Symmetry and exact solutions of multidimensional nonlinear Fokker–Planck equation

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Розглядається нелінійне рівняння Фоккера–Планка. За рахунок накладання на коефіцієнти функції нелінійної додаткової умови вдалось значно розширити симетрію рівняння Фоккера–Планка. Досліджено умовну симетрію, проведено редукцію та знайдено деякі точні розв'язки цього рівняння.

1. Let us consider equation

$$\frac{\partial p}{\partial t} = -\sum_{k=1}^{n} \frac{\partial}{\partial x_k}(A_k\rho) + \frac{1}{2}\sum_{i,k=1}^{n} \frac{\partial^2}{\partial x_i \partial x_k}(B_{ik}\rho) + F(\rho), \tag{1}$$

where $\rho(t,\vec{x})$, $\vec{x} = (x_1,\dots,x_n)$, $A_k(t,\vec{x})$, $B_{ik}(t,\vec{x})$, $F(\rho)$ are smooth real functions. If $F(\rho) = 0$, (1) coincides with classical linear Fokker–Planck equation (FPE), which finds broad application in the theory of Markov processes. In this case [1] ρ is the conditional probability density, $\vec{A} = (A_1, A_2, \dots A_n)$ is a drift velocity vector, B_{ik} are elements of diffusion matrix $B(t,\vec{x}) = \|B_{ik}\|_{i,k=1}^{n}$.

In the cases, when (1) (for $F(\rho) = 0$) is equivalent to the linear heat equation, it is possible to use effectively group-theoretical analysis methods to construct solutions of the linear FPE [2]. In other cases equation (1) for fixed A_k and B_{ik}, as a rule, has no nontrivial symmetry. Thus, it is impossible to apply to it symmetry methods [3].

In [4] a new interpretation for the equations like (1) was proposed, it opens wide possibilities for application of group-theoretical methods. The idea consists in complementing (1) with equations for coefficient functions A_k and B_{ik}. That is we add to (1) some system of equations for A_k and B_{ik}, (1) turns out then to be a nonlinear system (even if $F(\rho) = 0$). Such an extended system, as we show, can have a nontrivial symmetry which is used to construct exact solutions of equation (1).

Therefore our paper is based on the idea of nonlinear extension of equation (1).

2. We require the components of the vector $\vec{A}$ to satisfy conditions having the form of Euler's equation for the ideal liquid

$$\frac{\partial A_k}{\partial t} + A_l \frac{\partial A_k}{\partial x_l} = F_k(\rho). \tag{2}$$

For the potential flow when $A_k = \dfrac{\partial \varphi}{\partial x_k}$, $F_k = \frac{\partial F_1(\rho)}{\partial x_k}$, and $B_{ik} = D^{ik}$ ($D = \text{const} \geq 0$) equations (1) and (2) can be written as

$$\rho_0 + \rho_a\varphi_a + \rho\Delta\varphi - \frac{D}{2}\Delta\rho = F(\rho), \tag{3}$$

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$$\varphi_0 + \frac{1}{2}\varphi_a\varphi_a = F_1(\rho), \tag{4}$$

where $\rho_0 = \frac{\partial\rho}{\partial x_0}$, $x_0 \equiv t$, $\varphi_a = \frac{\partial\varphi}{\partial x_a}$, $a = \overline{1,n}$. Thus we tend to investigate symmetry properties and to construct families of solutions for (3), (4).

3. We assume D to be nonvanishing.

Theorem 1. *Equation (3) for $D > 0$ is invariant under the following algebras:*

1) $A_1 = \langle P_0, J_{ab}, X_a, Y, P_a\rangle$, *where*

$$P_0 = \frac{\partial}{\partial x_0}, \quad P_a = \frac{\partial}{\partial x_a}, \quad J_{ab} = x_aP_b - x_bP_a, \quad \{a,b\} = \overline{1,n},$$
$$X_a = g_a(x_0)P_a + g'_a(x_0)x_a\frac{\partial}{\partial\varphi}, \quad Y = h(x_0)\frac{\partial}{\partial\varphi},$$

(g_a, h are arbitrary smooth functions) for an arbitrary $F(\rho)$;

2) $A_2 = \langle A_1, D\rangle$, *where the operator of scale transformations D has the form*

$$D = 2x_0P_0 + x_aP_a - \frac{2}{k}I,$$

where $I = \rho\frac{\partial}{\partial\rho}$, for $F = \lambda\rho^{k+1}$, $k \neq 0$;

3) $A_3 = \langle A_2, A\rangle$, *where the operator of projective transformations A has the form*

$$A = x_0^2P_0 + x_0x_aP_a + \frac{\vec{x}^{\,2}}{2}\frac{\partial}{\partial\varphi} - nx_0I, \quad \textit{if} \quad F = \lambda\rho^{\frac{2}{n}+1},$$

where $\vec{x}^{\,2} = x_1^2 + x_2^2 + \cdots + x_n^2$;

4) $A_4 = \langle A_1, S\rangle$, *where*

$$S = f(x_0)P_0 + \frac{1}{2}f'(x_0)x_aP_a + \frac{1}{4}f''(x_0)\vec{x}^{\,2}\frac{\partial}{\partial\varphi} - \frac{n}{2}f'(x_0)I$$

(f is an arbitrary smooth function), if $F = 0$;

5) $A_5 = \langle A_1, C_0\rangle$, *where*

$$C_0 = \exp\{\lambda x_0\}I, \quad \textit{if} \quad F = \lambda\ln p.$$

Proof of this and following theorems can be made using Lie's algorithm (see, e.g. [5, 6]).

Remark 1. Algebra A_4 coincides with A_3, if we require condition $f''' = 0$ to be satisfied.

Remark 2. In the case $D = 0$ equation (1) turns into Liouville's equation. The question on the symmetry of (4) (if $F = 0$) then can be answered by the following theorem [7].

Theorem 2. *Equation (4) with $D = F = 0$ is invariant under infinitely-dimensional algebra which is generated by the operator*

$$X = 2f_1(x_0)P_0 + f'_1(x_0)x_aP_a + f_2(x_0)\left\{x_aP_a + 2\varphi\frac{\partial}{\partial\varphi}\right\} +$$
$$+ (f''_1 + f'_2)\left\{\frac{\vec{x}^{\,2}}{2}\frac{\partial}{\partial\varphi} - \frac{n}{2}x_0I\right\} + f_{3a}P_a + f'_{3a}(x_0)x_a\frac{\partial}{\partial\varphi} + dI + c_{ab}J_{ab}.$$

where $f_1'''(x_0) + f_2''(x_0) = 0$, $c_{ab} = -c_{ba}$, $\{d, c_{ab}\} \subset \mathbb{R}$, $f_1(x_0)$, $f_2(x_0)$, $f_{3a}(x_0)$, $a = \overline{1,4}$, $f_4(x_0)$ *are an arbitrary smooth functions. Operators* X_a *lead to the following finite transformations:*

$$x_a' = x_a + g_a(x_0)\theta, \quad \varphi' = \varphi + \dot g_a(x_0)x_a\theta + \frac{1}{2}g_a(x_0)\dot g_a(x_0)\theta^2,$$

$\rho' = \rho$, $x_0' = x_0$, $x_b' = x_b$, *where* $\dot g_a = \frac{dg_a}{dx_0}$, θ *is a group parameter.*

4. Let us now require the condition (4) on φ to be satisfied.

Theorem 3. *The system of equations (3), (4) for* $D \neq 0$ *and arbitrary* F, F_1 *is invariant under the algebra*

1) $AG(1,n) = \langle P_0, P_a, J_{ab}, G_a, Q\rangle$, *where* $G_a = x_0P_a + x_aQ$, $Q = \frac{\partial}{\partial\varphi}$ *and additionally is invariant under the following algebras:*

2) $AG_1(1,n) = \langle AG(1,n), D\rangle$, *if* $F = \lambda\rho^{k+1}$, $F_1 = \lambda_1\rho^k$, $k \neq 0$;

3) $AG_2(1,n) = \langle AG_1(1,n), A\rangle$, *if* $F = \lambda\rho^{\frac{2}{n}+1}$, $F_1 = \lambda_1\rho^{\frac{2}{n}}$;

4) $AG_3(1,n) = \langle AG_1(1,n), B\rangle$, *where the operator* B *has the form* $B = I + \lambda_1x_0Q$, *if* $F_1 = \lambda_1\ln\rho$, $F = 0$;

5) $AG_4(1,n) = \langle AG(1,n), C\rangle$, *where* $C = \exp\{\lambda x_0\}\left(\frac{\lambda_1}{\lambda}Q + I\right)$, *if* $F = \lambda\rho\ln\rho$, $F_1 = \lambda_1\ln\rho$, $\lambda \neq 0$;

6) $AG_5(1,n) = \langle AG_2(1,n), I\rangle$, *if* $F = F_1 = 0$, *where* λ_i *are arbitrary real constants,* $i = 1, 2$.

Remark 3. Operator C with $\lambda_1 = 0$ coincides with C_0.

Remark 4. In the case $D = 0$ the system (3), (4) is employed in quanturn mechanics and is called there "the classical approximation of the Schrödinger equations" [8]. Its symmetry was investigated in [7].

5. Conditional symmetry. The system (3), (4) has conditional symmetry. The condition which allows to enlarge symmetry of this system has the form

$$\Delta\rho = F_2(\rho). \tag{5}$$

Then the system of equations (3), (4), (5) is equivalent to the following system:

$$\begin{aligned}
&\rho_0 + \rho_a\varphi_a + \rho\Delta\varphi = F(\rho),\\
&\varphi_0 + \frac{1}{2}\varphi_a\varphi_a = F_1(\rho),\\
&\Delta\rho = F_2(\rho).
\end{aligned} \tag{6}$$

Theorem 4. *The system of equations (6) for arbitrary* F, F_1, F_2 *is invariant under the algebra* $AG(1,n)$ *and additionally under the following algebras:*

1) $AG_6(1,n) = \langle AG(1,n), Q_1\rangle$, *where* $Q_1 = x_aP_a + 2\varphi Q$ *if* F *is arbitrary and* $F_1 = F_2 = 0$;

2) $AG_7(1,n) = \langle AG(1,n), Q_2\rangle$, *where* $Q_2 = x_0P_0 - \varphi Q$ *for an arbitrary* F_2 *and* $F = F_1 = 0$;

3) $AG_8(1,n) = \langle AG(1,n), Q_1 + Q_2\rangle$ *for an arbitrary* F_1 *and* $F = F_2 = 0$;

4) $AG_9(1,n) = \langle AG_1(1,n), Q_3\rangle$, *where the operator* Q_3 *has the form:* $Q_3 = x_aP_a + 2\varphi Q - \frac{2}{k}I$, *if* $F = 0$, $F_1 = \lambda_1\rho^{-k}$, $F_2 = 0$, $k \neq 0$;

5) $AG_{10}(1,n) = \langle AG_1(1,n), Q_2\rangle$, *if* $F = F_1 = 0$, $F_2 = \lambda_2\rho^{k+1}$, $k \neq 0$;

6) $AG_{11}(1,n) = \langle AG_1(1,n), Q_1\rangle$, *if* $F = \lambda\rho^{k+1}$, $F_1 = F_2 = 0$, $k \neq 0$;

7) $AG_{12}(1,n) = \langle AG(1,n), Q_3\rangle$, *if* $F = 0$, $F_1 = \lambda_1\rho^{-k}$, $F_2 = \lambda\rho^{k+1}$, $k \neq 0$;

8) $AG_{13}(1,n) = \langle AG_1(1,n), Q_4\rangle$, *where the operator* Q_4 *has the form:* $Q_4 = x_0P_0 - \varphi Q - \frac{2}{k}I$, *if* $F = \lambda\rho^{\frac{k+2}{2}}$, $F_1 = \lambda_1\rho^k$, $F_2 = 0$, $k \neq 0$;

9) $AG_1(1,n)$ *if* $F = \lambda\rho^{k+1}$, $F_1 = \lambda_1\rho^k$, $F_2 = \lambda_2\rho^{k+1}$, $k \neq 0$;

10) $AG_{14}(1,n) = \langle AG(1,n), Q_3 + mQ_4\rangle$, $m \in \mathbb{R}$, *if* $F = \lambda\rho^{\frac{mk+2}{2}}$, $F_1 = \lambda_1\rho^{mk-k}$, $F_2 = \lambda_2\rho^{k+1}$, $k \neq 0$;

11) $AG_2(1,n)$, *if* $F = \lambda\rho^{\frac{2+n}{n}}$, $F_1 = \lambda_1\rho^{\frac{2}{n}}$, $F_2 = \lambda_2\rho^{\frac{2+n}{n}}$;

12) $AG_{15}(1,n) = \langle AG_2(1,n), Q_1\rangle$, *if* $F = \lambda_2\rho^{\frac{2+n}{n}}$, $F_1 = F_2 = 0$;

13) $AG_{16}(1,n) = \langle AG_2(1,n), Q_2\rangle$, *if* $F = F_1 = 0$, $F_2 = \lambda_2\rho^{\frac{2+n}{n}}$;

14) $AG_{17}(1,n) = \langle AG_2(1,n), Q_1 + Q_2\rangle$, *if* $F_1 = \lambda_1\rho^{\frac{2}{n}}$, $F = F_2 = 0$;

15) $AG_{18}(1,n) = \langle AG_2(1,n), Q_1, Q_2\rangle$, *if* $F = F_1 = F_2 = 0$;

16) $AG_{19}(1,n) = \langle AG_8(1,n), B\rangle$, *if* $F = F_2 = 0$, $F_1 = \lambda_1 \ln\rho$;

17) $AG_4(1,n) = \langle AG(1,n), C\rangle$, *if* $F = \lambda\rho\ln\rho$, $F_2 = 0$;

18) $AG_{20}(1,n) = \langle AG_6(1,n), C_0\rangle$, *if* $F = \lambda\rho\ln\rho$, $F_1 = F_2 = 0$, $\lambda \neq 0$.

Remark 5. It follows from the commutation equalities that some of above mentioned algebras coincide (for instance, $AG(1,n)$ and $AG_{12}(1,n)$, $AG_7(1,n)$ and $AG_{13}(1,n)$).

6. Reduction of the system (3), (4). Using the operators mentioned in Theorems 3 and 4 we have constructed ansatzes and have obtained corresponding reduced systems of equations. Some of them are adduced below (for the case of three spatial variables, $n = 3$):

1) Ansatz $\rho = \exp\left\{\frac{2x_0}{\alpha}\right\}\Phi(\omega_1,\omega_2)$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$,

$$\varphi = \frac{\lambda_1}{2\alpha}x_0^2 - \frac{x_0^3}{3\alpha^2} + \frac{x_0x_1}{\alpha} + g(\omega_1,\omega_2), \quad \omega_1 = \frac{x_0^2}{\alpha} - x_1, \quad \omega_2 = (\vec{x}^{\,2})^{\frac{1}{2}},$$

reduces (3), (4), if $F = 0$, $F_1 = \lambda_1\ln\rho$ to the following system:

$$\left(\frac{2}{\alpha} + g_{11} + g_{22}\right)\Phi + g_1\Phi_1 + g_2\omega_2^{-1}\Phi + \frac{D}{2}(\omega_2^{-1} + \Phi_{11} + \Phi_{22}) = 0,$$

$$g_1^2 + g_2^2 = \lambda\ln\Phi + \frac{2}{\alpha}\omega_1, \quad \text{where } g_i = \frac{\partial g}{\partial\omega_i}, \quad \Phi_i = \frac{\partial\Phi}{\partial\omega_i}, \quad i = 1,2.$$

2) Ansatz $\rho = \exp\left\{\frac{2x_0}{\alpha}\right\}\Phi(\omega)$, $\omega = (\vec{x}^{\,2})^{\frac{1}{2}}$, $\alpha \neq 0$

$$\varphi = \frac{\lambda_1}{\alpha}x_0^2 + g(\omega), \quad \text{with } F = 0, \ F_1 = \lambda_1\ln\rho$$

reduces (3), (4) to the system:

$$\left(\frac{2}{\alpha} + \frac{n-1}{\omega}g' + g''\right)\Phi + g'\Phi' + \frac{D}{2}\left(\Phi'' + \frac{n-1}{\omega}\Phi'\right) = 0,$$

$$(g')^2 = 2\lambda_1\ln\Phi, \quad \text{where } g' = \frac{dg}{d\omega}, \quad \Phi' = \frac{d\Phi}{d\omega}.$$

3) $F = 0$, $F_1 = \lambda_1\ln\rho$,

$$\rho = \exp\left\{\frac{2x_0}{\alpha}\right\}\Phi(\omega,\omega_2), \quad \omega_1 = (x_1^2 + x_2^2)^{\frac{1}{2}}, \quad \omega_2 = \operatorname{arctg}\frac{x_2}{x_1} - \frac{x_3}{\beta},$$

$$\varphi = \frac{\lambda_1}{2\alpha}x_0^2 + g(\omega_1,\omega_2), \quad \{\alpha,\beta\} \neq 0,$$

$$\left(\frac{2}{\alpha}+g_1\omega_1^{-1}+g_{11}\right)\Phi+g_1\Phi_1+g_2\Phi_2(\omega_1^{-2}+\beta^{-2})+g_{22}\Phi(\omega_1^{-2}+\beta^{-2})+$$
$$+\frac{D}{2}(\Phi_{11}+\omega_1^{-1}\Phi_1+\Phi_{22}(\omega_1^{-2}+\beta^{-2}))=0,$$
$$g_1^2+g_2^2(\omega_1^{-2}+\beta^{-2})=\frac{\lambda_1}{2}\ln\Phi.$$

4) Ansatz

$$\rho=\exp\left\{\frac{2}{\lambda\alpha}\exp(\lambda x_0)\right\}\Phi(x_3),\quad \varphi=2g(x_3),\quad \{\lambda,\alpha\}\neq 0,\quad \alpha\in\mathbb{R},$$

reduces (3), (4), if $F=\lambda\rho\ln\rho$, $F_1=0$, to the following system:

$$\Phi''+\frac{2\lambda}{D}\Phi\ln\Phi=0,$$
$$g'=0,\quad D\neq 0.$$

5) Ansatz $\rho=\exp\{2x_1\exp(\lambda x_0)\}\Phi(x_0)$,

$$\varphi=\frac{\lambda_1}{\lambda}x_1\exp\{\lambda x_0\}+\frac{x_2^2+x_3^2}{2x_0}+g(x_0),\quad \lambda\neq 0$$

reduces (3), (4), if $F=\lambda\rho\ln\rho$, $F_1=\lambda_1\ln\rho$, to the system ODE:

$$\Phi'+2x_0^{-1}\Phi+2\left(D+\frac{\lambda_1}{\lambda}\exp\right)\exp\{2\lambda x_0\}\Phi=\lambda\Phi\ln\Phi,$$
$$g'+\frac{\lambda_1^2}{2\lambda^2}\exp\{2\lambda x_0\}=\lambda_1\ln\Phi.$$

6) Ansatz $\rho=\exp\left\{\frac{2}{\alpha}x_3\exp(\lambda x_0)\right\}\Phi(\omega_1,\omega_2)$, $\{\alpha,\lambda\}\neq 0$, $\alpha\in\mathbb{R}$,

$$\varphi=\frac{\lambda_1}{\lambda}\exp\{\lambda x_0\}x_3+g(\omega_1,\omega_2),\quad \omega_1=x_0,\quad \omega_2=(x_1^2+x_2^2)^{\frac{1}{2}},$$

reduces (3), (4), if $F=\lambda\rho\ln\rho$, $F_1=\lambda_1\ln\rho$, to the following system:

$$\Phi_1+g_2\Phi_2+\left(\omega_2^{-1}g_2+\frac{\lambda_1}{\lambda\alpha^2}\exp\{2\lambda\omega_1\}\right)\Phi+\frac{D}{2}(\Phi_{22}+\omega_2^{-1}\Phi_2)=\lambda\Phi\ln\Phi,$$
$$2g_1+g_2^2=\lambda_1\ln\Phi-\frac{\lambda_1^2}{2\lambda^2\alpha^2}\exp\{2\lambda\omega_1\}.$$

7) $F=\lambda\rho\ln\rho$, $F_1=\lambda_1\ln\rho$,

$$\rho=\exp\left\{2\,\mathrm{arctg}\,\frac{x_2}{x_1}\exp(\lambda x_0)\right\}\Phi(\omega_1,\omega_2),\quad \omega_1=x_0,\quad \omega_2=(x_1^2+x_2^2)^{\frac{1}{2}},$$
$$\varphi=\frac{\lambda_1}{\lambda}\exp(\lambda x_0)\,\mathrm{arctg}\,\frac{x_2}{x_1}+\frac{x_3^2}{2x_0}+g(\omega_1,\omega_2),\quad \lambda\neq 0,$$
$$\Phi_1+\omega_2^{-2}\left(g_2+\left(D+\frac{2\lambda_1}{\lambda}\right)\exp\{2\lambda\omega_1\}\right)\Phi+g_2\Phi_2+\frac{D}{2}\Phi_{22}+$$
$$+\frac{D}{2}\omega_2^{-1}\Phi_2+g_{22}\Phi=\lambda\Phi\ln\Phi,$$
$$2g_1+g_2^2=\lambda_1\ln\Phi-\left(\frac{\lambda_1}{\lambda}\omega_2^{-1}\exp\{\lambda\omega_1\}\right)^2.$$

8) $F=\lambda\rho\ln\rho$, $F_1=\lambda_1\ln\rho$,

$$\rho=\exp\left\{\frac{2}{\lambda\alpha}\exp(\lambda x_0)\right\}\Phi(\omega),\quad \varphi=\frac{\lambda_1}{\lambda\alpha}\exp(\lambda x_0)+g(\omega),\quad \omega=(\vec{x}^{\,2})^{\frac12},$$

$$g'\Phi'+\Phi\left(g''+\frac{n-1}{\omega}g'\right)+\frac{D}{2}\left(\Phi''+\frac{n-1}{\omega}\Phi'\right)=\lambda\Phi\ln\Phi,$$
$$(g')^2=\lambda_1\ln\Phi.$$

9) Ansatz $\rho=\Phi(\omega)$, $\varphi=x_0x_3-\frac{x_0^3}{3}+g(\omega)$, $\omega=x_3-\frac{x_0^2}{2}$ for arbitrary $F(\rho)$, $F_1(\rho)$ reduces (3), (4) to the following system:

$$g'\Phi'+g''\Phi+\frac{D}{2}\Phi''=F(\Phi),$$
$$(g')^2+2\omega=F_1(\Phi).$$

10) $\rho=\Phi(\omega_1,\omega_2)$, $\varphi=\frac{x_3^2}{2x_0}+g(\omega_1,\omega_2)$, $\omega_1=x_0$, $\omega_2=(x_1^2+x_2^2)^{\frac12}$, $F(\rho)$, $F_1(\rho)$ are arbitrary functions,

$$\Phi_1+g_2\Phi_2+\Phi(g_{22}+\omega_2^{-1}g_2)+\frac{D}{2}(\Phi_{22}+\omega_2^{-1}\Phi_2)=F(\Phi)-\omega_1^{-1}\Phi,$$
$$g_1+g_2^2=F_1(\Phi).$$

11) $\rho=\Phi(\omega)$, $\varphi=g(\omega)-x_0-\sqrt{2}\,\mathrm{arctg}\,\frac{x_2}{x_1}$, $\omega=(x_1^2+x_2^2)^{\frac12}$, $F(\rho)$, $F_1(\rho)$ are arbitrary functions,

$$g'(\Phi'+\omega^{-1}\Phi)+g''\Phi+\frac{D}{2}(\Phi''+\omega^{-1}\Phi')=F(\Phi),$$
$$(g')^2=2F_1(\Phi).$$

12) $\rho=\Phi(x_0)$, $\varphi=\frac{x_3^2}{2x_0}+g(x_0)$,

$$\Phi'+\omega^{-1}\Phi=F(\Phi),$$
$$(g')=F_1(\Phi),$$

13) $\rho=\Phi(x_3)$, $\varphi=g(x_3)-x_0$,

$$g'\Phi'+g''\Phi+\frac{D}{2}\Phi''=F(\Phi),$$
$$2+(g')^2=2F_1(\Phi).$$

14) Ansatz $\rho=x_0^m\Phi(\omega)$, $\varphi=\frac{x_1^2+\cdots+x_k^2}{2x_0}+g(\omega)$, where $\omega=\frac{x_{k+1}^2+\cdots+x_{k+l}^2}{2x_0}$, $0\le k\le 2$, $1\le l\le 3-k$ reduces (3), (4), if $F=\lambda\rho^{\frac{m-1}{m}}$, $F_1=\lambda\rho^{-\frac1m}$, to the system ODE:

$$(k+m)\Phi+\Phi'(2\omega g'-\omega)+\Phi(lg'+2\omega g'')+D(l\Phi'+2\omega\Phi'')=\lambda\Phi^{\frac{m-1}{m}},$$
$$(g')^2-g'=\lambda_1\omega^{-1}\Phi^{-\frac1m}.$$

15) Ansatz $\rho = x_0^2\Phi(\omega_1;\omega_2)$, $\omega_1 = \frac{x_1}{x_0}$, $\omega_2 = \frac{x_2}{x_0}$, $\varphi = 2\lambda_1 x_0 \ln x_0 + \alpha x_3 + x_0 g(\omega_1,\omega_2)$, $\alpha \in \mathbb{R}$ reduces system (6), if $F = 0$, $F_1 = \lambda_1 \ln\rho$ to the following system:

$$\begin{aligned}
&2\lambda_1 + g - \omega_1 g_1 - \omega_2 g_2 + \frac{\alpha^2}{2} = \lambda_1 \ln\Phi,\\
&\Phi_{11} + \Phi_{22} = 0,\\
&2\Phi - \omega_1\Phi_1 - \omega_2\Phi_2 + g_1\Phi_1 + g_2\Phi_2 + (g_{11} + g_{22})\Phi = 0.
\end{aligned} \tag{7}$$

16) $F = 0$, $F_1 = \lambda_1 \ln\rho$, $F_2 = 0$,

$$\begin{aligned}
&\rho = x_0^2\Phi(\omega_1,\omega_2), \quad \varphi = \frac{x_1^2}{2x_0} + \lambda_1 x_0 \ln x_0 + x_0 g(\omega_1,\omega_2),\\
&\omega_1 = \frac{x_1}{x_0} - \operatorname{arctg}\frac{x_3}{x_2}, \quad \omega_2 = x_0^{-1}(x_2^2 + x_3^2)^{\frac12},
\end{aligned}$$

$$\begin{aligned}
&\lambda_1 + g - \omega_2 g_2 + \frac12 g_1^2(1 + \omega_2^{-2}) + g_2^2 = \frac{\lambda_1}{2}\ln\Phi,\\
&\Phi_{11}(1 + \omega_2^{-2}) + \Phi_{22}\omega_2^2 + \Phi_2\omega_2 = 0,\\
&3\Phi - \omega_2\Phi_2 + \omega_1(1 + \omega_2^{-2})\Phi_1 + g_2\Phi_2 + g_{11}(1 + \omega_2^{-2})\Phi + (g_{22} + \omega_2 g_2)\Phi = 0.
\end{aligned} \tag{8}$$

17) Ansatz $\rho = x_0^2\Phi(\omega_1,\omega_2)$, $\omega_1 = \frac{x_2}{x_0}$, $\omega_2 = \frac{x_3}{x_0}$, $\varphi = \frac{x_1^2}{2x_0} + \lambda_1 x_0 \ln x_0 + x_0 g(\omega_1,\omega_2)$ reduces system (6), if $F = 0$, $F_1 = \lambda_1 \ln\rho$, to the following system:

$$\begin{aligned}
&\lambda_1 + g - \omega_1 g_1 - \omega_2 g_2 + \frac12(g_1^2 + g_2^2) = \frac{\lambda_1}{2}\ln\Phi,\\
&\Phi_{11} + \Phi_{22} = 0,\\
&3\Phi + g_1\Phi_1 + g_2\Phi_2 + (g_{11} + g_{22})\Phi - \omega_1\Phi_1 - \omega_2\Phi_2 = 0.
\end{aligned} \tag{9}$$

18) Ansatz $\rho = \exp\{\exp(\lambda x_0)\ln x_1\}\Phi(\omega_1,\omega_2)$, $\varphi = \frac{x_2^2}{2x_0} + x_1^2 g(\omega_1,\omega_2)$, $\omega_1 = x_0$, $\omega_2 = \frac{x_2}{x_1} - \frac{x_0x_3}{x_1}$ reduces system (6), if $F = \lambda\rho\ln\rho$, $F_1 = 0$, to the system:

$$\begin{aligned}
&2g_1 + 2\omega_2 g_2(\omega_1^{-1} - 2) + g^2 + g_2^2(1 + \omega_1^2 + \omega_2^2) = 0,\\
&\exp(\lambda\omega_1)\Phi + \frac12\Phi_{22}(1 + \omega_1^2 + \omega_2^2) + \frac12\omega_2\Phi_2(1 - 4\exp(\lambda\omega_1)) +\\
&\quad + 2\Phi\exp(2\lambda\omega_1) = 0,\\
&\Phi_1 + \Phi_2(\omega_1^{-1} - 4\omega_2 g) + g\Phi(2 + 4\exp(\lambda\omega_1)) - 2(1 + \exp(\lambda\omega_1))\omega_2 g_2\Phi +\\
&\quad + (g_{22}\Phi + g_2\Phi_2)(1 + \omega_1^2 + \omega_2^2) + \omega_1^{-1}\Phi = \lambda\Phi\ln\Phi.
\end{aligned} \tag{10}$$

19) Ansatz $\rho = \exp\left\{\frac{2}{\alpha}\exp(\lambda x_0)\right\}\Phi(\omega)$, $\alpha \neq 0$,

$$\varphi = \exp\left\{\frac{2x_0}{\alpha}\right\}g(\omega), \quad \omega = (\vec{x}^{\,2})^{\frac12}\exp\left\{-\frac{x_0}{\alpha}\right\}, \quad F = \lambda\rho\ln\rho, \quad F_1 = 0$$

reduces (6) to the system ODE:

$$\begin{aligned}
&\Phi' g' + g'\frac{n-1}{\omega}\Phi - \frac{1}{\alpha}\omega\Phi + \Phi g'' = \lambda\Phi\ln\Phi,\\
&\Phi'' + \frac{n-1}{\omega}\Phi' = 0,\\
&\frac{2}{\alpha}g - \frac{1}{\alpha}\omega g' + \frac12(g')^2 = 0.
\end{aligned} \tag{11}$$

Remark 6. Systems of reduced equations (7)–(11) are overdetermined.

7. Exact solutions of nonlinear Fokker–Planck equations. Below we list some exact solutions of the FPEs in case of three spatial variables.

1) Equation (1) has a solution $\rho = x_1^{-3}$, if $A_k = \frac{x_k}{x_0}$, $F(\rho) = -6D\rho^{\frac{5}{3}}$;

2) $\rho = (x_1^2 + x_2^2)^{-\frac{3}{2}}, \quad A_k = \dfrac{x_k}{x_0}, \quad F(\rho) = -\dfrac{9}{2}D\rho^{\frac{5}{3}};$

3) $\rho = x_1^{-2}, \quad A_1 = \dfrac{x_1}{x_0}, \quad A_2 = \dfrac{x_2}{x_0}, \quad A_3 = 0, \quad F(\rho) = -3D\rho^2;$

4) $\rho = x_1^{-1}, \quad A_k = \delta^{1k}\dfrac{x_k}{x_0}, \quad F(\rho) = -D\rho^3;$

5) $\rho = (x^2)^{-\frac{3}{2}}, \quad A_k = \dfrac{x_k}{x_0}, \quad F(\rho) = -3D\rho^{\frac{5}{3}};$

6) $\rho = (x_1^2 + x_2^2)^{-\frac{3}{2}}, \quad A_1 = \dfrac{x_2}{x_0}, \quad A_2 = \dfrac{x_2}{x_0}, \quad A_3 = 0, \quad F = -2D\rho^3;$

7) $$\rho = \exp\left\{\frac{2}{\alpha}x_0 + \frac{D^2}{\lambda}\left(c \pm \frac{\lambda}{D^2}\vec{x}^{\,2}\right)^2\right\}, \quad \{\alpha, \lambda, D\} \neq 0, \quad \alpha \in \mathbb{R},$$
$$A_k = -D\left(c \pm \frac{\lambda}{D^2}(\vec{x}^{\,2})^{\frac{1}{2}}\right)x_k(\vec{x}^{\,2})^{-\frac{1}{2}}, \quad F = \frac{2}{\alpha}\rho, \quad c \in \mathbb{R};$$

8) $$\rho = \exp\left\{\frac{2}{\alpha}x_0 + \frac{2}{D}y(\omega_1) + \frac{2}{D}z(\omega_2)\right\}, \quad A_k = \frac{\partial\varphi}{\partial x_k}, \quad F = \frac{2}{\alpha}\rho,$$
$$\varphi = \frac{\lambda}{2\alpha}x_0^2 - \frac{x_0^3}{3\alpha^2} + \frac{x_0 x_1}{\alpha} + y(\omega_1) + z(\omega_2),$$
$$\omega_1 = \frac{x_0^2}{2\alpha} - x_1, \quad \omega_2 = (x_2^2 + x_3^2)^{\frac{1}{2}},$$

where $z = \dfrac{D}{2\lambda}\left(c_3 + \dfrac{2}{D}\omega_2\right)^2$ and y can be determined implicitly via relations

$$\pm D\left(\frac{2\lambda}{D}y + \frac{2}{\alpha}\omega_1\right)^{\frac{1}{2}} + \frac{D^2}{\lambda\alpha}\ln\left|\lambda\left(\frac{2\lambda}{D}y + \frac{2}{\alpha}\omega_1\right)^{\frac{1}{2}} \pm \frac{D}{\alpha}\right| = c - \lambda\omega_1, \quad \{\lambda, \alpha\} \neq 0;$$

9) $$\rho = \exp\left\{\frac{2}{D}y(\omega_1) + \frac{2}{D}z(\omega_2)\right\}, \quad A_k = \frac{\partial\varphi}{\partial x_k}, \quad F = 0,$$
$$\varphi = x_0 x_3 - \frac{x_0^3}{3} + y(\omega_1) + z(\omega_2), \quad \omega_1 = x_3 - \frac{x_0^2}{2}, \quad \omega_2 = (x_1^2 + x_2^2)^{\frac{1}{2}};$$

10) $$\rho = \exp\left\{\frac{2x_1}{x_0} - (3 + 2\lambda)\ln x_0 + 2c\right\},$$
$$A_k = \lambda\delta^{1k} + \frac{x_k}{x_0}, \quad c \in \mathbb{R}, \quad F = 0;$$

11) $$\rho = \exp\{(2x_3 - x_0^2)^2\}, \quad A_k = (x_0 + 1)\delta^{3k},$$
$$F = \rho(2\sqrt{2}\ln^{\frac{1}{2}}\rho - 4D\ln\rho - 2D);$$

12) $$\rho = \exp\left\{\frac{2}{\alpha}x_3\exp(\lambda x_0) - \exp(\lambda x_0)\Phi_{-\lambda}(x_0)\right\}, \quad \alpha \in \mathbb{R},$$
$$A_1 = \frac{x_1}{x_0}, \quad A_2 = \frac{x_2}{x_0}, \quad A_3 = 0, \quad F = \lambda\rho\ln\rho,$$

and $\Phi_{-\lambda}(x_0)$ can be determined via relation:

$$\Phi_\gamma(x_0) = \int \frac{\exp(\gamma x_0)}{x_0} dx_0; \tag{12}$$

13) $$\rho = \exp\left\{2\exp(\lambda x_0)\operatorname{arctg}\frac{x_2}{x_1} - \exp(\lambda x_0)\Phi_{-\lambda}(x_0)\right\},$$
$$A_1 = \frac{x_1}{x_0}, \quad A_2 = \frac{x_2}{x_0}, \quad A_3 = 0, \quad F(p) = \lambda\rho\ln\rho,$$

where $\Phi_{-\lambda}(x_0)$ is determined in (12);

14) $$\rho = \exp\left\{2x_1\exp(\lambda x_0) - \exp(\lambda x_0)\Phi_{-\lambda}(x_0) - 2\left(\frac{c}{\lambda^2} - \frac{D}{\lambda}\right)\exp(2\lambda x_0)\right\},$$
$$A_1 = \frac{c}{\lambda}\exp(\lambda x_0), \quad A_2 = \frac{x_2}{x_0}, \quad A_3 = 0, \quad F = \lambda\rho\ln\rho, \quad c \in \mathbb{R}; \quad \lambda \neq 0;$$

15) $$\rho = \exp\left\{2x_1\exp(\lambda x_0) - 2\exp(\lambda x_0)\Phi_{-\lambda}(x_0) - 2\left(\frac{c}{\lambda^2} - \frac{D}{\lambda}\right)\exp(2\lambda x_0)\right\},$$
$$A_1 = \frac{c}{\lambda}\exp(\lambda x_0), \quad A_2 = \frac{x_2}{x_0}, \quad A_3 = \frac{x_3}{x_0}, \quad F = \lambda\rho\ln\rho; \quad \lambda \neq 0;$$

16) $$\rho = \exp\left\{\frac{2}{\lambda\alpha}\exp(\lambda x_0) \pm 2\left(c + 2\left(-\frac{\lambda}{D}x_3\right)^{\frac{1}{2}}\right)^2\right\}, \quad \{\lambda, \alpha\} \neq 0,$$
$$A_k = 0, \quad F = \lambda\rho\ln\rho, \quad \{c, \alpha\} \subset \mathbb{R}.$$

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