On the new approach to variable separation in the two-dimensional Schrödinger equation

R.Z. ZHDANOV, I.V. REVENKO, W.I. FUSHCHYCH

Для двовимірного рівняння Шредінгера з потенціалом, який не залежить від часової змінної, повністю розв'язано задачу класифікації потенціалів, при яких воно допускає розділення змінних. Для кожного з потенціалів описано всі системи координат, в яких розділюється відповідне рівняння Шредінгера.

There is a lot of papers devoted to separation of variables (SV) in the two-dimensional Schrödinger equation

$$iu_1 + u_{x_1x_2} + u_{x_2x_2} = V(x_1, x_2)u \tag{1}$$

with some specific $V(x_1, x_2)$ (see, e.g. [1–3] and references therein). Saying about the problem of SV in the Eq. (1), we imply two mutually connected problems. The first one is to describe all functions $V(x_1, x_2)$ such that the equation (1) admits separation of variables (classification problem). The second problem is to construct for each function $V(x_1, x_2)$ all coordinate systems making it possible to separate corresponding Schrödinger equation.

As far as we know, the first problem has been solved provided $V = 0$ [3] and $V = \alpha x_1^{-2} + \beta x_2^{-2}$ [1] and the second one has not been considered in the literature at all. We guess that a possible reason for this was absence of an adequate mathematical technique to handle the classification problem. In the paper [4] we suggested a new approach to SV in partial differential equations which enabled us to solve the problem of SV in two-dimensional wave equation with time independent potential [4]. In the present paper we give the complete solution of the problem of SV in the Schrödinger equation (1) obtained within the framework of the above said approach.

Solution with separated variables is looked for in the form of the ansatz [4]

$$u = Q(t, \vec{x})\varphi_0(t)\varphi_1(\omega_1(t, \vec{x}))\varphi_2(\omega_2(t, \vec{x})), \tag{2}$$

where $\varphi_0(t)$, $\varphi_1(\omega_1(t, \vec{x}))$, $\varphi_2(\omega_2(t, \vec{x}))$ are smooth functions satisfying ordinary differential equations (ODE)

$$\begin{aligned} &\frac{d\varphi_0}{dt} = U_0(t, \varphi_0, \lambda_1, \lambda_2), \\ &\frac{d^2\varphi_a}{d\omega_a^2} = U_a\left(\omega_a, \varphi_a, \frac{d\varphi_a}{d\omega_a}; \lambda_1, \lambda_2\right), \quad a = \overline{1, 2}, \end{aligned} \tag{3}$$

and Q, ω_1, ω_2 are functions to be determined from the requirement that ansatz (2) reduces Eq. (1) to ODE, $\lambda_1, \lambda_2 \subset \mathbb{R}^1$ are arbitrary parameters (separation constants). It is important to emphasize that functions Q, ω_1, ω_2 do not depend on the parameters λ_1, λ_2.

Доповіді НАН України, 1994, № 11, P. 38–44.

Because of the lack of space we have no possibility to adduce all necessary computations. That is why, we shall restrict ourselves by pointing out main steps of realization of the approach to SV suggested in [4].

First of all, we note that the substitution

$$\omega_1 \to \omega_1' = \Omega_1(\omega_1), \quad \omega_2 \to \omega_2' = \Omega_2(\omega_2), \quad Q \to Q' = Q\psi_1(\omega_1)\psi_2(\omega_2) \tag{4}$$

does not alter the structure of relations (2), (3). That is why, we introduce the following equivalence relation: (ω_1, ω_2, Q) is equivalent to $(\omega_1', \omega_2', Q')$ provided (4) holds with some Ω_a, ψ_a.

Substituting (2) into (1) with account of equalities (3) and splitting obtained relation with respect to independent variables φ_0, φ_a, φ_{aa}, λ_a, $a = 1, 2$ we conclude that up to the equivalence relation (4) equations (3) take the form

$$\frac{d\varphi_0}{dt} = (\lambda_1 R_1(t) + \lambda_2 R_2(t) + R_0(t))\varphi_0,$$
$$\frac{d^2\varphi_a}{d\omega_a^2} = (\lambda_1 B_{1a}(\omega_a) + \lambda_2 B_{2a}(\omega_a) + B_{0a}(\omega_a))\varphi_a$$

and what is more, functions ω_1, ω_2, Q satisfy the over-determined system of nonlinear partial differential equations

$$\begin{aligned}
&1)\quad \sum_{b=1}^{2} \omega_{1x_b}\omega_{2x_b} = 0,\\
&2)\quad \sum_{b=1}^{2} [B_{a1}(\omega_1)\omega_{1x_b}\omega_{2x_b} + B_{a2}(\omega_2)\omega_{2x_b}\omega_{2x_b}] + R_a(t) = 0, \quad a = 1, 2,\\
&3)\quad 2\sum_{b=1}^{2} Q_{x_b}\omega_{ax_b} + Q\left(i\omega_{at} + \sum_{b=1}^{2}\omega_{ax_bx_b}\right) = 0, \quad a = 1, 2,\\
&4)\quad \sum_{b=1}^{2} [B_{01}(\omega_1)\omega_{1x_b}\omega_{2x_b} + B_{02}(\omega_2)\omega_{2x_b}\omega_{2x_b}]Q + iQ_i +\\
&\qquad + \sum_{b=1}^{2} Q_{x_bx_b} + R_0(t)Q - V(\vec{x})Q = 0.
\end{aligned} \tag{5}$$

Thus, to solve the problem of SV for the linear Schrödinger equation it is necessary to construct the general solution of the system of nonlinear equations (5). Roughly speaking, to solve a linear equation one has to solve a system of nonlinear equations! This is the reason why so far there is no complete description of all coordinate systems providing separability of the four-dimensional d'Alembert equation.

But in the case involved we have succeeded in integration of nonlinear system (5) for ω_1, ω_2, Q. First, we have established that the general solution of equations 1–3 from (5) determined up to the equivalence relation (4) splits into four inequivalent classes

$$\begin{aligned}
&1)\quad \omega_1 = A(t)x_1 + W_1(t), \quad \omega_2 = B(t)x_2 + W_2(t),\\
&\qquad Q(t, \vec{x}) = \exp\left[-\frac{i}{4}\left(\frac{\dot{A}}{A}x_1^2 + \frac{\dot{B}}{B}x_2^2\right) - \frac{i}{2}\left(\frac{\dot{W}_1}{A}x_1 + \frac{\dot{W}_2}{B}x_2\right)\right],
\end{aligned}$$

$$
\begin{aligned}
&2)\quad \omega_1=\frac{1}{2}\ln(x_1^2+x_2^2)+W(t),\quad \omega_2=\operatorname{arctg}\frac{x_1}{x_2},\\
&\qquad Q(t,\vec{x})=\exp\left[-\frac{I\dot{W}}{4}(x_1^2+x_2^2)\right],\\
&3)\quad x_1=\frac{1}{2}W(t)(\omega_1^2-\omega_2^2)+W_1(t),\quad x_2=W(t)\omega_1\omega_2+W_2(t),\\
&\qquad Q(t,\vec{x})=\exp\left[\frac{i}{4}\frac{\dot{W}}{W}[(x_1-W_1)^2+(x_2-W_2)^2]+\frac{i}{2}(\dot{W}_1x_1+\dot{W}_2x_2)\right],\\
&4)\quad x_1=W(t)\operatorname{ch}\omega_1\cos\omega_2+W_1(t),\quad x_2=W(t)\operatorname{sh}\omega_1\sin\omega_2+W_2(t),\\
&\qquad Q(t,\vec{x})=\exp\left[\frac{i}{4}\frac{\dot{W}}{W}[(x_1-W_1)^2+(x_2-W_2)^2]+\frac{i}{2}(\dot{W}_1x_1+\dot{W}_2x_2)\right].
\end{aligned}
\tag{6}
$$

Here A, B, W, W_1, W_2 are arbitrary smooth functions on t. Dot means differentiation with respect to t.

Substituting obtained expressions for Q, ω_1, ω_2 into the last equation from the system (5) and splitting with respect to the variables x_1, x_2 we get explicit forms of potentials $V(x_1,x_2)$ and systems of nonlinear ODE for functions $A(t)$, $B(t)$, $W(t)$, $W_1(t)$, $W_2(t)$. We have suceeded in integrating these and in constructing all coordinate systems providing SV in the initial equation (1). Complete list of these systems takes two dozens of pages, so we are to restrict ourselves to adducing explicit forms of potentials $V(x_1,x_2)$ such that the Schrödinger equation (1) admits SV.

$$
\begin{aligned}
&1)\quad V(\vec{x})=V_1(x_1^2+x_2^2)+V_2\left(\frac{x_1}{x_2}\right)(x_1^2+x_2^2)^{-1};\\
&2)\quad V(\vec{x})=V_2\left(\frac{x_1}{x_2}\right)(x_1^2+x_2^2)^{-1};\\
&3)\quad V(\vec{x})=[V_1(\omega_1)+V_2(\omega_2)](\omega_1^2+\omega_2^2)^{-1},\\
&\qquad \text{where } x_1=\frac{1}{2}(\omega_1^2-\omega_2^2),\quad x_2=\omega_1\omega_2;\\
&4)\quad V(\vec{x})=[V_1(\omega_1)+V_2(\omega_2)](\operatorname{sh}^2\omega_1+\sin^2\omega_2)^{-1},\\
&\qquad \text{where } x_1=\operatorname{ch}\omega_1\cos\omega_2,\quad x_2=\operatorname{sh}\omega_1\sin\omega_2;\\
&5)\quad V(\vec{x})=V_1(x_1)+V_2(x_2);\\
&6)\quad V(\vec{x})=kx_1^2+V_2(x_2);\\
&7)\quad V(\vec{x})=k_1x_1^2+k_2x_1^{-2}+V_2(x_2),\quad k_2\neq 0;\\
&8)\quad V(\vec{x})=kx_1^2,\quad k\neq 0;\\
&9)\quad V(\vec{x})=k_1x_1^2+k_2x_2^2,\quad k_1k_2\neq 0;\\
&10)\quad V(\vec{x})=k_1x_1^2+k_2x_1^{-2},\quad k_1k_2\neq 0;\\
&11)\quad V(\vec{x})=k_1x_1^2+k_2x_2^2+k_3x_2^{-2},\quad k_1k_3\neq 0;\\
&12)\quad V(\vec{x})=k_1x_1^2+k_2x_2^2+k_3x_1^{-2}+k_4x_2^{-2},\quad k_3k_4\neq 0,\quad k_1^2+k_2^2\neq 0;\\
&13)\quad V(\vec{x})=k_1x_1^{-2}+k_2x_2^{-2};\\
&14)\quad V(\vec{x})=0.
\end{aligned}
\tag{7}
$$

In the above formulae V_1, V_2 are arbitrary smooth functions, k, k_1, k_2, k_3, k_4 are arbitrary real constants.

Note 1. The Schrödinger equation with the potential

$$V(\vec{x}) = k(x_1^2 + x_2^2) + V_1\left(\frac{x_1}{x_2}\right)(x_1^2 + x_2^2)^{-1}, \quad k = \text{const}, \tag{8}$$

is reduced to the Schrödinger equation with the potential

$$V'(\vec{x}') = V_1'\left(\frac{x_1'}{x_2'}\right)(x_1'^2 + x'^2)^{-1} \tag{9}$$

by the change of variables

$$t' = \alpha(t), \quad \vec{x}' = \beta(t)\vec{x}, \quad u' = \exp(i\gamma(t)\vec{x}^2 + i\delta(t)).$$

(explicit form of the functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$ depends on the sign of the parameter k in (8)). Since the above change of variables does not alter the structure of ansatz (2), when classifying potentials $V(x_1, x_2)$ providing separability of Eq. (1) we consider potentials (7), (8) as equivalent.

Note 2. It is well-known (see, e.g. [5, 6]) that the general form of the invariance group admitted by Eq. (1) is as follows:

$$t' = f(t, \vec{\theta}), \quad x_a' = g_a(t, \vec{x}, \vec{\theta}), \quad a = 1, 2, \quad u' = h(t, \vec{x}, \vec{\theta})u, \tag{10}$$

where $\vec{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ are group parameters.

Since transformations (10) do not alter the structure of the ansatz (2), systems of coordinates t', x_1', x_2' and t, x_1, x_2 are considered as equivalent.

Thus, there exist fourteen inequivalent types of the Schrödinger equations of the form (1) admitting SV. Consequently, the classification problem for Eq. (1) is solved.

Next, we shall obtain all coordinate systems providing separability of the Schrödinger equation having the potential $V = k_1 x_1^2 + k_2 x_2^2$ (the harmonic oscillator type equation). Explicit forms of the coordinate systems to be found depend essentially on the signs of the parameters k_1, k_2. Here we consider the case, when $k_1 < 0$, $k_2 > 0$ (the cases $k_1 > 0$, $k_2 > 0$ and $k_1 < 0$, $k_2 < 0$ will be considered in a separate publication). It means that Eq. (1) can be written in the form

$$iu_t + u_{x_1x_1} + u_{x_2x_2} + \frac{1}{4}(a^2x_1^2 - b^2x_2^2)u = 0, \tag{11}$$

where a, b, are arbitrary real constants (the factor 1/4 is introduced for further convenience).

We have proved above that to describe all coordinate systems t, ω_1, ω_2 providing separability of Eq. (11) one has to construct the general solution of system (5). The general solution of equations 1–3 from (5) splits into four inequivalent classes listed in (6).

Analysis shows that only solutions belonging to the first class can satisfy equation 4 from (5). Substituting corresponding formulae for ω_1, ω_2, Q into equation 4 from (5) with $V = \frac{1}{4}(a^2x_1^2 - b^2x_2^2)$ and splitting with respect to x_1, x_2 one gets

$$B_{01}(\omega_1) = \alpha_1\omega_1^2 + \alpha_2\omega_1, \quad B_{02}(\omega_2) = \beta_1\omega_2^2 + \beta_2\omega_2,$$

$$\left(\frac{\dot{A}}{A}\right)^{\cdot} - \left(\frac{\dot{A}}{A}\right)^2 - 4\alpha_1A^4 - a^2 = 0, \tag{12a}$$

$$\left(\frac{\dot B}{B}\right)^{\cdot} - \left(\frac{\dot B}{B}\right)^2 - 4\beta_1 B^4 + b^2 = 0, \tag{12b}$$

$$\ddot\theta_1 - 2\frac{\dot A}{A}\dot\theta_1 - 2(2\alpha_1\theta_1 + \alpha_2)A^4 = 0, \tag{12c}$$

$$\ddot\theta_2 - 2\frac{\dot A}{A}\dot\theta_2 - 2(2\beta_1\theta_2 + \beta_2)B^4 = 0, \tag{12d}$$

Here α_1, α_2, β_1, β_2 are arbitrary real constants.

Integration of the system of nonlinear ODE (12a–d) is carried out in the Appendix. Substitution of the formulae (A.4)–(A.9) into expressions 1 from (5) yields the complete list of coordinate systems providing separability of the Schrödinger equation (11). These systems can be reduced to the canonical form if we use the Note 2. The invariance group of Eq. (11) is generated by the following basis operators [6]:

$$\begin{aligned}
&P_0 = \partial_t, \quad I = u\partial_u, \quad M = iu\partial_u, \quad P_1 = \operatorname{ch} at\partial_{x_1} + \frac{ia}{2}(x_1 \operatorname{sh} at)u\partial_u,\\
&P_2 = \cos bt\partial_{x_2} - \frac{ib}{2}(x_2 \sin bt)u\partial_u, \quad G_1 = \operatorname{sh} at\partial_{x_1} + \frac{ia}{2}(x_1 \operatorname{ch} at)u\partial_u,\\
&G_2 = \sin bt\partial_{x_1} + \frac{ib}{2}(x_2 \cos bt)u\partial_u.
\end{aligned} \tag{13}$$

Using the finite transformations generated by the infinitesimal operators (13) and the Note 2 we may choose in the formulae (A.4)–(A.9) $C_3 = C_4 = D_1 = 0$, $C_2 = D_2 = 1$, $D_3 = D_4 = 0$. As a result we come to the following assertion.

Theorem. *The Schrödinger equation (11) admits SV in 21 inequivalent coordinate systems of the form*

$$\omega_0 = t, \quad \omega_1 = \omega_1(t,\vec x), \quad \omega_2 = \omega_2(t,\vec x), \tag{14}$$

where ω_1 is given by one of the following formulae:

$$\begin{aligned}
&x_1(\operatorname{sh} a(t+C))^{-1} + a(\operatorname{sh} a(t+C))^{-2}, \quad x_1(\operatorname{ch} a(t+C))^{-1} + a(\operatorname{ch} a(t+C))^{-2},\\
&x_1 \exp(\pm a(t+C)) + a\exp(\pm 4a(t+C)), \quad x_1(a + \operatorname{sh} 2a(t+C))^{-1/2},\\
&x_1(a + \operatorname{ch} 2a(t+C))^{-1/2}, \quad x_1(a + \exp(\pm 2a(t+C)))^{-1/2}, \quad x_1
\end{aligned} \tag{15}$$

and ω_2 is given by one of the following formulae:

$$x_2(\sin bt)^{-1} + \beta(\sin bt)^{-2}, \quad x_2(\beta + \sin 2bt)^{-1/2}, \quad x_2. \tag{16}$$

In the above formulae C, α, β are arbitrary real parameters.

It is important to note that explicit form of the coordinate systems providing separability of Eq. (11) depends essentially on the parameters a, b contained in the potential $V(x_1,x_2)$. It means that in the free case ($V = 0$) the Schrödinger equation does not admit SV in such coordinate systems. Consequently, they are essentially new.

Appendix. Integration of nonlinear ODE (12a–d).

Evidently, equations (12a–d) can be rewritten in the following unified form:

$$\left(\frac{\dot y}{y}\right)^{\cdot} - \left(\frac{\dot y}{y}\right)^2 - \alpha y^4 = k, \quad \ddot z - \frac{\dot y}{y}\dot z - (\alpha z + \beta)y^4 = 0. \tag{A.1}$$

Provided $k = -a^2 < 0$, system (A.1) coincides with equations (12a,c) and under $k = b^2 > 0$ — with equations (12b,d).

First of all, we note that the function $z = z(t)$ is determined up to addition of an arbitrary constant. Really, the coordinate functions ω_a has the following structure:

$$\omega_a = yx_a + z, \quad a = 1, 2.$$

But the coordinate system t, ω_1, ω_2 is equivalent to the coordinate system t, $\omega_1 + C_1$, $\omega_2 + C_2$, $C_a \in \mathbb{R}^1$. Hence, it follows that the function $z(t)$ is equivalent to the function $z(t) + C$ with arbitrary real constant C. Consequently, provided $\alpha \neq 0$, we may choose in (A.1) $\beta = 0$.

The case 1. $a = 0$. On making the change of variables in (A.1)

$$w = \frac{\dot y}{y}, \quad v = \frac{z}{y} \tag{A.2}$$

we get

$$\dot w = w^2 + k, \quad \ddot v + kv = \beta y^3. \tag{A.3}$$

First, we consider the case $k = -a^2 < 0$. Then the general solutions of the first equation from (A.3) is given by the formulae $w = -a \operatorname{cth} a(t + C_1)$, $w = -a \operatorname{th} a(t + C_1)$, $w = \pm a$, $C_1 \in \mathbb{R}^1$, whence

$$\begin{aligned} &y = C_2 \operatorname{sh}^{-1} a(t + C_1), \quad y = C_2 \operatorname{ch}^{-1} a(t + C_1), \\ &y = \exp[\pm a(t + C_1)], \quad C_2 \in \mathbb{R}^1. \end{aligned} \tag{A.4}$$

The second equation of system (A.3) is linear inhomogeneous ODE. Its general solution after being substituted into (A.2) yields:

$$\begin{aligned} &z = (C_3 \operatorname{ch} at + C_4 \operatorname{sh} at) \operatorname{sh}^{-1} a(t + C_1) + \frac{\beta C_2^4}{2a^2} \operatorname{sh}^{-2} a(t + C_1), \\ &z = (C_2 \operatorname{ch} at + C_4 \operatorname{sh} at) \operatorname{ch}^{-1} a(t + C_1) + \frac{\beta C_2^4}{2a^2} \operatorname{ch}^{-2} a(t + C_1), \\ &z = (C_3 \operatorname{ch} at + C_4 \operatorname{sh} at) \exp[\pm a(t + C_1)] + \frac{\beta}{8a^2} \exp[\pm 4a(t + C_1)], \\ &C_3, C_4 \in \mathbb{R}^1. \end{aligned} \tag{A.5}$$

The case $k = b^2 > 0$ is treated in the analogous way, the general solution of (A.3) being given by the formulae

$$\begin{aligned} &y = D_2 \sin^{-1} b(t + D_1), \\ &z = (C_3 \cos bt + C_4 \sin bt) \sin^{-1} b(t + D_1) + \frac{\beta D_2^4}{2b^2} \sin^{-2} b(t + D_1), \end{aligned} \tag{A.6}$$

The case 2. $\alpha \neq 0$, $\beta = 0$. On making the change of variables in (A.1)

$$y = \exp w, \quad v = \frac{z}{y}$$

we get

$$\ddot{w} - \dot{w}^2 = k + \alpha \exp 4w, \quad \ddot{v} + kv = 0. \tag{A.1a}$$

The first ODE from (A.1a) is reduced to the first-order linear ODE

$$\frac{1}{2}p'(w) - p(w) = k + \alpha \exp 4w$$

to by the substitution $\dot{w} = (p(w))^{1/2}$, whence

$$p(w) = \alpha \exp 4w + \gamma \exp 2w - k, \quad \gamma \in \mathbb{R}^1.$$

Equation $\dot{w} = p(w)$ has a singular solution $w = C = \mathrm{const}$ such that $p(C) = 0$. If $\dot{w} \neq 0$ then integrating equation $\dot{w} = p(w)$ and returning to the initial variable y, we get

$$\int^{y(t)} \frac{d\tau}{\tau(\alpha\tau^4 + \gamma\tau^2 - k)^{1/2}} = t + C_1.$$

Taking the integral in the left-hand side of the above equality we obtain the general solution of the first ODE from (A.1). It is given by the following formulae:

under $k = -a^2 < 0$

$$\begin{aligned} &y = C_2(\alpha + \operatorname{sh} 2a(t + C_1))^{-1/2}, \quad y = C_2(\alpha + \operatorname{ch} 2a(t + C_1))^{-1/2}, \\ &y = C_2(\alpha + \exp[\pm 2a(t + C_1)])^{-1/2}, \end{aligned} \tag{A.7}$$

under $k = b^2 > 0$

$$y = D_2(\alpha + \sin 2b(t + D_1))^{-1/2}. \tag{A.8}$$

Here C_1, C_2, D_1, D_2 are arbitrary real constants.

Integrating the second ODE from (A.1a) and returning to the initial variable z we have

under $k = -a^2 < 0$

$$z = y(t)(C_3 \operatorname{sh} at + C_4 \operatorname{ch} at) \tag{A.9}$$

under $k = b^2 > 0$

$$z = y(t)(D_3 \cos bt + D_4 \sin bt)$$

where C_3, C_4, D_3, D_4 are arbitrary real constants.

Thus, we have constructed the general solution of the system of nonlinear ODE (A.1) which is given by formulae (A.5)–(A.9).

1. Boyer C., Lie theory and separation of variables for the equation $iU_t + U_{x_1x_1} + U_{x_2x_2} - (\alpha x_1^{-2} + \beta x_2^{-2})U = 0$, *SIAM J. Math. Anal.*, 1976, **7**, № 3, 230–263.
2. Miller W., Symmetry and separation of variables, Massachusetts, Addison-Wesley, 1977, 285 p.
3. Boyer C., Kalnins E., Miller W., Lie the theory and separation of variables, 6: The equation $iU_t + U_{x_1x_1} + U_{x_2x_2} = 0$, *J. Math. Phys.*, 1975, **16**, № 3, 499–511.
4. Fushchych W.I., Zhdanov R.Z., Revenko I.V., On the new approach to the variable separation in the wave equation with potential, *Докл. АН Украины*, 1993, № 1, 27–32.
5. Niederer U., The maximal kinematical group of the free Schödinger equation, *Helv. Phys. Acta*, 1972, **45**, № 5, 802–810.
6. Boyer C., The maximal kinematical invariance group for an arbitrary potential, *Helv. Phys. Acta*, 1974, **47**, № 5, 589–605.