Nonlinear representations for Poincaré and Galilei algebras and nonlinear equations for electromagnetic fields

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We construct nonlinear representations of the Poincaré, Galilei, and conformal algebras on a set of the vector-functions $\Psi = (\vec{E}, \vec{H})$. A nonlinear complex equation of Euler type for the electromagnetic field is proposed. The invariance algebra of this equation is found.

1. Introduction

It is well known that the linear representations of the Poincaré algebra $AP(1,3)$ and conformal algebra $AC(1,3)$, with the basis elements

$$P_\mu = ig^{\mu\nu}\partial_\nu, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + S_{\mu\nu}, \tag{1}$$

$$D = x_\nu P^\nu - 2i, \tag{2}$$

$$K_\mu = 2x_\mu D - (x_\nu x^\nu)P_\mu + 2x^\nu S_{\mu\nu}, \tag{3}$$

is realized on the set of solutions of the Maxwell equations for the electromagnetic field in vacuum (see e.g. [1, 2])

$$\frac{\partial \vec{E}}{\partial t} = \operatorname{rot} \vec{H}, \quad \frac{\partial \vec{H}}{\partial t} = -\operatorname{rot} \vec{E}, \tag{4}$$

$$\operatorname{div} \vec{E} = 0, \quad \operatorname{div} \vec{H} = 0. \tag{5}$$

Here $S_{\mu\nu}$ realize the representation $D(0,1) \oplus D(1,0)$ of the Lorentz group.

Operators (1)–(3) satisfy the following commutation relations:

$$[P_\mu, P_\nu] = 0, \quad [P_\mu, J_{\alpha\beta}] = i(g_{\mu\alpha}P_\beta - g_{\mu\beta}P_\alpha), \tag{6}$$

$$[J_{\alpha\beta}, J_{\mu\nu}] = i(g_{\beta\mu}J_{\alpha\nu} + g_{\alpha\nu}J_{\beta\mu} - g_{\alpha\mu}J_{\beta\nu} - g_{\beta\nu}J_{\alpha\mu}), \tag{7}$$

$$[D, P_\mu] = -iP_\mu, \quad [D, J_{\mu\nu}] = 0, \tag{8}$$

$$[K_\mu, P_\alpha] = i(2J_{\alpha\mu} - 2g_{\mu\alpha}D), \quad [K_\mu, J_{\alpha\beta}] = i(g_{\mu\nu}K_\beta - g_{\mu\beta}K_\alpha), \tag{9}$$

$$[K_\mu, D] = -iK_\mu, \quad [K_\mu, K_\nu] = 0, \quad \mu,\nu,\alpha,\beta = 0,1,2,3. \tag{10}$$

In this paper the nonlinear representations of the Poincaré, Galilei, and conformal algebras for the electromagnetic field $\vec{E}$, $\vec{H}$ are constructed. In particular, we prove that the continuity equation for the electromagnetic field is not invariant under the Lorentz group if the velocity of the electromagnetic field is taken in accordance with

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the Poynting definition. Conditional symmetry of the continuity equation is studied. The complex Euler equation for the electromagnetic field is introduced. The symmetry of this equation is investigated.

2. Formulation of the main results

The operators, realizing the nonlinear representations of the Poincaré algebras $AP(1,3) = \langle P_\mu, J_{\mu\nu}\rangle$, $AP_1(1,3) = \langle P_\mu, J_{\mu\nu}, D\rangle$, and conformal algebra $AC(1,3) = \langle P_\mu, J_{\mu\nu}, D,\ K_\mu\rangle$, have the structure

$$P_\mu = \partial_{x_\mu}, \tag{11}$$

$$J_{kl} = x_k\partial_{x_l} - x_l\partial_{x_k} + S_{kl}, \tag{12}$$

$$J_{0k} = x_0\partial_{x_k} + x_k\partial_{x_0} + S_{0k}, \quad k,l = 1,2,3, \tag{13}$$

$$D = x_\mu\partial_{x_\mu}, \tag{14}$$

$$K_0 = x_0^2\partial_{x_0} + x_0x_k\partial_{x_k} + (x_k - x_0E^k)\partial_{E^k} - x_0H^k\partial_{H^k}, \tag{15}$$

$$\begin{aligned}K_l = {} & x_0x_l\partial_{x_0} + x_lx_k\partial_{x_k} + [x_kE^l - x_0(E^lE^k - H^lH^k)]\partial_{E^k} + \\ & + [x_kH^l - x_0(H^lE^k + E^lH^k)]\partial_{H^k},\end{aligned} \tag{16}$$

where

$$\begin{aligned}&S_{kl} = E^k\partial_{E^l} - E^l\partial_{E^k} + H^k\partial_{H^l} - H^l\partial_{H^k},\\ &S_{0k} = \partial_{E^k} - (E^kE^l - H^kH^l)\partial_{E^l} - (E^kH^l + H^kE^l)\partial_{H^l}.\end{aligned}$$

The operators, realizing the nonlinear representations of the Galilei algebras $AG^{(2)}(1,3) = \langle P_\mu, J_{kl}, G_k^{(2)}\rangle$, $AG_1^{(2)}(1,3) = \langle P_\mu, J_{kl}, G_k^{(2)}, D\rangle$ have the form:

$$P_\mu = \partial_{x_\mu}, \quad J_{kl} = x_k\partial_{x_l} - x_l\partial_{x_k} + S_{kl}, \tag{17}$$

$$G_k^2 = x_k\partial_{x_0} - (E^kE^l - H^kH^l)\partial_{E^l} - (E^kH^l + H^kE^l)\partial_{H^l}, \tag{18}$$

$$D = x_0\partial_{x_0} + 2x_k\partial_{x_k} + E^k\partial_{E^k} + H^k\partial_{H^k}. \tag{19}$$

We see by direct verification that all represented operators satisfy the commutation relations of the algebras $AP(1,3)$, $AC(1,3)$, $AG(1,3)$.

3. Construction of nonlinear representations

In order to construct the nonlinear representations of Euclid-, Poincaré-, and Galilei groups and their extensions the following idea was proposed in [2, 3]: to use nonlinear equations invariant under these groups; it is necessary to find (point out, guess) the equations, which admit symmetry operators having a nonlinear structure. Such equation for the scalar field $u(x_0, x_1, x_2, x_3)$ is the eikonal equation

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} = 0, \quad \mu = 0,1,2,3 \tag{20}$$

which is invariant under the conformal algebra $AC(1,3)$ with the nonlinear operator K_μ [2, 3].

The nonlinear Euler equation for an ideal fluid

$$\frac{\partial v_k}{\partial t} + v_l \frac{\partial v_k}{\partial x_l} = 0, \quad k = 1, 2, 3 \tag{21}$$

which is invariant under nonlinear representation of the $AP(1,3)$ algebra, with basis elements

$$P_\mu = \partial_{x_\mu}, \quad J_{kl} = x_k \partial_{x_l} - x_l \partial_{x_k} + v_k \partial_{v_l} - v_l \partial_{v_k}, \tag{22}$$

$$J_{0k} = x_k \partial_0 + x_0 \partial_{x_k} + \partial_{v_k} - v_k v_l \partial_{v_l}, \tag{23}$$

was proposed in [3] to construct the nonlinear representation for the vector field. Note that equation (21) is also invariant with respect to the Galilei algebra $AG(1,3)$ with the basis elements

$$P_\mu = \partial_{x_\mu}, \quad J_{kl} = x_k \partial_{x_l} - x_l \partial_{x_k} + v_k \partial_{v_l} - v_l \partial_{v_k}, \quad G_a = x_0 \partial_{x_a} + \partial_{v_a}. \tag{24}$$

As mentioned in [2, 3] both the Lorentz–Poincaré–Einstein and Galilean principles of relativity are valid for system (21). We use the following nonlinear system of equations [4]

$$\frac{\partial E^k}{\partial x_0} + H^l \frac{\partial E^k}{\partial x_l} = 0, \quad \frac{\partial H^k}{\partial x_0} + E^l \frac{\partial H^k}{\partial x_l} = 0, \tag{25}$$

for constructing a nonlinear representation of the $AP(1,3)$ and $AG(1,3)$ algebras for the electromagnetic field. To construct the basis elements of the $AP(1,3)$ and $AG(1,3)$ algebras in explicit form we investigate the symmetry of system (25). We search for the symmetry operators of equations (25) in the form:

$$X = \xi^\mu \partial_{x_\mu} + \eta^l \partial_{E^l} + \beta^l \partial_{H^l}, \tag{26}$$

where $\xi^\mu = \xi^\mu(x, \vec{E}, \vec{H})$, $\eta^l = \eta^l(x, \vec{E}, \vec{H})$, $\beta^l = \beta^l(x, \vec{E}, \vec{H})$.

Theorem 1. *The maximal invariance algebra of system (25) in the class of operators (26) is the 20-dimensional algebra, whose basis elements are given by the formulas*

$$P_\mu = \partial x_\mu, \tag{27}$$

$$J^{(1)}_{kl} = x_k \partial_{x_l} - x_l \partial_{x_k} + E^k \partial_{E^l} - E^l \partial_{E^k} + H^k \partial_{H^l} - H^l \partial_{H^k}, \tag{28}$$

$$J^{(2)}_{kl} = x_k \partial_{x_l} + x_l \partial_{x_k} + E^k \partial_{E^l} + E^l \partial_{E^k} + H^k \partial_{H^l} + H^l \partial_{H^k}, \tag{29}$$

$$G^{(1)}_a = x_0 \partial_{x_a} + \partial_{E^a} + \partial_{H^a}, \tag{30}$$

$$G^{(2)}_a = x_a \partial_{x_0} - E^a E^k \partial_{E^k} - H^a H^k \partial_{H^k}, \tag{31}$$

$$D_0 = x_0 \partial_{x_0} - E^l \partial_{E^l} - H^l \partial_{H^l}, \tag{32}$$

$$D_1 = x_1 \partial_{x_1} + E^1 \partial_{E^1} + H^1 \partial_{H^1}, \tag{33}$$

$$D_2 = x_2 \partial_{x_2} + E^2 \partial_{E^2} + H^2 \partial_{H^2}, \tag{34}$$

$$D_3 = x_3 \partial_{x_3} + E^3 \partial_{E^3} + H^3 \partial_{H^3}. \tag{35}$$

Proof. To prove theorem 1 we use Lie's algorithm. The condition of invariance of the system $L(\vec{E},\vec{H})$, i.e. (25), with respect to operator X has the form

$$\underset{1}{X} L\Big|_{L=0} = 0, \tag{36}$$

where

$$\underset{1}{X} = X + [D_\alpha(\eta^l) - E^l_j D_\alpha(\xi^j)]\partial_{E^l_\alpha} + [D_\alpha(\beta^l) - H^l_j D_\alpha(\xi^j)]\partial_{H^l_\alpha},$$
$$E^l_\alpha = \frac{\partial E^l}{\partial x_\alpha}, \quad H^l_\alpha = \frac{\partial H^l}{\partial x_\alpha}, \quad l = 1,2,3; \quad \alpha = 0,1,2,3$$

is the prolonged operator. From the invariance condition (36) we obtain the system of equations which determine the coefficient functions ξ^μ, η^l, β^l of the operator (26):

$$\begin{array}{l}\eta^l_k = 0, \quad \eta^l_0 = 0, \quad \beta^l_k = 0, \quad \beta^l_0 = 0, \quad \xi^\mu_{\alpha\nu} = 0, \quad \xi^\mu_{E^a} = 0, \quad \xi^\mu_{H^a} = 0,\\ \eta^k = -E^k\xi^0_0 + \xi^k_0 + E^a\xi^k_a - E^aE^k\xi^0_a,\\ \beta^k = -H^k\xi^0_0 + \xi^k_0 + H^a\xi^k_a - H^aH^k\xi^0_a,\end{array} \tag{37}$$

where

$$\eta^l_k = \frac{\partial \eta^l}{\partial x_k}, \quad \eta^l_0 = \frac{\partial\eta^l}{\partial x_0}, \quad \xi^\mu_{E^a} = \frac{\partial \xi^\mu}{\partial E^a}, \quad \xi^\mu_{\alpha\nu} = \frac{\partial^2\xi^\mu}{\partial x_\alpha \partial x_\nu}.$$

Having found the general solution of system (37), we get the explicit form of all the linear independent symmetry operators of system (25), which have the structure (27)–(35). Operators of Lorentz rotations J_{0k} is given by the linear combination of the Galilean operators $G^{(1)}_k$ and $G^{(2)}_k$:

$$J_{0k} = G^{(1)}_k + G^{(2)}_k. \tag{38}$$

All the following statements, given here without proofs, can be proved in analogy with the above-mentioned scheme.

4. The finite transformations and invariants

We present some finite transformations which are generated by the operators J_{0k}:

$$\begin{array}{ll} J_{01}: & x_0 \to x'_0 = x_0 \operatorname{ch}\theta_1 + x_1 \operatorname{sh}\theta_1,\\ & x_1 \to x'_1 = x_1 \operatorname{ch}\theta_1 + x_0 \operatorname{sh}\theta_1,\\ & x_2 \to x'_2 = x_2, \quad x_3 \to x'_3 = x_3,\end{array} \tag{39}$$

$$\begin{array}{ll} E^1 \to E^{1'} = \dfrac{E^1\operatorname{ch}\theta_1 + \operatorname{sh}\theta_1}{E^1\operatorname{sh}\theta_1 + \operatorname{ch}\theta_1}, & H^1 \to H^{1'} = \dfrac{H^1\operatorname{ch}\theta_1 + \operatorname{sh}\theta_1}{H^1\operatorname{sh}\theta_1 + \operatorname{ch}\theta_1},\\ E^2 \to E^{2'} = \dfrac{E^2}{E^1\operatorname{sh}\theta_1 + \operatorname{ch}\theta_1}, & H^2 \to H^{2'} = \dfrac{H^2}{H^1\operatorname{sh}\theta_1 + \operatorname{ch}\theta_1},\\ E^3 \to E^{3'} = \dfrac{E^3}{E^1\operatorname{sh}\theta_1 + \operatorname{ch}\theta_1}, & H^3 \to H^{3'} = \dfrac{H^3}{H^1\operatorname{sh}\theta_1 + \operatorname{ch}\theta_1}.\end{array} \tag{40}$$

The operators J_{02}, J_{03} generate analogous transformations. θ_1 is the real group parameter of the geometric Lorentz transformation. Operators $G_k^{(2)}$ generate the following transformations:

$$G_1^{(2)}: \quad x_0 \to x_0' = x_0 + \theta_1 x_1, \quad x_k \to x_k' = x_k,$$
$$E^k \to E^{k'} = \frac{E^k}{1+\theta_1 E^1}, \quad H^k \to H^{k'} = \frac{H^k}{1+\theta_1 H^1}.$$

Analogous transformations are generated by the operators $G_2^{(2)}$, $G_3^{(2)}$. Operators $G_k^{(1)}$ generate the following transformations:

$$\begin{aligned} G_1^{(1)}: \quad & x_0 \to x_0' = x_0, \quad x_1 \to x_1' = x_1 + x_0\theta_1, \\ & x_2 \to x_2' = x_2, \quad x_3 \to x_3' = x_3, \\ & E^1 \to E^{1'} = E^1 + \theta_1, \quad H^1 \to H^{1'} = H^1 + \theta_1, \\ & E^2 \to E^{2'} = E^2, \quad E^3 \to E^{3'} = E^3, \\ & H^2 \to H^{2'} = H^2, \quad H^3 \to H^{3'} = H^3. \end{aligned}$$

The operators $G_2^{(1)}$, $G_3^{(1)}$ generate analogous transformations.

It is easy to verify that

$$I_1 = \frac{\left(1 - \vec{E}\vec{H}\right)^2}{\left(1-\vec{E}^2\right)\left(1-\vec{H}^2\right)}, \quad \vec{E}^2 \neq 1, \quad \vec{H}^2 \neq 1 \tag{41}$$

is invariant with respect to the nonlinear transformations of the Poincaré group which are generated by representations (28), (38).

The invariant of the Galilei group which is generated by representations (28), (31) has the form:

$$I_2 = \frac{\vec{E}^2\vec{H}^2}{\left(\vec{E}\vec{H}\right)^2}, \tag{42}$$

whereas the Galilei group which is generated by representations (28), (30) has the invariant

$$I_3 = (\vec{E} - \vec{H})^2. \tag{43}$$

5. Complex Euler equation for the electromagnetic field

Let us consider the system of equations

$$\frac{\partial \Sigma^k}{\partial x_0} + \Sigma^l \frac{\partial \Sigma^k}{\partial x_l} = 0, \quad \Sigma^k = E^k + iH^k. \tag{44}$$

The complex system (44) is equivalent to the real system of equations for $\vec{E}$ and $\vec{H}$

$$\begin{aligned} &\frac{\partial E^k}{\partial x_0} + E^l\frac{\partial E^k}{\partial x_l} - H^l\frac{\partial H^k}{\partial x_l} = 0, \\ &\frac{\partial H^k}{\partial x_0} + H^l\frac{\partial E^k}{\partial x_l} + E^l\frac{\partial H^k}{\partial x_l} = 0. \end{aligned} \tag{45}$$

The following statement has been proved with the help of Lie's algorithm.

Theorem 2. *The maximal invariance algebra of the system (45) is the 24-dimensional Lie algebra whose basis elements are given by the formulas*

$$
\begin{aligned}
& P_\mu = \partial_{x_\mu}, \\
& J^{(1)}_{kl} = x_k\partial_{x_l} - x_l\partial_{x_k} + E^k\partial_{E^l} - E^l\partial_{E^k} + H^k\partial_{H^l} - H^l\partial_{H^k}, \\
& J^{(2)}_{kl} = x_k\partial_{x_l} + x_l\partial_{x_k} + E^k\partial_{E^l} + E^l\partial_{E^k} + H^k\partial_{H^l} + H^l\partial_{H^k}, \\
& G^{(1)}_a = x_0\partial_{x_a} + \partial_{E^a}, \\
& G^{(2)}_a = x_a\partial_{x_0} - (E^aE^k - H^aH^k)\partial_{E^a} - (E^aH^k + H^aE^k)\partial_{H^k}, \\
& D_0 = x_0\partial_{x_0} - E^k\partial_{E^k} - H^k\partial_{H^k}, \\
& D_a = x_a\partial_{x_a} + E^a\partial_{E^a} + H^a\partial_{H^a} \quad \text{(no sum over } a), \\
& K_0 = x_0^2\partial_{x_0} + x_0x_k\partial_{x_k} + (x_k - x_0E^k)\partial_{E^k} - x_0H^k\partial_{H^k}, \\
& K_a = x_0x_a\partial_{x_0} + x_ax_k\partial_{x_k} + [x_kE^a - x_0(E^aE^k - H^aH^k)]\partial_{E^k} + \\
& \qquad + [x_kH^a - x_0(H^aE^k + E^aH^k)]\partial_{H^k}.
\end{aligned}
\tag{46}
$$

The algebra, engendered by the operators (46), include the Galilei algebras $AG^{(1)}(1,3)$, $AG^{(2)}(1,3)$ and Poincaré algebra $AP(1,3)$, and conformal algebra $AC(1,3)$ as subalgebras. Operators $G^{(2)}_a$ generate the linear geometrical transformations in $\mathbb{R}(1,3)$

$$
x_0 \to x_0' = x_0 + \theta_a x_a \quad \text{(no sum over } a), \quad x_l \to x_l', \tag{47}
$$

as well as the nonlinear transformations of the fields

$$
\begin{aligned}
& E^l + iH^l \to E^{l'} + iH^{l'} = \frac{E^l + iH^l}{1 + \theta_a(E^a + iH^a)} \quad \text{(no sum over } a), \\
& E^l - iH^l \to E^{l'} - iH^{l'} = \frac{E^l - iH^l}{1 + \theta_a(E^a - iH^a)}.
\end{aligned}
\tag{48}
$$

The invariant of the group $G^{(2)}(1,3)$ is

$$
I_4 = \frac{\big(\vec{E}^2 - \vec{H}^2\big) + 4\big(\vec{E}\vec{H}\big)^2}{\big(\vec{E}^2 + \vec{H}^2\big)^2}. \tag{49}
$$

Operators J_{0k} generate the linear transformations in $\mathbb{R}(1,3)$

$$
\begin{aligned}
& x_0 \to x_0' = x_0 \operatorname{ch}\theta_k + x_0 \operatorname{sh}\theta_k, \\
& x_k \to x_k' = x_k \operatorname{ch}\theta_k + x_0 \operatorname{sh}\theta_k \quad \text{(no sum over } k), \\
& x_l \to x_l' = x_l, \quad \text{if } l \neq k,
\end{aligned}
\tag{50}
$$

as well as the nonlinear transformations of the fields

$$
\begin{aligned}
& E^k + iH^k \to E^{k'} + iH^{k'} = \frac{(E^k + iH^k)\operatorname{ch}\theta_k + \operatorname{sh}\theta_k}{(E^k + iH^k)\operatorname{sh}\theta_k + \operatorname{ch}\theta_k}, \\
& E^k - iH^k \to E^{k'} - iH^{k'} = \frac{(E^k - iH^k)\operatorname{ch}\theta_k + \operatorname{sh}\theta_k}{(E^k - iH^k)\operatorname{sh}\theta_k + \operatorname{ch}\theta_k}.
\end{aligned}
$$

If $l \neq k$, then

$$E^l + iH^l \to E^{l'} + iH^{l'} = \frac{E^l + iH^l}{(E^k + iH^k)\,\mathrm{sh}\,\theta_k + \mathrm{ch}\,\theta_k},$$
$$E^l - iH^l \to E^{l'} - iH^{l'} = \frac{E^l - iH^l}{(E^k - iH^k)\,\mathrm{sh}\,\theta_k + \mathrm{ch}\,\theta_k} \quad \text{(no sum over } k). \tag{51}$$

The invariant of group $P(1,3)$ is

$$I_5 = \frac{1 - 2\left[(\vec{E}^2 - \vec{H}^2) - \frac{1}{2}(\vec{E}^2 - \vec{H}^2)^2 - 2(\vec{E}\vec{H})^2\right]}{\left[1 - (\vec{E}^2 + \vec{H}^2)\right]^2}, \quad \vec{E}^2 + \vec{H}^2 \neq 1. \tag{52}$$

The operator K_0 generates the following nonlinear transformations in $\mathbb{R}(1,3)$ and linear transformations of the fields

$$x_\mu \to x'_\mu = \frac{x_\mu}{1 - \theta_0 x_0},$$
$$E^k \to E^{k'} = E^k + \theta_0(x_k - x_0 E^k),$$
$$H^k \to H^{k'} = H^k(1 - \theta_0 x_0). \tag{53}$$

The operators K_a generate nonlinear transformations in both $\mathbb{R}(1,3)$ and of the fields

$$x_0 \to x'_0 = \frac{x_0}{1 - x_a\theta_a}, \quad x_a \to x'_a = \frac{x_a}{1 - x_a\theta_a}.$$

If $k \neq a$, then

$$x_k \to x'_k = \frac{x_k}{1 - x_a\theta_a},$$
$$E^a + iH^a \to E^{a'} + iH^{a'} = \frac{E^a + iH^a}{1 + \theta_a[x_0(E^a + iH^a) - x_a]},$$
$$E^a - iH^a \to E^{a'} - iH^{a'} = \frac{E^a - iH^a}{1 + \theta_a[x_0(E^a - iH^a) - x_a]}.$$

If $k \neq a$, then

$$E^k + iH^k \to E^{k'} + iH^{k'} = \frac{E^k + iH^k + \theta_a(E^a + iH^a)x_k}{1 + \theta_a[x_0(E^a + iH^a) - x_a]},$$
$$E^k - iH^k \to E^{k'} - iH^{k'} = \frac{E^k - iH^k + \theta_a(E^a - iH^a)x_k}{1 + \theta_a[x_0(E^a - iH^a) - x_a]} \quad \text{(no sum over } a). \tag{54}$$

Note 1. Setting $\vec{\Sigma} = a\vec{E} + ib\vec{H}$, where a, b are arbitrary functions of the invariants $\vec{E}^2$, $\vec{H}^2$, $\vec{E}\vec{H}$, we obtain more general form of the equation (44). The equation

$$\frac{\partial \Sigma^k}{\partial x_0} + \Sigma^l \frac{\partial \Sigma^k}{\partial x_l} = F(\vec{E}\vec{H}, \vec{E}^2, \vec{H}^2)\Sigma^k$$

is invariant only under some subalgebras of algebra (46) depending on the choice of function F.

Note 2. If we analyse the symmetry of the following equations

$$\begin{aligned} &\left(\frac{\partial}{\partial x_0} + E^l\frac{\partial}{\partial x_l} + H^l\frac{\partial}{\partial x_l}\right)E^k = 0,\\ &\left(\frac{\partial}{\partial x_0} + E^l\frac{\partial}{\partial x_l} + H^l\frac{\partial}{\partial x_l}\right)H^k = 0; \end{aligned} \tag{$*$}$$

or

$$\begin{aligned} &\frac{\partial E^k}{\partial x_0} = \pm\left(E^l\frac{\partial}{\partial x_l} + H^l\frac{\partial}{\partial x_l}\right)H^k,\\ &\frac{\partial H^k}{\partial x_0} = \pm\left(E^l\frac{\partial}{\partial x_l} + H^l\frac{\partial}{\partial x_l}\right)E^k, \end{aligned} \tag{$**$}$$

we obtain concrete examples of nonlinear representations for the Poincaré and Galilei algebras. This problem will be considered in a future paper.

6. Symmetry of the continuity equation and the Poynting vector

Let us consider the continuity equation for the electromagnetic field

$$L(\vec{E},\vec{H}) \equiv \frac{\partial\rho}{\partial x_0} + \operatorname{div}\rho\vec{v} = 0. \tag{55}$$

According to the Poynting definition ρ and ρv^k have the forms

$$\rho = \frac{1}{2}(\vec{E}^2 + \vec{H}^2), \quad \rho v^k = \varepsilon_{kln}E^lH^n. \tag{56}$$

Theorem 3. *The nonlinear system (55), (56) is not invariant under the Lorentz algebra, with basis elements:*

$$\begin{aligned} &J_{kl} = x_k\partial_{x_l} - x_l\partial_{x_k} + E^k\partial_{E^l} - E^l\partial_{E^k} + H^k\partial_{H^l} - H^l\partial_{H^k},\\ &J_{0k} = x_k\partial_{x_0} + x_0\partial_{x_k} + \varepsilon_{kln}(E^l\partial_{H^n} - H^l\partial_{E^n}), \quad k,l,n = 1,2,3. \end{aligned} \tag{57}$$

To prove theorem 3 it is necessary to substitute ρ and ρv^k, from formulas (56), to equation (55) and to apply Lie's algorithm, i.e., it is necessary to verify that the invariance condition

$$\underset{1}{J}_{\mu\nu}\left(L(\vec{E},\vec{H})\right)\Big|_{L=0} \equiv 0 \tag{58}$$

is not satisfied, where $\underset{1}{J}_{\mu\nu}$ is the first prolongation of the operator $J_{\mu\nu}$.

Theorem 4. *The continuity equation (55), (56) is conditionally invariant with respect to the operators $J_{\mu\nu}$, given in (57) if and only if $\vec{E}$, $\vec{H}$ satisfy the Maxwell equation (4), (5).*

Thus the continuity equation, which is the mathematical expression of the conservation law of the electromagnetic field energy and impulse is not Lorentz-invariant if $\vec{E}$, $\vec{H}$ does not satisfy the Maxwell equation. A more detailed discussion on conditional symmetries can be found in [1, 2].

The following statement can be proved in the case when

$$\rho = F^0(\vec{E},\vec{H}) \quad \text{and} \quad \rho v^k = F^k(\vec{E},\vec{H}), \tag{59}$$

where F^0, F^k are arbitrary smooth functions $F^0 \not\equiv 0$, $F^k \not\equiv 0$.

Theorem 5. *The continuity equation (55), (59) is invariant with respect to the classic geometrical Lorentz transformations if and only if*

$$r(B) = 4, \tag{60}$$

where $r(B)$ is the rank of the Jacobi matrix of functions F^μ.

In conclusion we present some statements about the symmetry of the following systems:

$$\begin{gathered}\frac{\partial \vec{E}}{\partial x_0} = \mathrm{rot}\,\vec{H} + \vec{F}_1(\vec{E},\vec{H}), \quad \frac{\partial \vec{H}}{\partial x_0} = -\mathrm{rot}\,\vec{E} + \vec{F}_2(\vec{E},\vec{H}),\\ \mathrm{div}\,\vec{E} = R_1(\vec{E},\vec{H}), \quad \mathrm{div}\,\vec{H} = R_2(\vec{E},\vec{H}),\end{gathered} \tag{61}$$

$$\begin{gathered}\frac{\partial (R\vec{E})}{\partial x_0} = \mathrm{rot}\,(R\vec{H}), \quad \frac{\partial N\vec{H}}{\partial x_0} = -\mathrm{rot}\,(N\vec{E}),\\ \mathrm{div}\,(R\vec{E}) = 0, \quad \mathrm{div}\,(N\vec{H}) = 0.\end{gathered} \tag{62}$$

Here

$$R = R(W_1, W_2), \quad N = N(W_1, W_2), \quad W_1 = \vec{E}^2 - \vec{H}^2, \quad W_2 = \vec{E}\vec{H}.$$

$$\mathrm{div}\,(R\vec{E} + N\vec{H}) = 0. \tag{63}$$

Theorem 6. *The system of equations (61) is invariant under the Lorentz algebra with the basis elements (57) if and only if*

$$\vec{F}_1 \equiv \vec{F}_2 \equiv 0, \quad R_1 \equiv R_2 \equiv 0.$$

Theorem 7. *The system of equations (62) is invariant under the Lorentz algebra (57) if R and N are arbitrary functions of the invariants $W_1 = \vec{E}^2 - \vec{H}^2$, $W_2 = \vec{E}\vec{H}$.*

Theorem 8. *The equation (63) is invariant under the Lorentz algebra with the basis elements (57) if and only if $\vec{E}$, $\vec{H}$ satisfy the system of equations*

$$\frac{\partial (R\vec{E} + N\vec{H})}{\partial x_0} = \mathrm{rot}\,(R\vec{H} - N\vec{E}).$$

Thus it is established that, besides the generally recognized linear representation of the Lorentz group discovered by Henry Poincaré in 1905 [5], there exists the nonlinear representation constructed by using the nonlinear equations of hydrodynamical type [4]. It is obvious that for instance the linear superposition principle does not hold for a non-Maxwell electrodynamic theory based on the equation (25) or (45).

The nonlinear representations for the algebras $AG(1,3)$, $A\tilde{P}(1,2)$, $A\tilde{P}(2,2)$, $AC(1,2)$, $AC(2,2)$ for a scalar field have been considered in [6], $AP(1,1)$ in [7], and $AP(1,2)$ in [8].

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