Conditional symmetries of the equations of mathematical physics

W.I. FUSHCHYCH

We briefly present the results of research in conditional symmetries of equations of mathematical and theoretical physics: the Maxwell, D'Alembert, Schrödinger and KdV equations, as well as the equations of heat conduction and acoustics. Exploiting conditional symmetry, we construct a wide class of exact solutions of these equations, which cannot be obtained by the classical method of Sophus Lie.

1. Introduction

The concept and terminology of conditional symmetry and conditional invariance were introduced and developed in the series of articles [1–11] (see also Mathematical Reviews for the years 1983–1993). Later, this concept was exploited by other authors for the construction of solutions of various non-linear equations of mathematical physics. It turned out that nearly all the basic non-linear equations of mathematical physics have non-trivial conditional symmetry [2, 9, 10].

We understand the conditional symmetry of an equation as being a symmetry (local or non-local) of some non-trivial subset of its solution set (the formal definition of the idea of conditional symmetry can be found in Appendix 4 of [2] and in the article [3]). The general definition of conditional symmetry as the symmetry of a subset of the set of solutions is non-constructive and requires further specification: the analytical description of a condition (as an equation) on the solutions of the given equation, which extend or alter the symmetry of the starting equation. Therefore, the basic problem in the investigation of conditional symmetries is that of describing those supplementary equations which increase or change the symmetry of the beginning equation. This is very complex, non-linear problem in general (even in the case of quite simple non-linear equations), which can often be significantly more complicated than constructing solutions of the equation at hand. It is thus meaningful to talk of the conditional symmetry of some class of equations.

Non-trivial conditional symmetries of a PDE (partial differential equation) allows us to obtain in explicit form such solutions which can not be found by using the symmetries of the whole set of solutions of the given PDE. Moreover, conditional symmetries increase significantly the class of PDEs for which we can construct ansatzes which reduce these equations to (systems of) ODEs (ordinary differential equations). As a rule, the reduced equations one obtains from conditional symmetries are significantly simpler than those found by reduction using symmetries of the full set of solutions. This allows us to construct exact solutions of the reduced equations.

Looking back, we can say today, that many mathematicians, mechanicians and physicists, such as Euler, D'Alembert, Poincaré, Volterra, Whittaker, Bateman, implicitly used conditional symmetries for the construction of exact solutions of the linear

in Proceedings of the International Workshop "Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics", Editors N.H. Ibragimov, M. Torrisi, A. Valenti, Dordrecht, Kluwer Academic Publishers, 1993, P. 231–239.

wave equation. Some well-known solutions of this equation can not be obtained by using only Lie symmetries of the full solution set.

2. Conditional symmetry of Maxwell's equation

We shall first consider the first pair of Maxwell's equations

$$\frac{\partial \boldsymbol{E}}{\partial t} = \operatorname{rot} \boldsymbol{H}, \quad \frac{\partial \boldsymbol{H}}{\partial t} = -\operatorname{rot} \boldsymbol{E}. \tag{1}$$

The maximal invariance algebra (in the sense of Lie) of these equations is studied in [2]. The basis elements of this algebra $\langle \partial_0, \partial_a, J_{ab}, D \rangle$ are

$$\begin{aligned} &\partial_0 = \frac{\partial}{\partial x^0}, \quad \partial_a = \frac{\partial}{\partial x^a}, \quad J_{ab} = x_a \partial_b - x_b \partial_a + s_{ab}, \quad a, b = 1, 2, 3, \\ &D = x^\mu \partial_\mu + \mathrm{const}, \end{aligned} \tag{2}$$

s_{ab} are 6×6 matrices realizing a representation of the group $O(3)$. Thus the system (1) is invariant under the four-dimensional translations ∂_μ, the rotations J_{ab} and scale transformations D, but it is not invariant under the Lorentz boosts

$$J_{0a} = x_0 \partial_a - x_a \partial_0 + s_{0a}, \quad x_0 = t, \tag{3}$$

the matrices $\langle s_{0a}, s_{ab} \rangle$ realizing a representation of the Lorentz group $O(1,3)$.

Theorem 1 ([2] 1983, [15] 1987). *The system (1) is conditionally invariant under the Lorentz boosts (3) if and only if the solutions of (1) satisfy the conditions*

$$\operatorname{div} \boldsymbol{E} = 0, \quad \operatorname{div} \boldsymbol{H} = 0. \tag{4}$$

It is evident from this theorem, that the concept of conditional invariance of a PDE is natural, and leads us, by purely group-theoretic means, to the fundamental, overdetermined system of Maxwell's equations.

3. Conditional symmetry of the wave equation

We now examine the non-linear D'Alembert equation

$$\Box u = F(u), \quad u = u(x_0, x_1, x_2, x_3), \tag{5}$$

$F(u)$ being an arbitrary, smooth function. Equation (5) has conformal symmetry $C(1,3)$ if and only if $F = \lambda u^3$ or $F = 0$ (see for instance [8, 10]). This is the maximal symmetry of all of the solution set of equation (5). For an arbitrary function, (5) admits only the symmetry groups $P(1,3)$.

Theorem 2 ([5], 1985). *Equation (5), with $F = 0$ is conditionally invariant under the infinite-dimensional algebra with basis elements*

$$X = \xi^\mu(x,u) \frac{\partial}{\partial x^\mu} + \eta(x,u) \frac{\partial}{\partial u}, \tag{6}$$

$$\xi^\mu(x,u) = c^{00}(u) x^\mu + c^{\mu\nu}(u) x_\nu + d^\mu(u), \quad \eta(x,u) = \eta(u), \tag{7}$$

where $c^{00}(u)$, $c^{\mu\nu}(u)$, $d^\mu(u)$, $\eta(u)$ are arbitrary functions of u, if one imposes the condition

$$\frac{\partial u}{\partial x_\mu} \frac{\partial u}{\partial x^\mu} = 0. \tag{8}$$

In this way, the eikonal equation (8), significantly increases the symmetry of the starting equation (5). The system of equations (5), (8), with $F = 0$, is consistent.

Theorem 3 ([10, 15], 1988, 1989). *The equation (5) is conditionally invariant under the conformal group, if*

$$F = \frac{3\lambda}{u + c}, \tag{9}$$

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} = \lambda, \tag{10}$$

where λ, c are arbitrary constants. The operators of conformal symmetry are

$$\begin{aligned} &K_\mu = 2x_\mu D - (x_\alpha x^\alpha - u^2)\frac{\partial}{\partial x^\mu}, \quad \mu = 0, 1, 2, 3, \\ &D = x^\mu \frac{\partial}{\partial x^\mu} + u\frac{\partial}{\partial u}. \end{aligned} \tag{11}$$

Remark. It is important to note, that the operators (11) differ principally from the conformal operators for equation (5), when $F = 0$ or $F = \lambda u^3$. In those cases, the conformal operators are

$$\hat{K}_\mu = 2x_\mu D - x_\alpha x^\alpha \frac{\partial}{\partial x^\mu}, \quad D = x^\mu \frac{\partial}{\partial x^\mu}. \tag{12}$$

The operators (11) are *non-linear*, whereas those in (12) are linear.

Thus the wave equation (5), (9), with non-linear condition (10), has a symmetry possessed by neither the solution set for the linear equation, nor that for the nonlinear equation.

4. Criteria for conditional symmetry

Let us consider some PDE

$$\begin{aligned} &L(x, u_{(1)}, u_{(2)}, \ldots, u_{(n)}) = 0, \\ &u_{(1)} = (u_0, u_1, \ldots, u_n), \quad u_{(2)} = (u_{01}, u_{02}, \ldots, u_{nn}), \quad \ldots, \\ &u_\mu = \frac{\partial u}{\partial x^\mu}, \quad u_{\mu\nu} = \frac{\partial^2 u}{\partial x^\mu \partial x^\nu}, \quad \ldots. \end{aligned} \tag{13}$$

Definition 1 (S. Lie, 1884). *Equation (13) is invariant with respect to the operator (6) if*

$$X_s L = \lambda L, \tag{14}$$

where X_s is the s-th prolongation of (6), and $\lambda = \lambda(x, u)$ is an arbitrary function.

Let us denote by the symbol

$$Q = \langle Q_1, Q_2, \ldots, Q_r \rangle \tag{15}$$

some set of operators which does not belong to the invariance algebra (IA) of equation (13).

Definition 2 ([2], 1987). *Equation(13) is said to be conditionally invariant under the operators Q from (15), if there exists a supplementary condition on the solutions of (13) of the form*

$$L_1(x,u,u_{(1)},\dots,u_{(n)})=0 \tag{16}$$

such that (13) together with (16) is invariant under the Q.

Thus one has the following conditions

$$Q_sL=\lambda_0L+\lambda_1L_1, \tag{17}$$

$$Q_sL_1=\lambda_2L+\lambda_3L_1 \tag{18}$$

or

$$Q_sL\Big|_{\substack{L=0\\ L_1=0}}=0,\quad Q_sL_1\Big|_{\substack{L=0\\ L_1=0}}=0. \tag{19}$$

An important class of supplementary conditions (16) is that for which the equation $L_1=0$ is a quasi-linear equation of first order

$$L_1(x,u,u_{(1)})\equiv Qu=0, \tag{20}$$

$$Q=y^\mu(x,u)\frac{\partial}{\partial x^\mu}+z(x,u)\frac{\partial}{\partial u} \tag{21}$$

with y^μ, z being smooth functions. In this case, we shall say that (13) is Q-conditionally invariant.

In this way, the problem of finding the conditional symmetry of (13) reduces to the solution of the equations (17), (18). The conditions (16), (20) can be considered as equations for the construction of ansatzes for the starting equation (13). The problem of calculating the conditional symmetry is far more complicated than the usual method of Lie for finding the symmetry of the full solution set. In the case of conditional symmetries, the defining equations are, as a rule, non-linear equations which can be solved in only some cases. Fortunately, for most of the equations of non-linear mathematical physics, one can construct partial solutions of the defining equations.

5. A list of equations with non-trivial conditional symmetry

Conditional symmetries began to be exploited only quite recently, and the first publications appeared only in 1983 [1, 2]. Now, the number of articles in this are is increasing rapidly with each year, and therefore it is difficult to give a complete list (for 1992) of important equations of mathematical physics possessing conditional symmetry. So I shall only give those equations which we have studied and which are interested from our Kievan point of view. We have put in brackets the year(s) when the conditional symmetry of the given equation was found. More detailed information about ansatzes and solutions of the above equations are to be found in the original articles, a list of which are given in [2, 9, 11].

$$1.\quad u_0+u_{11}=F(u)=\begin{cases}\lambda u(u^2-1),\\ \lambda(u^3-3u+2),\\ \lambda u^3,\\ \lambda u(u^3+1).\end{cases} \qquad (1988,\ 1990)$$

2. $iu_0 + \Delta u + F(|u|)u = 0,$
$F(|u|) = \lambda_1 |u|^{4/r} + \lambda_2 |u|^{-4/r}, \quad F(|u|) = \lambda_3 \ln(u^* u),$ (1990)
$\lambda_1,\ \lambda_2,\ r$ arbitrary, real; $\quad \lambda_3$ arbitrary, complex.

3. $u_{00} = u\Delta u, \quad u_{00} = c(x, u, u_{(1)})\Delta u.$ (1987, 1988)

4. $u_{01} - (F(u)u_1)_1 - u_{22} - u_{33} = 0.$ (1990)

5. $u_0 + \nabla(F(u)\nabla u) = 0.$ (1988)

6. $u_0 + F(u)u_1^k + u_{111} = 0.$ (1991)

7. $u_0 + (\varphi(u))_{11} + \dfrac{N}{x_1}(\varphi(u))_1 = F(u),$
$u_0 + u_{11} + \dfrac{3}{2x_1}u_1 = \lambda u^3,$ (1992)
$u_0 + uu_{11} + \dfrac{N}{x_1}uu_1 = \lambda u + \lambda_2.$

8. $\boldsymbol{u}_0 + (\boldsymbol{u}\nabla)\boldsymbol{u} = -\dfrac{1}{\rho}\nabla p,$
$\rho_0 + \operatorname{div}(\rho \boldsymbol{u}) = 0, \quad p = f(\rho), \quad p = \dfrac{1}{2}\rho^2.$ (1992)

9. $\gamma^\mu \partial_\mu \Psi + F(\bar{\Psi}\Psi)\Psi = 0.$ (1989)

10. $(1 - u_\alpha u^\alpha)\Box u + u^\mu u^\nu u_{\mu\nu} = 0.$ (1989)

6. Conditional symmetry and exact solutions of KdV type equations

To illustrate the constructive nature of conditional symmetries, we shall examine the equation

$$u_0 + F(u)u_1^k + u_{111} = 0, \tag{22}$$

where $F(u)$ is a smooth function, $k \neq 0$ is an arbitrary, real parameter. When $F(u) = u$, $k = 1$, equation (22) coincides with the standard KdV equation.

Theorem 4 ([11], 1991). *Equation (22) is Q-conditionally invariant with respect to the following operators*

$$Q = x_0^r \partial_1 + H(x, u)\partial_u \tag{23}$$

with r an arbitrary, real parameter, in the following cases

1. $F(u) = \lambda_1 u^{(2-k)/k} + \lambda_2 u^{(1-k)/2}, \quad H(x, u) = \left(\dfrac{\lambda_1 k}{2}\right)^{-1/k} u^{1/2};$ (24)

2. $F(u) = (\lambda_1 \ln u)^{1-k}, \quad H(x, u) = (k\lambda_1)^{-1/k};$ (25)

3. $F(u) = (\lambda_1 \arcsin u + \lambda_2)(1 - u^2)^{(1-k)/2},$
$H(x, u) = (k\lambda_1)^{-1/k}(1 + u^2)^{1/2};$ (26)

$$4.\quad F(u) = (\lambda_1 \sinh^{-1} u + \lambda_2)(1+u^2)^{(1-k)/2},\quad H(x,u) = (k\lambda_1)^{-1/k}(1+u^2)^{1/2}; \tag{27}$$

$$5.\quad F(u) = \lambda_1 u, \quad H(x,u) = (k\lambda_1)^{-1/k}, \tag{28}$$

where $r = 1/k$, $k \neq 0$, λ_1, λ_2 are arbitrary, real parameters.

Exploiting the operator of conditional symmetry (23), one can construct ansatzes for the solutions of equation (22), some of which I now exhibit.

The ansatz

$$u = \left(\frac{x_1}{2}\left(\frac{k\lambda_1 x_0}{2}\right)^{-1/k} + \varphi(x_0)\right)^2$$

gives the solution

$$u = \left(\frac{x_1}{2}\left(\frac{k\lambda_1 x_0}{2}\right)^{-1/k} + \lambda x_0^{-1/k} - \lambda_2/\lambda_1\right)^2$$

when $F(u)$ is as in (24). The ansatz

$$u = \exp\left(\varphi(x_0) + (k\lambda_1 x_0)^{-1/k} x_1\right)$$

gives the solution

$$u = \exp\left(-\frac{k(k\lambda_1)^{-3/k}}{k-2} x_0^{1-3/k} + \lambda x_0^{-1/k} + (k\lambda_1 x_0)^{-1/k} x_1 - \lambda_2/\lambda_1\right)$$

when $F(u)$ is as in (25) with $k \neq 2$. The ansatz

$$u = \sin\left(\varphi(x_0) + (k\lambda_1 x_0)^{-1/k} x_1\right)$$

gives the solution

$$u = \sin\left(\frac{k(k\lambda_1)^{-3/k}}{k-2} x_0^{1-3/k} + \lambda x_0^{-1/k} + (k\lambda_1 x_0)^{-1/k} x_1 - \lambda_2/\lambda_1\right)$$

for $k \neq 2$.

Theorem 5 ([12], 1990). *The equation*

$$u_{01} - (F(u)u_1)_1 - u_{22} - u_{33} = 0 \tag{29}$$

is invariant under under the infinite-dimensional algebra

$$X = a_i(u) R_i, \quad i = 1, \ldots, 12, \tag{30}$$

where $a_i(u)$ are arbitrary, smooth functions, if one adds to (29) the condition

$$u_0 u_1 - F(u) u_1^2 - u_2^2 - u_3^2 = 0. \tag{31}$$

The operators R_i are given as follows:

$$R_{\mu+1} = \partial_\mu, \quad \mu = 0, \ldots, 3, \quad R_5 = x_3\partial_2 - x_2\partial_3, \quad R_6 = x_2\partial_1 + 2x_0\partial_2,$$
$$R_7 = x_3\partial_1 + 2x_0\partial_3, \quad R_8 = x^\mu \partial_\mu,$$

$$R_9 = x_0\partial_0 + 2x_1\partial_1 + 3x_2\partial_2 + 3x_3\partial_3 - 2\frac{F(u)}{\dot F(u)}\partial_u, \quad R_{10} = \dot F(u)x_0\partial_1 - \partial_u,$$
$$R_{11} = x_2\partial_0 + 2(x_1 + F(u)x_0)\partial_2, \quad R_{12} = x_3\partial_0 + 2(x_1 + F(u)x_0)\partial_3.$$

7. Antireduction

In [10], we have begun work on antireduction. By the term antireduction of a PDE we understand the finding of such ansatzes which transform the given PDE into a system of equations for some (unknown, and to be found) functions. In this process, the number of independent variables may remain the same, or be reduced (dimensional reduction), but the number of dependent variables increases. As a rule, one usually exploits the converse of this, that is, one reduces to a system with fewer dependent variables (reduction of components). To illustrate the effectiveness of antireduction, we consider the equation for short waves in gas dynamics

$$2u_{01} - 2(x_1 + u_1)u_{11} + u_{22} + 2\lambda u_1 = 0. \tag{32}$$

We impose the condition

$$\left[u_{111}x_1^{3/2}\right]_1 = 0 \tag{33}$$

on (32). The general solution of (33) is

$$u = x_1^{3/2}\varphi^1 + x_1^2\varphi^2 + x_1\varphi^3 + \varphi^4 \tag{34}$$

with $\varphi^i = \varphi^i(x_0, x_2)$, $i = 1, 2, 3, 4$ being arbitrary functions. Using (34) as an ansatz, equation is reduced to a system with two independent variables

$$\begin{aligned} &\varphi^3 = 0, \quad \varphi^1_{22} = 0, \quad \varphi^2_{22} = 0, \quad \varphi^4_{22} = \frac{9}{4}(\varphi^2)^2, \\ &\varphi^1_0 = \varphi^1\left(3\varphi^2 + \frac{1}{2} - \lambda\right), \quad \varphi^2_0 = 2\varphi^2_2 - \varphi_2(1-\lambda). \end{aligned} \tag{35}$$

Solving the system (35), we found exact solutions of the starting equation (32) [11].

The above results are only a sample of those already obtained. They illustrate the very fruitful nature of conditional symmetry and conditional invariance, and I hope that I have been able to demonstrate that there are new aspects to this concept which are yet to be exploited fully.

Acknowledgements

I would like to express my heartfelt thanks to Professors M. Torrisi and A. Valenti for such a well-organized conference and to Professor M. Boffi for enabling me to attend it. I would also like to thank Dr. P. Basarab-Horwath, Mathematics Department, Linköping University, Sweden, for useful and fruitful discussions during the final preparation of my talk.

1. Fushchych W., On symmetry and partial solutions of some multi-dimensional equations of mathematical physics, in Algebraic-theoretical methods in the problems of mathematical physics, Kiev, Institute of mathematics, Ukrainian Academy of Sciences, 1983, 4–23.
2. Fushchych W., Nikitin A., Symmetries of Maxwell's equations, Kiev, Naukova Dumka, 1983; Dordrecht, Kluwer, 1987.

3. Fushchych W., How can one extend the symmetry of differential equations? in Symmetry and solutions of non-linear equations in mathematical physics, Kiev, Institute of mathematics, Ukrainian Academy of Sciences, 1987, 4–16.
4. Fushchych W., Tsyfra I., On reduction and solutions of the nonlinear wave-equation with broken symmetry, *J. Phys. A*, 1987, **20**, L45–L48.
5. Shulha M., Symmetry and some exact solutions of the D'Alembert equation with a nonlinear condition, in Symmetry and solutions of equations of mathematical physics, Kiev, Institute of Mathematics, Ukrainian Academy of Sciences, 1985, 36–38.
6. Fushchych W., Serov M., Chopyk W., Conditional invariance and non-linear heat equations, *Proc. Ukr. Acad. Sci., Ser. A*, 1988, № 9, 17–21.
7. Fushchych W., Serov M., Conditional invariance and exact solutions of a non-linear equation of acoustics, *Proc. Ukr. Acad. Sci., Ser. A*, 1988, № 10, 27–31.
8. Fushchych W., Shtelen W., Serov M., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kiev, Naukova dumka, 1989; Dordrecht, Kluwer, 1993.
9. Fushchych W., Chopyk W., Conditional invariance of a non-linear Schrödinger equation, *Proc. Ukr. Acad. Sci., Ser. A*, 1990, № 4, 30–33.
10. Fushchych W., Conditional symmetry of the equations of mathematical physics, *Ukr. Math. J.*, 1991, **43**, № 11, 1456–1470.
11. Fushchych W., Serov M., Amerov T., On the conditional symmetry of the generalized Korteweg-de Vries equation, *Proc. Ukr. Acad. Sci., Ser. A*, 1991, № 12, 28–30.
12. Fushchych W., Chopyk W., Myronyuk P., Conditional invariance and exact solutions of three-dimensional non-linear equations in acoustics, *Proc. Ukr. Acad. Sci. Ser. A*, 1990, № 9, 25–28.
13. Fushchych W., Shtelen W., Serov M., Popowych R., Conditional symmetry of the linear heat equation, *Proc. Ukr. Acad. Sci., Ser. A*, 1992, № 12, 20–26.
14. Fushchych W., Repeta W., Exact solutions of the equations of gas dynamics and non-linear acoustics, *Proc. Ukr. Acad. Sci., Ser. A*, 1991, № 8, 35–42.
15. Fushchych W., Zhdanov R., Symmetry and exact solutions of non-linear spinor equations, *Phys. Rep.*, 1989, **46**, 325–365.