On maximal subalgebras of the rank $n-1$ of the conformal algebra $AC(1,n)$

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Проведено класифікацію максимальних підалгебр рангу $n-1$ алгебри $AC(1,n)$, які належать алгебрі $A\tilde{P}(1,n)$.

Consider the multidimensional eikonal equation

$$\left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \cdots - \left(\frac{\partial u}{\partial x_{n-1}}\right)^2 = 1, \tag{1}$$

where $u = u(x)$ is a scalar function of the variable $x = (x_0, x_1, \ldots, x_{n-1})$, $n \geq 2$. In [1] it was established that the Lie algebra $AC(1,n)$ of the group $C(1,n)$ of the Minkowski $R_{1,n}$ space with the metric $x_0^2 - x_1^2 - \cdots - x_n^2$, where $x_n = u$, is a maximal algebra of the equation (1) invariance. The basis of the algebra $AC(1,n)$ is formed by such vector fields as:

$$P_\alpha = \partial_\alpha, \quad J_{\alpha\beta} = g^{\alpha\gamma} x_\gamma \partial_\beta - g^{\beta\gamma} x_\gamma \partial_\alpha, \quad D = -x^\alpha \partial_\alpha,$$
$$K_\alpha = -2(g^{\alpha\beta} x_\beta) D - (g^{\beta\gamma} x_\beta x_\gamma)\partial_\alpha,$$

where $g_{00} = -g_{11} = \cdots = -g_{nn} = 1$, $g_{\alpha\beta} = 0$, when $\alpha \neq \beta$ $(\alpha, \beta, \gamma = 0, 1, \ldots, n)$. The algebra $AC(1,n)$ contains the Poincaré algebra $AP(1,n)$ which is generated by vector fields P_α, $J_{\alpha\beta}$ and the extended Poincaré algebra $A\tilde{P}(1,n) = AP(1,n) \subset\!\!\!\!+ \langle D \rangle$.

In order to reduce the equation (1) by subalgebras of the algebra $AC(1,n)$, it is necessary to describe all $C(1,n)$-nonequivalent subalgebras of this algebra. The subalgebras K_1 and K_2 of the algebra $AC(1,n)$ are called as $C(1,n)$-equivalent ones if they have the same invariants with respect to $C(1,n)$-conjugation. Among $C(1,n)$-equivalent algebras there exists one (maximal) subalgebra containing all the other subalgebras. The maximal subalgebras K_1 and K_2 of the algebra $AC(1,n)$ are equivalent if and only if K_1 and K_2 are $C(1,n)$-conjugated.

The maximal subalgebras of the rank n of the algebra $AP(1,n)$ with respect to $P(1,n)$-conjugation are described in [2]. The maximal subalgebras of the rank n of the algebra $A\tilde{P}(1,n)$ with respect to $\tilde{P}(1,n)$-conjugation are described in [3, 4]. The present article is a continuation of researches which were realized in [3, 4]. The full classification of the maximal subalgebras of the rank $n-1$ of the algebra $AC(1,n)$ which are contained in the algebra $A\tilde{P}(1,n)$ has been carried out in the present article. Ansatzes corresponding to these subalgebras reduce the equation (1) to ordinary differential equations.

We will use the notations:

$$M = P_0 + P_n, \quad T = \frac{1}{2}(P_0 - P_n), \quad G_a = J_{0n} - J_{an},\ a = 1, \ldots, n-1,$$
$$AO[r,s] = \langle J_{ab} \,|\, a, b = r, \ldots, s \rangle, \quad r \leq s,$$

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$$AE[r,s] = \langle P_r,\ldots,P_s\rangle \supset\!\!\!\!+ \ AO[r,s], \quad r \le s,$$
$$AE_1[r,s] = \langle G_r,\ldots,G_s\rangle \supset\!\!\!\!+ \ AO[r,s], \quad r \le s.$$

If $s > r$ then $AO[r,s] = 0$, $AE[r,s] = 0$ by definition.

Let

$$\Phi(r,s,\gamma) = \langle G_r + \gamma P_r,\ldots,G_s + \gamma P_s\rangle \supset\!\!\!\!+ \ AO[r,s], \quad r,s \in \mathbb{N},\ r \le s,\ \gamma \in \mathbb{R}.$$

Let

$$\Gamma_{d,q} = U \supset\!\!\!\!+ \ F,$$

where F is the diagonal of $AO[1,d] \oplus AO[d+1,2d] \oplus \cdots \oplus AO[(q-1)d+1,qd]$, and U is the Abelian algebra which has the basis

$$G_1 + \gamma_1 P_1 + \lambda_1 P_{(q-1)d+1},\ \ldots,\ G_d + \gamma_1 P_d + \lambda_1 P_{qd},$$
$$G_{d+1} + \gamma_2 P_{d+1} + \lambda_2 P_{(q-1)d+1},\ \ldots,\ G_{2d} + \gamma_2 P_{2d} + \lambda_2 P_{qd},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$G_{(q-2)d+1} + \gamma_{q-1} P_{(q-2)d+1} + \lambda_{q-1} P_{(q-1)d+1},\ \ldots,\ G_{(q-1)d} + {} + \gamma_{q-1} P_{(q-1)d} + \lambda_{q-1} P_{qd},$$

where $0 \le \gamma_1 < \gamma_2 < \cdots < \gamma_{q-1}$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\ldots$, $\lambda_{q-1} > 0$.

Results of the work [5] reduce the problem constructing invariants of any subalgebra of the algebra $A\tilde{P}(1,n)$ to the problem of constructing invariants of the irreducible subalgebras of the orthogonal algebra $AO(k)$ for all $k \le n$. The latter problem has no solution in quadratures. Therefore, we shall restrict ourself considering of such subalgebras of the algebra $A\tilde{P}(1,n)$ which projections onto $AO[1,n]$ are subdirect sums on the algebras $AO[r,s]$. Moreover, to find real solutions of the equation (1) it is necessary to exclude from consideration such subalgebras of the algebra $AP(1,n)$ which with respect to equivalence contain $P_0 + P_n$ or P_0. Therefore we prove the following theorems.

Theorem 1. *Let L be the maximal subalgebra of the rank $n-1$ of the algebra $AP(1,n)$. Then L is $C(1,n)$-conjugated with one of the following algebras:*

1) $L_1 = AE[1,n-1]$;
2) $L_2 = AO[1,m] \oplus AE[m+1,n]$, $m = 1,\ldots,n$, $n \ge 2$;
3) $L_3 = AE_1[1,m] \oplus AE[m+1,n-1]$, $m = 1,\ldots,n-1$, $n \ge 2$;
4) $L_4 = AO[1,m] \oplus AE[m+1,n-1] \oplus \langle J_{0n}\rangle$, $m = 1,\ldots,n-1$, $n \ge 3$;
5) $L_5 = AO[0,m] \oplus AE[m+1,n-1]$, $m = 2,\ldots,n-1$, $n \ge 3$;
6) $L_6 = AO[0,m] \oplus AO[m+1,q] \oplus AE[q+1,n-1]$, $m = 2,\ldots,n-1$, $q = m+1,\ldots,n$, $n \ge 3$;
7) $L_7 = \langle G_1 + P_0 - P_n\rangle \oplus AE[2,n-1]$, $n \ge 2$;
8) $L_8 = \Phi(d_0+1,d_1,\gamma_1) \oplus \cdots \oplus \Phi(d_{t-1}+1,m,\gamma_t) \oplus AE[m+1,n-1]$, $m = 1,\ldots,n-1$, $n \ge 3$;
9) $L_9 = \langle J_{0n} + P_1\rangle \oplus AE[2,n-1]$, $n \ge 2$;
10) $L_{10} = (AE_1[1,m] \oplus \langle J_{0n} + P_{m+1}\rangle) \oplus AE[m+2,n-1]$, $m = 1,\ldots,n-2$, $n \ge 3$;
11) $L_{11} = \langle J_{12} + P_0\rangle \oplus AE[3,n]$, $n \ge 2$.

Theorem 2. *Let L be the maximal subalgebra of the rank $n-1$ of the algebra $A\tilde{P}(1,n)$ which has a nonzero projection onto $\langle D\rangle$. Then L is $C(1,n)$-conjugated with one of the following algebras:*

1) $L_1 = (AO[0,d] \oplus AO[d+1,m] \oplus AO[m+1,q] \oplus AE[q+1,n]) \subset\!\!\!+ \langle D\rangle$, $d = 2,\ldots,n-2$, $m = d+1,\ldots,n-2$, $q = m+1,\ldots,n-1$, $2n \le d+q$, $n \ge 4$;

2) $L_2 = (AO[0,m] \oplus AE[m+1,n-2]) \subset\!\!\!+ \langle D+\alpha J_{n-1,n}\rangle$, $m = 2,\ldots,n-2$, $n \ge 4$, $\alpha > 0$;

3) $L_3 = (AO[1,m] \oplus AO[m+1,q] \oplus AE[q+1,n]) \subset\!\!\!+ \langle D\rangle$, $m = 2,\ldots,n-2$, $q = m+2,\ldots,n$, $2m \le q$, $n \ge 2$;

4) $L_4 = (AE_1[1,m] \oplus AE[m+1,n-3]) \subset\!\!\!+ \langle J_{n-2,n-1} + cJ_{0n}, D+\alpha J_{0n}\rangle$, $m = 1,\ldots,n-3$, $n \ge 4$, $c > 0$, $\alpha \ge 0$;

5) $L_5 = (AO[1,m] \oplus AO[m+1,q] \oplus AE[q+1,n-1]) \subset\!\!\!+ \langle D, J_{0n}\rangle$, $m = 1,\ldots,n-2$, $q = m+1,\ldots,n-1$, $2m \le q$, $n \ge 3$;

6) $L_6 = AE[3,n-1] \subset\!\!\!+ \langle J_{12} + cJ_{0n}, D+\alpha J_{0n}\rangle$, $c > 0$, $\alpha \ge 0$, $n \ge 3$;

7) $L_7 = (AE_1[1,d] \oplus AO[d+1,m] \oplus AE[m+1,n-1]) \subset\!\!\!+ \langle D+\alpha J_{0n}\rangle$, $d = 1,\ldots,n-2$, $m = d+1,\ldots,n-1$, $n \ge 3$, $\alpha \ge 0$;

8) $L_8 = (AO[1,m] \oplus AE[m+1,n-1]) \subset\!\!\!+ \langle D+\alpha J_{0n}\rangle$, $m = 1,\ldots,n-1$, $n \ge 2$, $\alpha \ge 0$;

9) $L_9 = (\langle G_1 + 2T\rangle \oplus AO[2,m] \oplus AE[m+1,n-1]) \subset\!\!\!+ \langle 2D - J_{0n}\rangle$, $m = 2,\ldots,n-1$, $n \ge 3$;

10) $L_{10} = (AE_1[1,d] \oplus AO[d+1,m] \oplus AE[m+1,n-1]) \subset\!\!\!+ \langle D + J_{0n} + M\rangle$, $d = 1,\ldots,n-2$, $m = d+1,\ldots,n-1$, $n \ge 3$;

11) $L_{11} = (AO[1,m] \oplus AE[m+1,n-1]) \subset\!\!\!+ \langle D + J_{0n} + M\rangle$, $m = 1,\ldots,n-1$, $n \ge 2$;

12) $L_{12} = (AE_1[1,m] \oplus AE[m+1,n-3]) \subset\!\!\!+ \langle J_{n-2,n-1} + \alpha M, D + J_{0n} + M\rangle$, $m = 1,\ldots,n-3$, $n \ge 4$, $\alpha \ge 0$;

13) $L_{13} = (AE_1[1,m] \oplus AE[m+1,n-3]) \subset\!\!\!+ \langle J_{n-2,n-1} + M, D + J_{0n}\rangle$, $m = 1,\ldots,n-3$, $n > 4$;

14) $L_{14} = AE[3,n-1] \subset\!\!\!+ \langle J_{12} + \alpha M, D + J_{0n} + M\rangle$, $n \ge 3$, $\alpha \ge 0$;

15) $L_{15} = AE[3,n-1] \subset\!\!\!+ \langle J_{12} + M, D + J_{0n}\rangle$, $n \le 3$;

16) $L_{16} = (\Gamma_{d,q} \oplus AE[dq+1,n-1] \subset\!\!\!+ \langle D - J_{0n}\rangle$, $d \ge 2$, $n \ge 5$;

17) $L_{17} = (\Phi(d_0+1,d_1,\gamma_1) \oplus \Phi(d_1+1,d_2,\gamma_2) \oplus \cdots \oplus \Phi(d_{t-1}+1,d_t,\gamma_t) \oplus AO[d_t+1,m] \oplus AE[m+1,n-1]) \subset\!\!\!+ \langle D - J_{0n}\rangle$, where $d_0 = 0$, $\gamma_1 < \gamma_2 < \cdots < \gamma_t$, $t > 1$, $m = 1,\ldots,n-2$, $n \ge 3$;

18) $L_{18} = (\Gamma_{d,q} \oplus \Phi(l_0+1,l_1,\mu_1) \oplus \Phi(l_1+1,l_2,\mu_2) \oplus \cdots \oplus \Phi(l_{t-1}+1,l_t,\mu_t) \oplus AE[l_t+1,n-1]) \subset\!\!\!+ \langle D - J_{0n}\rangle$, where $\mu_1 < \mu_2 < \cdots < \mu_t$, $t \ge 1$, $l_0 = dq$.

L_1–L_{11} and L_1–L_{18} of the theorems 1 and 2 respectively and to carry out a reduction of the equation (1). Consider, for example, the subalgebra L_{17}. The ansatz

$$u^2 = \left[-(x_0+x_m) + \sum_{i=1}^{t} \frac{1}{x_0 - x_m + \gamma_i}\left(x^2_{d_{i-1}+1} + \cdots x^2_{d_i}\right)\right]\varphi(\omega) - \\ - x^2_{d_t+1} - \cdots - x^2_{m-1}, \quad \omega = x_0 - x_m,$$

corresponds to this subalgebra. This ansatz reduces the equation (1) to equation $\varphi\dot\varphi - \varphi = 0$. Using the solution of this equation we find the following solution of the equation (1):

$$u^2 = \left[-(x_0+x_m) + \sum_{i=1}^{t} \frac{1}{x_0 - x_m + \gamma_i}\left(x^2_{d_{i-1}+1} + \cdots + x^2_{d_i}\right)\right] \times \\ \times (x_0 - x_m + C) - x^2_{d_t+1} - \cdots - x^2_{m-1}.$$

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