Nonlocal ansatzes for nonlinear wave equation

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Запропоновано нелокальні анзаци, що редукують нелінійні хвильові рівняння до системи хвильових рівнянь з меншим числом незалежних змінних. Показано, що ці анзаци можна одержати, використовуючи оператори нелокальної симетрії рівняння.

1. In the present paper we suggest a nonlocal ansatz

$$\frac{\partial u}{\partial x_\mu} = a_{\mu\nu}(x,u)\varphi_\nu(\omega) + h_\mu(x,u), \quad \mu,\nu = 0,1,2,3, \tag{1}$$

for reduction of the second order nonlinear differential equation

$$g_{\mu\nu}(x,u)\frac{\partial^2 u}{\partial x^\mu \partial x^\nu} + F\left(x,u,\frac{\partial u}{\partial x^\mu}\right) = 0 \tag{2}$$

to the system of equations for some functions $\varphi_\nu(\omega)$, $\omega = (\omega_1,\omega_2,\omega_3)$. The functions $a_{\mu\nu}(x,u)$, $h_\mu(x,u)$ are determined from the condition that the equation (2) is reduced to the system of equations for $\varphi_\nu(\omega)$ (for more detail about the reduction method see [1, 2]).

To illustrate the efficiency of the ansatz (1) we consider two nonlinear two-dimensional equations of type (2)

$$u_{12} = u_1 F_1(u_1 - u), \tag{3}$$

$$u_{00} = F_2(u_{11}), \tag{4}$$

where $u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}$, $u_\mu = \frac{\partial u}{\partial x_\mu}$, F_1, F_2 are smooth functions.

2. For equation (3) we shall search for ansatz (1) in the form

$$\frac{\partial u}{\partial x_1} = \varphi_1(\omega) + h_1(x,u), \quad \frac{\partial u}{\partial x_2} = \varphi_2(\omega) + h_2(x,u), \tag{5}$$

h_1, h_2, ω has to be determined in the way that functions φ_1, φ_2 satisfy the system of the ordinary differential equations with a new independent variable ω [1, 2]. Substituting (5) into (3) and using the compatibility condition $u_{12} \equiv u_{21}$, we obtain

$$\begin{gathered}
\frac{\partial h_1}{\partial x_2} - \frac{\partial h_1}{\partial u}(\varphi_2 + h_2) + \frac{\partial \varphi_1}{\partial \omega}\frac{\partial \omega}{\partial x_2} = (\varphi_1 + h_1)[\varphi_1 + h_1 - u], \\
\frac{\partial h_2}{\partial x_1} + \frac{\partial h_2}{\partial u}(\varphi_1 + h_1) + \frac{\partial \varphi_2}{\partial \omega}\frac{\partial \omega}{\partial x_1} = (\varphi_1 + h_1)[\varphi_1 + h_1 - u], \\
\frac{\partial h_1}{\partial x_2} + \frac{\partial h_1}{\partial u}h_2 = h_1 F_1[h_1 + \varphi_1 - u],
\end{gathered} \tag{6}$$

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$$\begin{aligned}
&\varphi_2\frac{\partial h_1}{\partial u}+\frac{\partial \varphi_1}{\partial \omega}\frac{\partial \omega}{\partial x_2}=\varphi_1F_1[h_1+\varphi_1-u]=R_1(\omega),\\
&\frac{\partial h_2}{\partial x_1}+\frac{\partial h_2}{\partial u}h_1=h_1F_1[h_1+\varphi_1-u],\\
&\varphi_1\frac{\partial h_2}{\partial u}+\frac{\partial \varphi_2}{\partial \omega}\frac{\partial \omega}{\partial x_1}=\varphi_1F_1[h_1+\varphi_1-u]=R_2(\omega),
\end{aligned}\tag{7}$$

where $R_1(\omega)$, $R_2(\omega)$ are unknown functions. System (7) is a condition on functions h_1, h_2, $\omega(x_1,x_2)$ guaranteeing that system (6) depends on the ω only, i.e., ansatz (5) reduces partial differential equation to a system of ordinary differential equations for functions φ_1 and φ_2. Hence, in order to describe ansatzes of type (5) it is necessary to solve nonlinear system (7). Here, we get a particular solution of system (7) only, namely

$$h_1=u,\quad h_2=F_1[\varphi_1(x_2)]u,\quad \omega=x_2.\tag{8}$$

It is easy to verify that solution (8) satisfies system (7) and in this case reduced system (6) takes the form

$$\varphi_2(x_2)+\frac{\partial\varphi_1}{\partial x_2}=\varphi_1(x_2)F_1[\varphi_1(x_2)].\tag{9}$$

Having integrated the system

$$\frac{\partial u}{\partial x_1}=u+\varphi_1(x_2),\quad \frac{\partial u}{\partial x_2}=F_1[\varphi_1(x_2)]u+\varphi_1(x_2)F_1[\varphi_1(x_2)]-\frac{\partial\varphi_1}{\partial x_2}\tag{10}$$

one can obtain particular solutions of equation (3). The solution of equation (10) is given by the formula

$$u=-\varphi_1(x_2)+ce^{x_1+\int F_1(\varphi_1(x_2))dx_2},\tag{11}$$

where $\varphi_1(x_2)$ is an arbitrary smooth function and C is an arbitrary constant.

3. Now we suggest the method of construction of ansatzes (1), based on a nonlocal symmetry of the equation

$$\frac{\partial^2 u}{\partial x_1\partial x_2}=F\left(u,\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2}\right).\tag{12}$$

We consider the first order system

$$V_2^1+V_3^1V^2=F(x_3,V^1,V^2),\tag{13}$$

$$V_2^1+V_3^2V^1=F(x_3,V^1,V^2),\tag{14}$$

corresponding to the equation (12), where $V_i^k\equiv\frac{\partial V^k}{\partial x_i}$, $x_3\equiv u$, $\frac{\partial u}{\partial x_i}\equiv V^i$.

The problem of construction of all ansatzes from the class (1) for equation (12) is equivalent to the problem of finding all operators of the Q-conditional symmetry [1, 2, 5].

Theorem 1. *The system (13), (14) is Q-conditionally invariant under the operators $Q_1 = \partial_{x_1}$, $Q_2 = \partial_{x_3} + \eta^1\partial_{V_1} + \eta^2\partial_{V_2}$ if and only if the functions η^1, η^2 satisfy the following equation*

$$\begin{gathered}\eta^1_{V_2} = 0, \quad \eta^1_{x_2} = \eta^1_{x_1} = \eta^2_{x_1} = 0, \quad \eta^2_{V^1} = \frac{F}{V^1}, \\ \eta^1_{x_2} - \eta^1_{V^1}F = F'_{x_3} + \eta^1 F'_{V_1} + \frac{F}{V^1}F'_{V_2} - \eta^1\frac{F}{V^1} - \eta^1_{x_3}V^2.\end{gathered} \tag{15}$$

The correctness of Theorem 1 is easily verified with the help of the infinitesimal criterion of the Q-conditional invariance [1, 5]. Thus, arbitrary setting $\eta^1(x_3, V^1)$ as a function on x_3, V^1 we get classes of nonlinearities $F(x^3, V^1, V^2)$ with which equation (12) admits operators $\{Q_1, Q_2\}$. In the case of equation (3) η^1, η^2 are as follows: $\eta^1 = 1$, $\eta^2 = F_1(V^1 - x_3)$, $F = V^1F_1(V^1 - x_3)$. It should be noted that Q_2 is not a prolongation of Lie operator, but it is the nonlocal symmetry operator of the equation (12). Operators $\{Q_1, Q_2\}$ lead to the ansatz (10).

Then we consider the equation

$$u_{00} = F_2(u_{11}), \tag{16}$$

where F_2 is an arbitrary smooth function. Using the invariance of equation (16) under the operators ∂x_0, ∂x_1, ∂_u, $x_1\partial_u$, $x_2\partial_u$ we write it in the form of the following system

$$V^0_1 = V^1_0, \quad V^2_0 = V^1_1, \quad V^0 = F_2(V^2), \tag{17}$$

where $u_{00} \equiv V^0$, $u_{01} \equiv V^1$, $u_{11} \equiv V^2$.

Theorem 2. *The system (17) is invariant with respect to the continuous group of transformations with the infinitesimal operator*

$$Q = \xi^0\partial_{x_0} + \xi^1\partial_{x_1} \tag{18}$$

if ξ^0, ξ^1 are a solution of the system of equations

$$\begin{gathered}\frac{\partial\xi^0}{\partial x_0} = \frac{\partial\xi^0}{\partial x_1} = \frac{\partial\xi^1}{\partial x_0} = \frac{\partial\xi^1}{\partial x_1} = 0, \\ \frac{\partial\xi^0}{\partial V^1} = \frac{\partial\xi^1}{\partial V^0}F'_2(V^2) + \frac{\partial\xi^1}{\partial V^2}, \\ \frac{\partial\xi^1}{\partial V^1} = \frac{\partial\xi^0}{\partial V^0}F'_2(V^2) + \frac{\partial\xi^0}{\partial V^2}.\end{gathered} \tag{19}$$

The finite transformations

$$x'_0 = x_0 + a\xi^0, \quad x'_1 = x_1 + a\xi^1 \tag{20}$$

correspond to the operator (18). Formulae (18), (19) give the operator of the nonlocal symmetry of equation (16). With the help of this operator, one can construct nonlocal ansatzes reducing the equation (16) to the system of three ordinary differential equations for three unknown functions. The analogous procedure has been called an ati-reduction in [6].

Furthermore, the finite transformations (20) can be used for generating new solutions. The transformations (20) are more general than contact ones since ξ^0, ξ^1 are the functions on u_{00}, u_{01}, u_{11}.

For example, we shall take $F_2(u_{11}) = \sin u_{11}$. In this case one of the solutions of system (19) is

$$\xi^0 = \frac{c}{2}(V^1)^2 - C\cos V^2 + D, \quad \xi^1 = cV^1 \sin V^2 + DV^1, \tag{21}$$

where $C, D = \mathrm{const}$. We start from a solution of the equation $u = \frac{x_0 x_1}{2} - \sin x_0$. Then

$$V^0 = \sin x_0, \quad V^1 = x_1, \quad V^2 = x_0. \tag{22}$$

Using the transformations (20) we obtain the system

$$\begin{aligned} V^0 &= \sin\left[x_0 + a\left(\frac{c}{2}(V^1)^2 - C\cos V^2 + D\right)\right], \\ V^1 &= x_1 + a[CV^1\sin V^2 + DV^1], \\ V^2 &= x_0 + a\left[\frac{c}{2}(V^1)^2 - C\cos V^2 + D\right]. \end{aligned} \tag{23}$$

Thus, in order to find new solutions of equations (16) it is necessary to solve the overdetermined but compatible system

$$\begin{aligned} V_{00} &= \sin\left[x_0 + a\left(\frac{c}{2}(u_{01})^2 - C\cos u_{11} + D\right)\right], \\ u_{01} &= x_1 + a[cu_{01}\sin u_{11} + Du_{01}], \\ u_{11} &= x_0 + a\left[\frac{c}{2}(u_{01})^2 - C\cos u_{11} + D\right]. \end{aligned} \tag{24}$$

The maximal local invariance group of equation (16) is the 7-parameter group. The basic elements of the corresponding algebra are

$$\begin{aligned} &P_0 = \partial_{x_0}, \quad P_1 = \partial_{x_1}, \quad P_2 = \partial_u, \quad D = x_0\partial_{x_0} + x_1\partial_{x_1} + 2u\partial_u, \\ &Q_1 = x_1\partial_u, \quad Q_2 = x_2\partial_u, \quad Q_3 = x_1x_2\partial_u. \end{aligned} \tag{25}$$

It can be shown, that the system (24) has no solutions invariant under the operator $\alpha_0 P_0 + \alpha_1 P_1 + \alpha_2 P_2 + dD + \beta_1 Q_1 + \beta_2 Q_2 + \beta_3 Q_3$, where α_0, α_1, α_2, d, β_1, β_2, β_3 are arbitrary constants. Therefore, no solution of system (24) is invariant one for equation (16).

Further, we consider the equation

$$[F(u)]_1 = u_{22}. \tag{26}$$

We write the equation (26) in the form of the system

$$F(u) = \theta_{x_2}, \quad u_{x_2} = \theta_{x_1}. \tag{27}$$

Theorem 3. *The system (27) is invariant with respect to the one-parameter Lie group generated by an operator of the form*

$$Q = -x_1\partial_{x_1} + \theta\partial_{x_2} + u\partial_u \tag{28}$$

if $F = \frac{1}{\ln u}$.

Operator (28) is a nonlocal symmetry operator of equation (26). We use It to construct the nonlocal ansatz and exact solutions of the equation

$$\left(\frac{1}{\ln u}\right)_1 = u_{22}. \tag{29}$$

The ansatz

$$u = \frac{f(\theta)}{x_1}, \quad \theta = \frac{x_2}{\varphi(\theta) - \ln x_1} \tag{30}$$

corresponds to operator (28), where $f(\theta)$, $\varphi(\theta)$ are unknown functions.

Substituting (30) into (27) we obtain the reduced system of ordinary differential equations

$$\theta\varphi' + \varphi = \ln f, \quad f' = \theta. \tag{31}$$

The solution of system (31) has the form

$$\varphi = \frac{\theta^2}{2}\ln\frac{\theta^2+2c}{2} - \frac{\theta^2}{2} + C\ln(\theta^2+2c) + c_1, \quad f = \frac{\theta^2}{2} + c. \tag{32}$$

Using the formula (30) and the substitution $\frac{1}{\ln u} = z$ we obtain the solution of the equation

$$z_1 + \left(\frac{1}{z^2}e^{\frac{1}{z}}z_2\right)_2 = 0 \tag{33}$$

namely

$$\begin{gathered} e^{\frac{1}{z}} = \frac{\theta^2}{2} + c, \\ \theta = \frac{x^2}{\frac{\theta^2}{2}\ln\frac{\theta^2+2c}{2} - \frac{\theta^2}{2} + C\ln(\theta^2+2c) - \ln x_1 + c_1}. \end{gathered} \tag{34}$$

Formulae (34) give the solution of the nonlinear diffusion equation (33). In conclusion, we emphasize that the finite transformations

$$x_1' = e^{-a}x_1, \quad x_2' = x_2 + a\theta, \quad z' = \frac{z}{1+az}, \quad \theta' = 0 \tag{35}$$

can be used for the nonlocal generating of solutions of equation (33), since the system (27) admits the operator (28) in Lie sense. In this case the formula of generating solutions takes the form

$$z'' = \frac{z'(e^{-a}x_1, x_2 + a\theta)}{1 - az'(e^{-a}x_1, x_2 + a\theta)}, \tag{36}$$

where θ is the solution of the system

$$\theta_{x_1} = -z'_{x_2}\left(\frac{1}{(z')^2}e^{\frac{1}{z'}}\right), \quad \theta_{x_2} = z', \tag{37}$$

z' is the initial solution and z'' is the new solution of the equation (33), a is an arbitrary constant.

Suggested approach can be effectively applied for the nonlocal generating of solutions of equations which are invariant with respect to the group of contact transformations.

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