Invariants of one-parameter subgroups of the conformal group $C(1, n)$

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Досліжена структура інваріантів одної з основних груп симетрії математичної фізики — конформної групи $C(1,n)$ у просторі Мінковського $R(1,n)$ для $n \geq 3$. Зокрема, побудовані повні системи інваріантів однопараметричних підгруп групи $C(1,n)$.

The conformal group $C(1,n)$ of transformations in the Minkovsky space $R(1,n)$ takes the central place among the invariance Lie groups of mathematical physics [1]. Our interest in the functional invariants of one-parameter subgroups has been motivated by their application in finding solutions of diferential equations. In Ref. [2] substitution (ansatz)

$$u(x) = f(x)\varphi(\omega) + g(x) \tag{1}$$

for construction of exact solutions of multi-dimensional equations is proposed. In ansatz (1) the unknown function $\varphi(\omega)$ depends on the complete system of functional invariants $\omega_1, \omega_2, \ldots, \omega_m$ of the one-parameter subgroups of the invariance Lie group of the given equation.

In this paper complete systems of functional invariants of one-parameter subgroups of the conformal group $C(1,n)$ ($n \geq 3$) are obtained. It should be noted that analogous problem for Poincaré groups $P(1,n)$ and $P(2,n)$ in Refs. [3] and [4] is determined.

It is known that to each local Lie group corresponds its Lie algebra, in particular, to group $C(1,n)$ corresponds Lie algebra $AC(1,n)$. Generators P_α, J_{0a}, J_{ab}, D, K_α ($\alpha = \overline{0,n}$, $a,b = \overline{1,n}$ generate the basis of Lie algebra $AC(1,n)$. We shall consider the Lie algebra $AC(1,n)$ as algebra of differential operators determined in the space of scalar functions $u(x)$ ($x \in R(1,n)$):

$$\begin{aligned} &P_\alpha = -\partial_\alpha = -\frac{\partial}{\partial x_\alpha}, \quad J_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha, \quad x_\alpha = g_{\alpha\beta} x^\beta, \\ &g_{\alpha\beta} = (1, -1, \ldots, -1) \times \delta_{\alpha\beta}, \quad D = -x^\alpha \partial_\alpha, \quad K_\alpha = 2x_\alpha D + s^2 \partial_\alpha \\ &(s^2 \equiv x^\alpha x_\alpha = x_0^2 - x_1^2 - \ldots - x_n^2), \quad \alpha, \beta = \overline{0,n}. \end{aligned} \tag{2}$$

It should be noted that the Lie algebra $AC(1,n)$ (2) is an invariance algebra of many differential equations [1].

The function $F(x)$ ($x \in R(1,n)$) is the invariant of the on-parameter subgroups of the group $C(1,n)$ if and only if it is the solution of the differential equation

$$Lu = 0, \tag{3}$$

where L is the corresponding one-dimensional subalgebra of the Lie algebra $AC(1,n)$ (2) (see, for example, [5]).

Consequently, the problem of finding of invariants of one-parameter subgroups, of the group $C(1,n)$ is reduced to finding the system of functionally-independent

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solutions of equation (3). Such systems of functionally-independent solutions of equation (3) will be called complete systems of invariants (CSI) of the corresponding one-dimensional subalgebras of the algebra $AC(1,n)$.

Lie algebra $AP(1,n) = \langle P_\alpha, J_{\alpha\beta} \,|\, \alpha = \overline{0,n},\ \beta = \overline{1,n}\rangle$ is subalgebra the algebra $AC(1,n)$. CSI of one-dimensional subalgebras of the algebra $AP(1,n)$ are constructed in [3]. Consequently, we shall describe CSI of the one-dimensional subalgebras of the factor algebra $AC(1,n)/AP(1,n)$.

One-dimensional subalgebras of the algebra $AC(1,n)/AP(1,n)$ are such algebras [3, 6]:

$$
\begin{aligned}
&L_1 = \langle D + \alpha J_{0n}\rangle\ (0 < \alpha \le 1); \quad L_2 = \langle D + J_{0n} + M\rangle;\\
&L_3 = \langle X_t + \alpha D + \beta J_{0n}\rangle\ (\alpha \ge \beta \ge 0,\ \alpha \ne 0);\\
&L_4 = \langle X_t + \alpha(D + J_{0n} + M)\rangle\ (\alpha > 0);\\
&L_5 = \langle J_{12} + \beta_1 J_{34} + \cdots + \beta_{\frac{n}{2}-1} J_{n-1,n} + \gamma D\rangle\\
&(n \equiv 0\ (\mathrm{mod}\,2),\ \gamma > 0,\ 0 \le \beta_1 \le \cdots \le \beta_{\frac{n}{2}-1} \le 1);\\
&L_6 = \langle S + T\rangle; \quad L_7 = \langle S + T \pm M\rangle;\\
&L_8 = \langle X_t + \alpha(S+T)\rangle\ (\alpha > 0); \quad L_9 = \langle S + T + \alpha Z\rangle\ (\alpha > 0);\\
&L_{10} = \langle X_t + \alpha(S+T) \pm M\rangle\ (\alpha > 0);\\
&L_{11} = \langle X_t + S + T + G_1 + P_2\rangle; \quad L_{12} = \langle X_t + \alpha(S+T) + \beta Z\rangle\ (\alpha, \beta > 0);\\
&L_{13} = \langle P_0 + K_0\rangle; \quad L_{14} = \langle \alpha(P_0 + K_0) + J_{12}\rangle\ (\alpha > 0);\\
&L_{15} = \langle \alpha(P_0 + K_0) + J_{12} + \beta_1 J_{34} + \cdots + \beta_s J_{2s+1,2s+2}\rangle\\
&(\alpha > 0;\ 0 < \beta_1 \le \ldots \le \beta_s \le 1;\ s = 1, 2, \ldots, [(n-2)/2]);\\
&L_{16} = \langle \alpha(P_0 + L_0) + J_{12} + \gamma_1 J_{34} + \cdots + \gamma_{\frac{n-3}{2}} J_{n-2,n-1} + \gamma_{\frac{n-1}{2}}(K_n - P_n)\rangle\\
&(\alpha > 0,\ 0 < \gamma_1 \le \cdots \le \gamma_{\frac{n-1}{2}} \le 1);\\
&L_{17} = \langle J_{12} + \gamma_1 J_{34} + \cdots + \gamma_{\frac{n-1}{2}}(K_n - P_n)\rangle\\
&(0 < \gamma_1 \le \cdots \le +\gamma_{\frac{n-1}{2}} \le 1),
\end{aligned}
\tag{4}
$$

where

$$
\begin{aligned}
&X_t = \alpha_1 J_{12} + \alpha_2 J_{34} + \cdots + \alpha_t J_{2t-1,2t}\\
&(\alpha_1 = 1,\ 0 \le \alpha_2 \le \cdots \le \alpha_t \text{ if } t \ne 1;\ t = 1, 2, \ldots, [(n-1)/2]),\\
&M = P_0 + P_n, \quad T = \tfrac{1}{2}(P_0 - P_n), \quad S = \tfrac{1}{2}(K_0 + K_n), \quad G_1 = J_{01} - J_{1n},
\end{aligned}
$$

$Z = J_{0n} - D$. In algebras L_{16} and L_{17} value n is an odd number.

Let $y = y(x) = x_0 + x_n$, $z = z(x) = x_0 - x_n$, $h_a = h_a(x) = x_{2a-1}^2 + x_{2a}^2$, $\varphi_a = \varphi_a(x) = \operatorname{arctg} \frac{x_{2a}}{x_{2a-1}}$, $\psi = \psi(x) = x_1^2 + x_2^2 + \cdots + x_{n-1}^2$.

Record $L : f_1(x), f_2(x), \ldots, f_s(x)$ designates that functions $f_1(x), f_2(x), \ldots, f_s(x)$ form CSI of the algebra L.

Theorem. *Following functions are CSI of one-dimensional subalgebras of the algebra $AC(1,n)/AP(1,n)$:*

$$
L_1\colon\quad z x_1^{-1-\alpha},\ y x_1^{\alpha-1},\ x_2 x_1^{-1},\ x_3 x_1^{-1},\ \ldots,\ x_{n-1} x_1^{-1};
$$

L_2: $y-\ln|z|,\ y-2\ln|x_1|,\ x_2x_1^{-1},\ x_3x_1^{-1},\ \ldots,\ x_{n-1}x_1^{-1};$

L_3: $z^{\alpha}x_{2t+1}^{-\alpha-\beta},\ y^{\alpha}x_{2t+1}^{\beta-\alpha},\ \alpha\varphi_1-\alpha_1\ln|x_{2t+1}|,\ \alpha\varphi_2-\alpha_2\ln|x_{2t+1}|,\ \ldots,$
$\alpha\varphi_t-\alpha_t\ln|x_{2t+1}|,\ h_1x_{2t+1}^{-2},\ h_2x_{2t+1}^{-2},\ \ldots,\ h_tx_{2t+1}^{-2},\ x_{2t+2}x_{2t+1}^{-1},$
$x_{2t+3}x_{2t+1}^{-1},\ \ldots,\ x_{n-1}x_{2t+1}^{-1};$

L_4: $zx_{2t+1}^{-2},\ y-2\ln|x_{2t+1}|,\ \alpha\varphi_1-\alpha_1\ln|x_{2t+1}|,\ \alpha\varphi_2-\alpha_2\ln|x_{2t+1}|,\ \ldots,$
$\alpha\varphi_t\ln|x_{2t+1}|,\ h_1x_{2t+1}^{-2},\ h_2x_{2t+1}^{-2},\ \ldots,\ h_tx_{2t+1}^{-2}, h_2x_{2t+1}^{-2},\ \ldots,\ h_tx_{2t+1}^{-2},$
$x_{2t+2}x_{2t+1}^{-1},\ x_{2t+3}x_{2t+1}^{-1},\ \ldots,\ x_{n-1}x_{2t+1}^{-1};$

L_5: $\ln h_1-2\gamma\varphi_1,\ \beta_1\ln h_1-2\gamma\varphi_2,\ \ldots,\ \beta_{\frac{n}{2}-1}\ln h_1-2\gamma\varphi_{\frac{n}{2}}\ln h_1-2\gamma\varphi_{\frac{n}{2}},$
$x_0^2h_1^{-1},\ h_2h_1^{-1},\ \ldots,\ h_3h_1^{-1},\ \ldots,\ h_{\frac{n}{2}}h_1^{-1};$

L_6: $(1+z^2)x_1^{-2},\ y-z\psi(1+z^2)^{-1},\ x_2x_1^{-1},\ x_3x_1^{-1},\ \ldots,\ x_{n-1}x_1^{-1};$

L_7: $y\pm 2\,\mathrm{arctg}\,z-z(1+z^2)^{-1}\psi,\ (1+z^2)x_1^{-2},\ x_2x_1^{-1},$
$x_3x_1^{-1},\ \ldots,\ x_{n-1}x_1^{-1};$

L_8: $y-z(1+z^2)^{-1}\psi,\ (1+z^2)h_1^{-1},\ \alpha\varphi_1-\alpha_1\,\mathrm{arctg}\,z,\ x_{2t+1}^2h_1^{-1},\ h_2h_1^{-1},$
$h_3h_1^{-1},\ \ldots,\ h_th_1^{1},\ \alpha_2\varphi_1-\alpha_1\varphi_2,\ \alpha_3\varphi_1-\alpha_1\varphi_3,\ \ldots,$
$\alpha_t\varphi_1-\alpha_1\varphi_t,\ x_{2t+2}x_{2t+1}^{-1},\ x_{2t+3}x_{2t+1}^{-1},\ \ldots,\ x_{n-1}x_{2t+1}^{-1};$

L_9: $2\alpha\,\mathrm{arctg}\,z+\ln(x_1^2(1+z^2)^{-1}),\ 2\alpha\,\mathrm{arctg}\,z+\ln(y+z\psi(1+z^2)^{-1}),$
$x_2x_1^{-1},\ x_3x_1^{-1},\ \ldots,\ x_{n-1}x_1^{-1};$

L_{10}: $\alpha y\pm 2\,\mathrm{arctg}\,z-\alpha z(1+z^2)^{-1}\psi,\ \alpha\varphi_1-\alpha_1\,\mathrm{arctg}\,z,\ (1+z^2)h_1^{-1},$
$x_{2t+1}^2h_1^{-1},\ h_2h_1^{-1},\ h_3h_1^{-1},\ \ldots,\ h_th_1^{-1},\ x_{2t+2}x_{2t+1}^{-1},\ x_{2t+3}x_{2t+1}^{-1},\ \ldots,$
$x_{n-1}x_{2t+1}^{-1},\ \alpha_2\varphi_1-\alpha_1\varphi_2,\ \alpha_3\varphi_1-\alpha_1\varphi_3,\ \ldots,\ \alpha_t\varphi_1-\alpha_1\varphi_t;$

L_{11}: $(1+z^2)x_{2t+1}^{-2},\ (x_1+zx_2)(1+z^2)^{-1},\ \varphi_2-\alpha_2\,\mathrm{arctg}\,z,$
$y+2(x_1+zx_2)(1+z^2)^{-1}\,\mathrm{arctg}\,z-z\psi(1+z^2)^{-1},$
$\mathrm{arctg}\,z-(x_2-zx_1)(1+z^2)^{-1},\ h_2x_{2t+1}^{-2},\ h_3x_{2t+1}^{-2},\ \ldots,\ h_tx_{2t+1}^{-2},$
$x_{2t+2}x_{2t+1}^{-1},\ x_{2t+3}x_{2t+1}^{-1},\ \ldots,\ x_{n-1}x_{2t+1}^{-1};$

L_{12}: $2\beta\,\mathrm{arctg}\,z+\alpha\ln|y-z\psi(1+z^2)^{-1}|,\ 2\beta\,\mathrm{arctg}\,z+\alpha\ln h_1(1+z^2)^{-1},$
$\alpha\varphi_1-\alpha_1\,\mathrm{arctg}\,z,\ \alpha_2\varphi_1-\alpha_1\varphi_2,\ \alpha_3\varphi_1-\alpha_1\varphi_3,\ \ldots,\ \alpha_t\varphi_1-\alpha_1\varphi_t,$
$h_2h_1^{-1},\ h_3h_1^{-1},\ \ldots,\ h_th_1^{-1},\ x_{2t+2}x_{2t+1}^{-1},\ x_{2t+3}x_{2t+1}^{-1},\ \ldots,\ x_{n-1}x_{2t+1}^{-1};$

L_{13}: $(yz-\psi+1)x_1^{-1},\ x_2x_1^{-1},\ x_3x_1^{-1},\ \ldots,\ x_nx_1^{-1};$

L_{14}: $2\alpha\varphi_1-\mathrm{arctg}((yz-\psi-1)(2x_0)^{-1}),\ (yz-\psi+1)x_3^{-1},$
$h_1x_3^{-1},\ x_4x_3^{-1},\ x_5x_3^{-1},\ \ldots,\ x_nx_3^{-1};$

L_{15}: $2\alpha\varphi_1-\mathrm{arctg}((yz-\psi-1)(2x_0)^{-1}),\ (yz-\psi+1)^2h_1^{-1},\ \beta_1\varphi_1-\varphi_2,$
$\beta_2\varphi_1-\varphi_3,\ \ldots,\ \beta_s\varphi_1-\varphi_{s+1},\ h_2h_1^{-1},\ h_3h_1^{-1},\ \ldots,\ h_{s+1}h_1^{-1},$
$x_{2s+3}^2h_1^{-1},\ x_{2s+4}x_{2s+3}^{-1},\ x_{2s+5}x_{2s+3}^{-1},\ \ldots,\ x_nx_{2s+3}^{-1};$

L_{16}: $2\alpha\varphi_1-\mathrm{arctg}((yz-\psi-1)(2x_0)^{-1}),$
$2\gamma_{\frac{n-1}{2}}\varphi_1-\mathrm{arctg}((yz-\psi+1)(2x_n)^{-1}),\ ((yz-\psi-1)^2+4x_0^2)(4h_1)^{-1},$
$h_2h_1^{-1},\ h_3h_1^{-1},\ \ldots,\ h_{\frac{n-1}{2}}h_1^{-1},\ \gamma_1\varphi_1-\varphi_2,\ \gamma_2\varphi_1-\varphi_3,\ \ldots,$
$\gamma_{\frac{n-3}{2}}\varphi_1-\varphi_{\frac{n-1}{2}};$

L_{17}: $2\gamma_{\frac{n-1}{2}}\varphi_1-\mathrm{arctg}((yz-\psi+1)(2x_n)^{-1}),\ (yz-\psi-1)x_0^{-1},\ \gamma_1\varphi_1-\varphi_2,$
$\alpha_2\varphi_1-\varphi_3,\ \ldots,\ \gamma_{\frac{n-3}{2}}\varphi_1-\varphi_{\frac{n-1}{2}},\ h_2h_1^{-1},\ h_3h_1^{-1},\ \ldots,\ h_{\frac{n-1}{2}}h_1^{-1},\ h_1x_0^{-2}.$

Values of numerical parameters are given in expression (4).

In order to prove the theorem it is sufficient to verify that each CSI satisfies the equation (3).

1. Фущич В.И., Штелень В.М., Серов Н.И., Симметрийный анализ и точные решения нелинейных уравнений математической физики, Киев, Наук. думка, 1989, 336 с.
2. Фущич В.И., Симметрия в задачах математической физики, в сб. Теоретико-алгебраические исследования в математической физике, Киев, Ин-т математики АН УССР, 1981, 6–28.
3. Фущич В.И., Баранник Л.Ф., Баранник А.Ф., Подгрупповой анализ групп Галилея, Пуанкаре и редукция нелинейных уравнений, Киев, Наук. думка, 1991, 304 с.
4. Баранник Л.Ф., Лагно В.И., Однопараметрические подгруппы обобщенной группы Пуанкаре $P(1,n)$ и их инварианты, *Укр. мат. журн.*, 1989, **41**, № 9, 1169–1172.
5. Овсянников Л.В., Групповой анализ дифференциальных уравнений, М., Наука, 1978, 400 с.
6. Баранник Л.Ф., Фущич В.И., О непрерывных подгруппах конформной группы пространства Минковского $R(1,n)$, Препринт № 88.34, Киев, Ин-т математики АН УССР, 1988, 48 с.