - g_2(f-g) = \dot{P}(f)\dot{R}(g)V(P+R). \tag{28}$$

Evidently, equation (28) is equivalent to the equation

$$(\partial_f^2 - \partial_g^2)[\dot{P}(f)\dot{R}(g)V(P+R)] = 0$$

or

$$(\dddot{P}\dot{P}^{-1} - \dddot{R}\dot{R}^{-1})V + 3(\ddot{P} - \ddot{R})\dot{V} + (\dot{P}^2 - \dot{R}^2)\ddot{V} = 0. \tag{29}$$

Thus, to integrate equation (25) it suffices to construct all functions $P(f)$, $R(g)$, $V(P+R)$ satisfying (29) and substitute them into equation (28).

Let us prove the following assertion.

Lemma. *The general solution of equation (29), determined up to transformations (4), is given by the one of the following formulas:*

(1) $V = V(x)$ *is an arbitrary function*, $\dot{P} = \alpha$, $\dot{R} = \alpha$;

$$(2)\quad V = mx, \quad \dot{P}^2 = \alpha P + \beta, \quad \dot{R}^2 = \alpha R + \gamma; \tag{30}$$

(3) $V = mx^{-2}$, $P = F(f)$, $R = G(g)$,

$$\dot{F}^2 = \alpha F^4 + \beta F^3 + \gamma F^2 + \delta F + \rho, \quad \dot{G}^2 = \alpha G^4 - \beta G^3 + \gamma G^2 - \delta G + \rho; \tag{31}$$

(4) $V = m\sin^{-2}x$, $P = \operatorname{arctg} F(f)$, $R = \operatorname{arctg} G(g)$, *and* F, G *are determined by (31)*;

(5) $V = m\,\mathrm{sh}^{-2}x$, $P = \operatorname{arth} F(f)$, $R = \operatorname{arth} G(g)$ *and* F, G *are determined by (31)*;

(6) $V = m\,\mathrm{ch}^{-2}x$, $P = \operatorname{arcth} F(f)$, $R = \operatorname{arcth} G(g)$ *and* F, G *are determined by (31)*;

(7) $V = m\exp x$,
$\dot{P}^2 = \alpha\exp 2P + \beta\exp P + \gamma$, $\dot{R}^2 = \alpha\exp 2R + \delta\exp R + \rho$;

$$(8)\quad V = \cos^{-2}x(m_1 + m_2\sin x), \quad \dot{P}^2 = \alpha\sin 2P + \beta\cos 2P + \gamma, \quad \dot{R}^2 = \alpha\sin 2R + \beta\cos 2R + \gamma; \tag{32}$$

$$(9)\quad V = \mathrm{ch}^{-2}x(m_1 + m_2\mathrm{sh}\,x), \quad \dot{P}^2 = \alpha\,\mathrm{sh}\,2P + \beta\mathrm{ch}\,2P + \gamma, \quad \dot{R}^2 = \alpha\,\mathrm{sh}\,2R - \beta\mathrm{ch}\,2R + \gamma; \tag{33}$$

$$(10)\quad V = \mathrm{sh}^{-2}x(m_1 + m_2\mathrm{ch}x), \quad \dot{P}^2 = \alpha\,\mathrm{sh}\,2P + \beta\mathrm{ch}\,2P + \gamma, \quad \dot{R}^2 = -\alpha\,\mathrm{sh}\,2R + \beta\mathrm{ch}\,2R + \gamma; \tag{34}$$

$$(11)\quad V = (m_1 + m_2\exp x)\exp x, \quad \ddot{P} = -\dot{P}^2 + \beta, \quad \ddot{R} = -\dot{P}^2 + \beta; \tag{35}$$

$$(12)\quad V = m_1 + m_2x^{-2}, \quad \dot{P}^2 = \alpha P^2 + \beta P + \gamma, \quad \ddot{R}^2 = \alpha R^2 - \beta R + \gamma, \tag{36}$$

(13) $V = m$,
$\dot{P}^2 = \alpha P^2 + \beta P + \gamma$, $\dot{R}^2 = \alpha R^2 + \delta R + \rho$.

Here α, β, γ, δ, ρ, m_1, m_2, m *are arbitrary real parameters;* $x = \xi + \eta = P + R$.

Proof. Since the functions P, R in (29) are arbitrary, equation (29) is equivalent to the following system of equations:

$$(H_{ffff}H_f^{-1} - H_{ggg}H_g^{-1})V(H) + 3(H_{ff} - H_{gg})\dot{V}(H) + (H_f^2 - H_g^2)\ddot{V}(H) = 0, \tag{37}$$

$$H_{fg} = 0; \tag{38}$$

here, $H = P(f) + R(g)$.

Taking differential consequences of equation (37), we have

$$\begin{aligned}
H_{ffff} &= H_{fff}H_{ff}H_f^{-1} + \dot{V}V^{-1}(H_{ggg}H_g^{-1}H_f^2 - 4H_{fff}H_f) + \\
&\quad + \ddot{V}V^{-1}(3H_{gg}H_f^2 - 5H_{ff}H_f^2) + \dddot{V}V^{-1}(H_g^2H_f^2 - H_f^4), \\
H_{gggg} &= H_{ggg}H_{gg}H_g^{-1} + \dot{V}V^{-1}(H_{fff}H_f^{-1}H_g^2 - 4H_{ggg}H_g) + \\
&\quad + \ddot{V}V^{-1}(3H_{ff}H_g^2 - 5H_{gg}H_g^2) + \dddot{V}V^{-1}(H_f^2H_g^2 - H_g^4).
\end{aligned} \tag{39}$$

For system (39) to be compatible, it is necessary that relations $H_{ffffg} = H_{ggggf} = 0$ hold. Differentiating the first equation in (39) with respect to g and taking into account relations (39), we get

$$\begin{aligned}
&(H_{fff}H_f^{-1} - H_{ggg}H_g^{-1})(5\dot{V}^2V^{-2} - 4\ddot{V}V^{-1}) + (H_{ff} - H_{gg}) \times \\
&\quad \times (8\ddot{V}\dot{V}V^{-2} - 5\dddot{V}V^{-1}) + (H_f^2 - H_g^2)(\dddot{V}\dot{V}V^{-2} - (\dddot{V}V^{-1})^{\cdot}) = 0.
\end{aligned} \tag{40}$$

Since equation (40) is a necessary compatibility condition for a system (39), one has to supplement the system under study (equations (37), (38)) by equation (40). To investigate the system of equations (37), (38), (40) it is necessary to consider several inequivalent cases.

Case 1. Let $\ddot{V} = 0$, $\dot{V} \neq 0$. Then equalities $H_{ff} = H_{gg} = 2\alpha$, $\alpha = \mathrm{const}$ hold. Hence, we have

$$\begin{aligned}
&V = m(H + C) \equiv m(x + C), \\
&P(f) = \alpha f^2 + \beta, \quad R(g) = \alpha g^2 + \gamma, \quad \beta, \gamma \subset \mathbb{R}^1,
\end{aligned}$$

i.e., we obtain the potential listed in the lemma under number 2.

Case 2. Let $\ddot{V} \neq 0$ and let equation (40) be a consequence of equation (37). In this case, the coefficients of V, $\dot{V}$, $\ddot{V}$ must be proportional

$$\begin{aligned}
&(5\dot{V}^2V^{-2} - 4\ddot{V}V^{-1}) = (8\ddot{V}\dot{V}V^{-2} - 5\dddot{V}V^{-1})(3V)^{-1} = \\
&\quad = (2\dddot{V}\dot{V}V^{-2} - \ddddot{V}V^{-1})(\ddot{V})^{-1}.
\end{aligned}$$

From the above equalities, we get a system of two ordinary differential equations for a function $V = V(H)$

$$\dddot{V} = 4\dot{V}V^{-1} - 3\dot{V}^3V^{-2}, \tag{41}$$

$$\ddddot{V} = 2\dddot{V}\dot{V}V^{-1} - 4\ddot{V}^2V^{-1} + 5\dot{V}^2\ddot{V}V^{-2}. \tag{42}$$

But equation (42) is the differential consequence of equation (41). The general solution of equation (41), determined up to equivalence relations (4), is given by one of the following formulas [17]:

$$\begin{aligned}
&V_1 = mH^{-2}, \quad V_2 = m\sin^{-2}H, \\
&V_3 = m\,\mathrm{sh}^{-2}H, \quad V_4 = m\,\mathrm{ch}^{-2}H, \quad V_5 = m\exp H;
\end{aligned} \tag{43}$$

i.e., we obtain potentials listed in the lemma under numbers 3–7.

By substituting $V = V_1 = mH^{-2}$ into (37) and replacing H by $P(f) + R(g)$, we get

$$(P+R)^2(\dddot{P}\dot{P}^{-1} - \dddot{R}\dot{R}^{-1}) - 6(P+R)(\ddot{P} - \ddot{R}) + 6(\dot{P}^2 - \dot{R}^2) = 0. \tag{44}$$

By differentiating (44) with respect to f and g, we obtain

$$(P+R)(\dot{h}_1\dot{P}^{-1} - \dot{h}_2\dot{R}^{-1}) = 2(h_1 - h_2),$$

where $h_1 = \dddot{P}\dot{P}^{-1}$ and $h_2 = \dddot{R}\dot{R}^{-1}$.

Differentiation of the above equation with respect to f and g yields the following relation:

$$(\dot{h}_1\dot{P}^{-1})^{\cdot}\dot{P}^{-1} = (\dot{h}_2\dot{R}^{-1})^{\cdot}\dot{R}^{-1}. \tag{45}$$

Since the functions $P(f)$, $R(g)$ are independent, it follows from (45) that the equalities

$$(\dot{h}_1\dot{P}^{-1})^{\cdot} = 12\alpha\dot{P}, \quad (\dot{h}_2\dot{R}^{-1})^{\cdot} = 12\alpha\dot{R}. \tag{46}$$

hold, where α is an arbitrary real parameter.

Integration of equations (46) yields

$$\dot{P} = \alpha P^4 + C_1P^3 + C_2P^2 + C_3P + C_4,$$
$$\dot{R}^2 = \alpha R^4 + D_1R^3 + D_2R^2 + D_3R + D_4,$$

where $C_1, \ldots, C_4$, $D_1, \ldots, D_4$ are arbitrary real constants. Substituting the above result into the initial equation (44), we get restrictions on the choice of arbitrary constants

$$C_1 = -D_1 = \beta, \quad C_2 = D_2 = \gamma, \quad C_3 = -D_3 = \delta, \quad C_4 = D_2 = \rho.$$

Thus, we have obtained the potential listed in the lemma under number 3.

It is straightforward to verify that the equations obtained by the substitution of functions $V = m\sin^{-2}H$, $V = m\,\mathrm{ch}^{-2}$ with $H = P(f) + R(g)$ into (37) are reduced to equation (44) by the following changes of variables:

$$P \to \mathrm{arctg}\,P, \quad R \to \mathrm{arctg}\,R,$$
$$P \to \mathrm{arth}\,P, \quad R \to \mathrm{arth}\,R,$$
$$P \to \mathrm{arcth}\,P, \quad R \to \mathrm{arcth}\,R;$$

i.e., the potentials listed in the lemma under numbers 4–6 are obtained.

Equation (1) with the potential $V = m\exp H$ is reduced to the Klein–Gordon–Fock equation (see case 4 and Note 2 below).

Case 3. Let $\ddot{V} \neq 0$ and assume, in addition, that equation (41) does not hold. In this case, we can exclude from equations (37), (40) the third derivatives of the function H

$$H_{ff} - H_{gg} + A(H)(H_f^2 - H_g^2) = 0, \tag{47a}$$

where

$$A(H) = (\ddddot{V} - 2\dddot{V}\dot{V}V^{-1} - 4\ddot{V}^2V + 5\ddot{V}\dot{V}^2V^{-2})(\dddot{V} - 4\ddot{V}\dot{V}V^{-1} + 3\dot{V}^3V^{-2})^{-1}.$$

It follows from (47a)

$$H_{fff} = \dot{A}H_f(H_g^2 - H_f^2) - 2H_{ff}H_f A, \quad H_{ggg} = \dot{A}H_g(H_f^2 - H_g^2) - 2H_{gg}H_g A,$$

(we have used equation (38)).

By taking the first differential consequence of the above equations with account of equation (38), we get

$$2\dot{A}(H_{ff} - H_{gg}) + \ddot{A}(H_f^2 - H_g^2) = 0. \tag{48}$$

Clearly, equations (47a) and (48) are consistent iff the function $A(H)$ satisfies the following ordinary differential equation:

$$\ddot{A} = 2\dot{A}A,$$

the general solution of which is given by one of the formulas (up to scaling $H \to CH$).

$$A = C, \quad A = \operatorname{tg}(H + C), \quad A = -\operatorname{th}(H + C),$$
$$A = -\operatorname{cth}(H + C), \quad A = -(H + C)^{-1}, \quad C \in \mathbb{R}^1.$$

Next, we consider the above cases separately.

Case 3.1. $A(H) = C$, $C \neq 0$. In this case, equation (47a) takes the form

$$P_{ff} - R_{gg} + C(P_f^2 - R_g^2) = 0 \tag{47b}$$

or

$$P_{ff} + CP_f^2 = R_{gg} + CR_g^2 = \beta, \quad \beta \in \mathbb{R}^1.$$

Finally, we get

$$P_{ff} = -CP_f^2 + \beta, \quad R_{gg} = -CR_g^2 + \beta. \tag{49}$$

Differentiating the first equation with respect to f, the second equation with respect to g, and subtracting, we get

$$P_{fff}P_f^{-1} - P_{ggg}P_g^{-1} = -2C(P_{ff} - R_{gg}). \tag{50}$$

Substituting (49), (50) into equation (37), we come to the equation for $V = V(H)$,

$$\ddot{V} - 3C\dot{V} + 2C^2V = 0$$

the general solution of which reads

$$V = m_1 \exp CH + m_2 \exp 2CH, \quad m_2, m_2 \subset \mathbb{R}^1. \tag{51}$$

It is not difficult to check that function (51) satisfies equation (47b) provided that $A(H) = C$. Consequently, if the potential is given by formula (51) (after rescaling $H \to CH$, we can choose $C = 1$), then the functions $P(f)$, $R(g)$ are determined by equations (35).

Case 3.2. $A = \operatorname{tg}(H + C)$. Multiplying equation (47) by $\operatorname{ctg}(H + C)$ and differentiating the obtained expression with respect to f and g, we arrive at the equation

$$(P_{fff}P_f^{-1} - P_{ggg}P_g^{-1}) - 2\operatorname{ctg}(H + C)(P_{ff} - R_{gg}) = 0. \tag{52}$$

After excluding the function $\operatorname{ctg}(H+C)$ from (47) and (52), we get an equation with separated variables

$$(P_{fff}P_f^{-1} - P_{ggg}P_g^{-1}) + 2(P_f^2 - R_g^2) = 0,$$

whence

$$P_{fff}P_f^{-1} + 2P_f^2 = \theta, \quad R_{ggg}R_g^{-1} + 2R_g^2 = \theta. \tag{53}$$

In (53), θ is an arbitrary real constant.

Substitution of formulas (52), (53) into equation (37) gives the equation for $V = V(H)$,

$$\ddot{V} - 3\operatorname{tg}(H+C)\dot{V} - 2V = 0$$

the general solution of which has the form [17]

$$V = \cos^{-2}(H+C)[m_1 + m_2\sin(H+C)]. \tag{54}$$

As a direct check shows, the function $V(H)$ (54) satisfies equation (47b) with $A = \operatorname{tg}(H+C)$.

Integrating equations (53), we get

$$P_f^2 = C_1\sin 2P + C_2\cos 2P + \gamma, \quad P_g^2 = D_1\sin 2R + D_2\cos 2R + \gamma, \tag{55}$$

where C_i, D_i, and γ are arbitrary real constants.

Substitution of (55) into (47) with $A = \operatorname{tg}(H+C)$ yields the following restrictions on the choice of the constants C_i, D_i: $C_1 = D_1 = \alpha$, $C_2 = D_2 = \beta$.

Thus, provided that the function $V(H)$ is given by (44), the functions $P(f)$, $R(g)$ are determined by equations (32).

Case 3.3. $A = -\operatorname{th}(H+C)$. In this case, one can obtain the following differential consequence of equation (47):

$$P_{fff}P_f^{-1} - P_{ggg}P_g^{-1} = 2\operatorname{cth}(P+R+C)(P_{ff} - R_{gg}). \tag{56}$$

Excluding the function $\operatorname{ctg}(H+C)$ from equations (47), (56), we get the equation

$$P_{fff}P_f^{-1} - P_{ggg}P_g^{-1} = 2(P_f^2 - R_g^2),$$

whence

$$P_{fff}P_f^{-1} - 2P_f^2 = \theta, \quad R_{ggg}R_g^{-1} - 2R_g^2 = \theta. \tag{57}$$

In (57), θ is an arbitrary real constant.

Integration of equations (57) gives

$$P_f^2 = C_1\operatorname{sh} 2P + C_2\operatorname{ch} 2P + \gamma, \quad R_g^2 = D_1\operatorname{sh} 2R + D_2\operatorname{ch} 2R + \gamma, \tag{58}$$

where C_i, D_i, and γ are arbitrary real constants.

Substituting expressions (56), (57) into (37), we obtain an equation for $V(H)$,

$$\ddot{V} + 3\operatorname{th}(H+C)\dot{V} + 2V = 0,$$

the general solution of which has the form [17]

$$V = \operatorname{ch}^{-2}(H+C)(m_1 + m_2\operatorname{sh}(H+C)), \quad m_i \in \mathbb{R}^1. \tag{59}$$

It is not difficult to become convinced of the fact that function (59) satisfies equation (47b) with $A = -\mathrm{th}\,(H + C)$.

At last, substituting (57) and (58) into (47), we get $C_1 = D_1 = \alpha$, $C_2 = -D_2 = \beta$. Consequently, if the potential $V(H)$ is given by formula (59), then functions $P(f)$ and $R(g)$ are determined by equations (33).

Case 3.4. $A = -\mathrm{cth}\,(H+C)$. In this case, one can obtain the following differential consequence of equation (47):

$$P_{fff}P_f^{-1} - R_{ggg}R_g^{-1} = 2\,\mathrm{th}\,(P + R + C)(P_{ff} - R_{gg}). \tag{60}$$

Using equations (37), (47), and (60), we get an equation for $V(H)$,

$$\ddot{V} + 3\,\mathrm{cth}\,(H + C)\dot{V} + 2V = 0,$$

the general solution of which has the form [17]

$$V = \mathrm{sh}^{-2}(H + C)(m_1 + m_2\,\mathrm{ch}(H + C)), \quad m_i \in \mathbb{R}^1. \tag{61}$$

By direct computation, one can check that function (61) satisfies equation (47b) with $A = -\mathrm{cth}\,(H + C)$.

Next, by eliminating the function $\mathrm{th}\,(H+C)$ from equations (47) and (60), we get an equation with separated variables

$$P_{fff}P_f^{-1} - P_{ggg}P_g^{-1} - 2P_f^2 + 2R_g^2 = 0,$$

whence

$$P_{fff}P_f^{-1} - 2P_f^3 = \theta, \quad R_{ggg}R_g^{-1} - 2R_g^2 = \theta.$$

Here, θ is an arbitrary real constant.

Integration of the above ordinary differential equations shows that the functions $P(f)$ and $R(g)$ are determined by equations (58), where C_i, D_i, and γ are arbitrary real constants. Substituting (58) into equation (47), we have the following restrictions on the choice of C_i, D_i:

$$C_1 = -D_1 = \alpha, \quad C_2 = D_2 = \beta.$$

Thus, if the function $V(H)$ is given by (61), then functions $P(f)$ and $R(g)$ are determined by equations (34).

Case 3.5. $A = -(H + C)^{-1}$. In this case, it follows from (47a) that the equality $P_{fff}P_f^{-1} = R_{ggg}R_g^{-1}$ holds. Hence, we get equations for $P(f)$, $R(g)$,

$$P_{fff} = \theta P_f, \quad R_{ggg} = \theta R_g \tag{62}$$

with arbitrary $\theta \in \mathbb{R}^1$. Moreover, the equation for $V(H)$ has the form $\ddot{V}+3(H+C)\dot{V} = 0$, whence

$$V = m_1 + m_2(H + C)^{-2}, \quad m_i \in \mathbb{R}^1. \tag{63}$$

It is not difficult to check that function (63) satisfies (47b) with $A = -(H+C)^{-1}$.

Integration of equations (62) yields the following result:

$$P_f^2 = \alpha P^2 + C_1P + C_2, \quad R_g^2 = \alpha R^2 + D_1R + D_2, \tag{64}$$

here α, C_i, and D_i are arbitrary real constants.

Next, substituting (64) into (47), we get $C_1 = -D_1 = \beta$, $C_2 = D_2 = \gamma$.

Thus, if the potential V is given by (63), then the functions $P(f)$, $R(g)$ are determined by equations (36).

Case 4. $V(H) = m = \text{const}$. In this case, equation (37) reads $P_{fff}P_f^{-1} = R_{ggg}R_g^{-1}$, whence

$$P_{fff} = \theta P_f, \quad R_{ggg} = \theta R_g, \tag{65}$$

where $\theta \in \mathbb{R}^1$ is an arbitrary constant.

Integrating (65), we get equations listed in the lemma under number 13.

Case 5. $V(H)$ is an arbitrary function. In this case, the coefficients of $\ddot{V}$, $\dot{V}$, V in (37) must vanish. Consequently, the relations

$$H_{fff}H_f^{-1} = H_{ggg}H_g^{-1}, \quad H_{gg} = H_{ff}, \quad H_f^2 = H_g^2$$

hold. Hence, we have $H_f = \alpha$, $H_g = \alpha$, $\alpha \in \mathbb{R}^1$. The lemma is proved.

Theorems 1, 2 are direct consequences of the above lemma. To prove Theorems 3–8, one has to integrate ordinary differential equations (30), (32)–(36) and substitute the obtained expressions into (27),

$$\frac{1}{2}(x+t) = P(f) \equiv P\left(\frac{\omega_1+\omega_2}{2}\right), \quad \frac{1}{2}(x-t) = R(g) \equiv R\left(\frac{\omega_1-\omega_2}{2}\right),$$

and (28).

Integration of equations (30), (32)–(36) is carried out in a standard way [17, 18], the obtained result depends essentially on relations between parameters α, β, γ, δ, ρ. This procedure demands very cumbersome computations; this is why we omit details.

With the above remarks, the proof of Theorems 1–8 is completed.

4. Discussion. Let us say a few words about intrinsic characterization of SV in equation (1). It is well known that the solution of a second-order partial differential equation with separated variables is a joint eigenfunction of mutually commuting second-order symmetry operators of the equation under study (for more details, see [6, 10, 14]). Below we construct, in an explicit form, a second-order symmetry operator of equation (1) such that the solution with separated variables is its eigenfunction and the parameter λ is an eigenvalue.

Making the change of variables (24) in equation (1), we get

$$u_{\omega_1\omega_1} - u_{\omega_2\omega_2} = V(\xi+\eta)(\dot{f}(\xi)\dot{g}(\eta))^{-1}u. \tag{66}$$

Provided that equation (1) admits SV, by virtue of equation (25), there exist functions $g_1(f+g)$ and $g_2(f-g)$ such that

$$V(\xi+\eta)(\dot{f}(\xi)\dot{g}(\eta))^{-1} = g_1(f+g) - g_2(f-g).$$

Since $f+g=\omega_1$ and $f-g=\omega_2$, equation (66) takes the form

$$u_{\omega_1\omega_1} - u_{\omega_2\omega_2} = (g_1(\omega_1) - g_2(\omega_2))u$$

or

$$Xu = 0, \quad X = \partial^2_{\omega_1} - \partial^2_{\omega_2} - g_1(\omega_1) + g_2(\omega_2).$$

It is evident that the operators $Q_i = \partial^2_{\omega_i} - g_i(\omega_i)$, $i = 1, 2$, commute with the operator X, i.e., they are symmetry operators of equation (1) and, moreover, the relations

$$Q_i u = Q_i \varphi_1(\omega_1)\varphi_2(\omega_2) = \lambda \varphi_1(\omega_1)\varphi_2(\omega_2) = \lambda u, \quad i = 1, 2$$

hold.

Thus, each solution of equation (1) with separated variables is an eigenfunction of some second-order symmetry operator admitted by equation (1).

Now, let us turn to partial differential equations related to equation (1). First, we consider the wave equation

$$\Box u + U(y_0^2 - y_1^2)u = 0. \tag{67}$$

It occurs [11] that equation (67) is reduced to the form (1) by the change of variables

$$t = \exp\left(\frac{1}{2}y_1\right)\operatorname{ch} y_0, \quad t = \exp\left(\frac{1}{2}y_1\right)\operatorname{sh} y_0$$

and, moreover, the potentials $V(\tau)$, $U(\tau)$ are connected by the relation

$$U(\tau) = \frac{1}{4\tau}V(\tau). \tag{68}$$

Consequently, to obtain all potentials $U(y_0^2 - y_1^2)$ such that equation (67) admits a nontrivial SV, one has to substitute potentials $V(x)$ listed in Theorem 2 into formula (68). The solution with separated variables has the form (7), where

$$y_1 + y_0 = \exp\{P((\omega_1 + \omega_2)/2)\}, \quad y_1 - y_0 = \exp\{R((\omega_1 - \omega_2)/2)\}.$$

The explicit form of the functions P and R is given in Theorems 3–8.

Another related equation is the following equation of hyperbolic type

$$v_{x_0x_0} - v_{x_1x_1}c^2(x_1) = 0, \tag{69}$$

which is widely used in various areas of mathematical physics (see, e.g. [19] and references therein).

Equation (69) is reduced to the form (1) by the change of variables

$$u(t, x) = [c(x_1)]^{-1/2}v(x_0, x_1) \quad t = x_0, \quad x = \int [c(x_1)]^{-1}dx_1,$$

and, moreover,

$$V(x) = -c^{3/2}(x_1)(c^{1/2}(x_1))^{\cdot\cdot}\big|_{x=\int \frac{dx_1}{c(x_1)}}. \tag{70}$$

Thus, to describe all functions $c(x_1)$ that provide separability of equation (69), it suffices to integrate the ordinary differential equation (70). Let us show how to reduce the nonlinear equation (70) to a linear one.

On making in (70) the change of the variable

$$c(x_1) = (\dot{y}(x_1))^{-1},$$

we get

$$\dddot{y} = \frac{3}{2}\ddot{y}(\dot{y})^{-1} + 2V(y)\dot{y}^3.$$

The above equation with the change of the variable $\dot{y} = z^2(y)$ is reduced to the form

$$z_{yy} - V(y)z = 0. \tag{71}$$

So, the general solution of the nonlinear equation (70) is given by the formula

$$c(x_1) = z^{-2}(y(x_1)), \tag{72}$$

where $z(y)$ is a general solution of the linear differential equation (71) and the function $y(x_1)$ is determined by the quadrature

$$\int^{y(x_1)} z^{-2}(\tau)d\tau = x_1 + C, \quad C \in \mathbb{R}^1. \tag{73}$$

Consequently, the problem of description of all functions $c(x)$ such that equation (69) admits a nontrivial SV is reduced to the integration of the linear ordinary differential equation (71), where V is given by (6). Solutions with separated variables have the form

$$\nu = \sqrt{c(x_1)}\varphi_1(\omega_1(x_0, x_1))\varphi_2(\omega_2(x_0, x_1)),$$

where the functions ω_i, are determined by the equalities

$$\frac{1}{2}\left(x_0 + \int \frac{dx_1}{c(x_1)}\right) = P\left(\frac{\omega_1+\omega_2}{2}\right), \quad \frac{1}{2}\left(x_0 - \int \frac{dx_1}{c(x_1)}\right) = R\left(\frac{\omega_1-\omega_2}{2}\right),$$

and the explicit form of P and R is given in Theorems 3–8.

Let us also note that, by using the corollary of Theorem 9 and formulas (71)–(73), it is not difficult to obtain the results of Bluman and Kumei [19]. In that paper, they have pointed out all the functions $c(x_1)$ that provide the extension of the symmetry group admitted by equation (69).

The third related equation is the nonlinear wave equation

$$U_{tt} - [c^{-2}(U)U_x]_x = 0. \tag{74}$$

By substitution $U = V_x$, equation (74) is reduced to the form

$$V_{tt} - c^{-2}(V_x)V_{xx} = 0.$$

Applying the Legendre transformation

$$x_0 = V_t, \quad x_1 = V_x, \quad v_{x_0} = t, \quad v_{x_1} = x, \quad v + V = tV_t + xV_x,$$

we get equation (69). Consequently, the method of SV in the linear equation (1) makes it possible to construct exact solutions of the nonlinear wave equation (74).

In conclusion, we suggest a possible generalization of the definition of SV in order to take into consideration nonlinear partial differential equations,

$$U\left(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{N}{u}\right) = 0, \tag{75}$$

where $x = (x_0, x_1, \dots, x_{n-1})$ and the symbol $\underset{k}{u}$ denotes the collection of k-th order derivatives of the function $u(x)$.

When speaking about a solution of equation (75) with separated variables $\omega_i = \omega_i(x, u)$, $i = \overline{1, n}$, we mean the ansatz

$$F(x, u, \varphi_1(\omega_1), \dots, \varphi_n(\omega_n)) = 0, \tag{76}$$

which reduces equation (75) to n ordinary differential equations

$$\varphi_i^{(N)} = f_i\bigl(\omega_i, \varphi_i, \dot{\varphi}_i, \dots, \varphi_i^{(N-1)}, \vec{\lambda}\bigr). \tag{77}$$

In the above formulas, $\omega_i \in C^N(\mathbb{R}^{n+1}, \mathbb{R}^1)$, f_i are some sufficiently smooth functions, and $\vec{\lambda} = (\lambda_1, \dots, \lambda_{n-1})$ are real parameters.

We say that equation (75) admits SV in the coordinates $\omega_i(x, u)$, $i = \overline{1, n}$, if the substitution of ansatz (76) into (75) with subsequent elimination of the N-th order derivatives $\varphi_i^{(N)}$, $i = \overline{1, n}$, yields an identity with respect to the variables φ_i, $\dot{\varphi}_i$, $\dots$, $\varphi_i^{(N-1)}$, $i = \overline{1, n}$, $\vec{\lambda}$ (considered as independent ones).

An application of the above approach to SV in nonlinear equations will be the topic of our future publications.

Here, we present without derivation some results on separation of variables in a two-dimensional nonlinear wave equation obtained with the use of the above described approach.

We have succeeded in separating variables in the following PDE:

1) $\Box_2 u = \lambda_1(\operatorname{ch} u + (\operatorname{sh} 2u) \operatorname{arctg} e^u) + \lambda_2 \operatorname{sh} 2u$;
2) $\Box_2 u = \lambda_1 e^u + \lambda_2 e^{-2u}$;
3) $\Box_2 u = \lambda_1(\operatorname{sh} u - (\operatorname{sh} 2u) \operatorname{arctg} e^u) + \lambda_2 \operatorname{sh} 2u$;
4) $\Box_2 u = \lambda_1 \left(2 \sin u + (\sin 2u) \ln \operatorname{tg} \dfrac{u}{2}\right) + \lambda_2 \sin 2u$;
5) $\Box_2 u = \lambda_1 u + \lambda_2 u \ln u$,

where λ_1 and λ_2 are arbitrary constants.

Below, we adduce ansatzes for $u(x)$ which provide a separation of equations 1–5 and corresponding reduced ordinary differential equations.

1) $u(x) = \ln \operatorname{tg}(\varphi_1(x_0) + \varphi_2(x_1))$,
$\dot{\varphi}_1^2 = C \cos 4\varphi_1 + A\varphi_1 + B_1, \quad \dot{\varphi}_2^2 = C \cos 4\varphi_2 - A\varphi_2 + B_2$,

where C, A, B_1 and B_2 are arbitrary constants satisfying the relations $A = \lambda_1/2$, $B_1 - B_2 = \lambda_2/2$;

2) $u(x) = \ln(\varphi_1(x_0) + \varphi_2(x_1))$,
$\dot{\varphi}_1^2 = 2A\varphi_1^3 + B\varphi_1^2 + C\varphi_1 + D_1, \quad \dot{\varphi}_2^2 = -2A\varphi_2^3 + B\varphi_2^2 - C\varphi_2 + D_2$,

where A, B, C, D_1 and D_2 are arbitrary constants satisfying relations $A = \lambda_1$, $D_2 - D_1 = \lambda_2/2$;

3) $u(x) = \ln \operatorname{th}(\varphi_1(x_0) + \varphi_2(x_1))$,
$\dot{\varphi}_1^2 = C \operatorname{ch} 4\varphi_1 + A\varphi_1 + B_1, \quad \dot{\varphi}_2^2 = C \operatorname{ch} 4\varphi_2 - A\varphi_2 + B_2$,

where C, A, B_1 and B_2 are arbitrary constants satisfying the relations $A = \lambda_1/2$, $B_1 - B_2 = \lambda_2/2$;

$$4)\quad u(x) = 2\,\mathrm{arctg}\exp(\varphi_1(x_0) + \varphi_2(x_1)),$$
$$\dot\varphi_1^2 = C\,\mathrm{sh}\,2\varphi_1 + 2A\varphi_1 + 2B_1, \quad \dot\varphi_2^2 = C\,\mathrm{sh}\,2\varphi_2 - 2A\varphi_2 + 2B_2,$$

where C, A, B_1, and B_2 are arbitrary constants satisfying the relations $A = \lambda_1$, $B_1 - B_2 = \lambda_2$;

$$5)\quad u(x) = \exp(\varphi_1(x_0) + \varphi_2(x_1)),$$
$$\dot\varphi_1^2 = C_1 e^{-2\varphi_1} + A\varphi_1 + B_1, \quad \dot\varphi_2^2 = C_2 e^{-2\varphi_2} - A\varphi_2 + B_2,$$

where C_1, C_2, A, B_1, and B_2 are arbitrary constants satisfying the relations $A = \lambda_1$, $B_1 - B_2 = \lambda_2 - \lambda_1$.

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