Orthogonal and non-orthogonal separation of variables in the wave equation $u_{tt} - u_{xx} + V(x)u = 0$

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We develop a direct approach to the separation of variables in partial differential equations. Within the framework of this approach, the problem of the separation of variables in the wave equation with time-independent potential reduces to solving an overdetermined system of nonlinear differential equations. We have succeeded in constructing its general solution and, as a result, all potentials $V(x)$ permitting variable separation have been found. For each of them we have constructed all inequivalent coordinate systems providing separability of the equation under study. It should be noted that the above approach yields both orthogonal and non-orthogonal systems of coordinates.

1. Introduction

Separation of variables (SV) in two- and three-dimensional Laplace, Helmholtz, d'Alembert and Klein–Gordon–Fock equations has been carried out in classical works by Bocher [1], Darboux [2], Eisenhart [3], Stepvanov [4], Olevsky [5], and Kalnins and Miller (see [6] and references therein). Nevertheless, a complete solution to the problem of sv in a two-dimensional wave equation with time-independent potential

$$(\Box + V(x))u \equiv u_{tt} - u_{xx} + V(x)u = 0 \tag{1}$$

has not been obtained yet. In (1) $u = u(t,x) \in C^2(\mathbb{R}^2, \mathbb{R}^1)$, $V(x) \in C(\mathbb{R}^1, \mathbb{R}^1)$.

Equations belonging to the class (1) are widely used in modern mathematical physics and can be related to other important linear and nonlinear partial differential equations (PDE). First, we mention the Lorentz-invariant wave equation

$$u_{y_0y_0} - u_{y_1y_1} + U(y_0^2 - y_1^2)u = 0. \tag{2}$$

The above equation can be reduced to the form (1) with the change of variables [7]

$$t = \exp(y_1/2)\cosh y_0, \quad x = \exp(y_1/2)\sinh y_0$$

and what is more, potentials $V(\tau)$, $U(\tau)$ are connected by the following relation:

$$U(\tau) = (4\tau)^{-1}V(\tau).$$

Another related equation is the hyperbolic type equation

$$v_{x_0x_0} - c^2(x_1)v_{x_1x_1} = 0 \tag{3}$$

that is widely used in various areas of mathematical physics.

Equation (3) is reduced to the form (1) by the change of variables

$$u(t,x) = [c(x_0)]^{-1/2}v(x_0,x_1), \quad t = x_0, \quad x = \int [c(x_1)]^{-1}dx_1$$

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and what is more

$$V(x) = -c^{3/2}(x_1)[c^{1/2}(x_1)], \tag{4}$$

where $x = \int [c(x_1)]^{-1} dx_1$.

The third related equation is the nonlinear wave equation

$$W_{tt} - [c^{-2}(W)W_x]_x = 0. \tag{5}$$

By substitution $W = R_x$, equation (5) is reduced to the form

$$R_{tt} - c^{-2}(R_x)R_{xx} = 0.$$

Applying to the above equation the Legendre transformation

$$x_0 = R_t, \quad x_1 = R_x, \quad v_{x_0} = t, \quad v_{x_1} = x, \quad v = tR_t + xR_x - R,$$

we obtain (3). Consequently, the method of SV in the linear equation (1) makes it possible to construct exact solutions of the nonlinear wave equation (5).

Let us also mention the Euler–Poisson–Darboux equation

$$v_{tt} - v_{xx} - x^{-1}v_x + m^2x^{-2}v = 0 \tag{6}$$

that is reduced to an equation of the form (1)

$$u_{tt} - u_{xx} + (m^2 - 1/4)x^{-2}u = 0$$

by the change of dependent variable $v(t,x) = x^{-1/2}u(t,x)$.

For the solution of (1) with separated variables $\omega_1(t,x)$, $\omega_2(t,x)$, we use the ansatz

$$u(t,x) = Q(t,x)\varphi_1(\omega_1)\varphi_2(\omega_2) \tag{7}$$

which reduces PDE (1) to two ordinary differential equations (ODE) for functions φ_1, φ_2.

There exist three possibilities for SV in (1). The first is to separate it into two second-order ODE. The second possibility is to separate (1) into first-order and second-order ODE, and the third possibility is to separate (1) into two first-order ODE. In the present paper we shall investigate in detail the first two possibilities. The third possibility requires special separate consideration and will be the topic of future publications.

Consider the following ODE:

$$\ddot{\varphi}_i = A_i(\omega_i,\lambda)\dot{\varphi}_i + B_i(\omega_i,\lambda)\varphi_i, \quad i = 1,2, \tag{8}$$

where $A_i, B_i \subset C^2(\mathbb{R}^1 \times \Lambda, \mathbb{R}^1)$ are some unknown functions, $\lambda \in \Lambda \subset \mathbb{R}^1$ is a real parameter (separation constant).

Definition 1 [7, 8]. *Equation (1) separates into two ODE if substitution of the ansatz (7) into (1) with subsequent exclusion of the second derivatives $\ddot{\varphi}_1$, $\ddot{\varphi}_2$ according to (8) yields an identity with respect to the variables $\dot{\varphi}_i$, φ_i, λ (considered as independent).*

On the basis of the above definition one can formulate a constructive procedure of SV in (1), suggested for the first time in [7]. At the first step, one has to substitute expression (7) into (1) and to express the second derivatives $\ddot{\varphi}_1$, $\ddot{\varphi}_2$ via functions $\dot{\varphi}_i$, φ_i

according to (8). At the second step, the equality obtained is split with respect to the independent variables $\dot{\varphi}_i$, φ_i, λ. As a result, one obtains an over-determinated system of partial differential equations for functions Q, ω_1 and ω_2 with undefined coefficients. The general solution of this system gives rise to all systems of coordinates providing separability of (1).

Definition 2. *Equation (1) separates into first- and second-order ODE*

$$\begin{aligned} &\dot{\varphi}_1 = A(\omega_1, \lambda)\varphi_1, \\ &\ddot{\varphi}_2 = B_1(\omega_2, \lambda)\dot{\varphi}_2 + B_2(\omega_2, \lambda)\varphi_2 \end{aligned} \tag{9}$$

if substitution of the ansatz (7) into (1) with subsequent exclusion of derivatives $\dot{\varphi}_1$, $\ddot{\varphi}_2$ *according to (9) yields an identity with respect to the variables* φ_1, $\dot{\varphi}_2$, φ_2, λ *(considered as independent).*

Let us emphasize that the above approach to SV in (1) has much in common with the non-Lie method of reduction of nonlinear PDE suggested in [9–11]. It is also important to note that the idea to represent solutions of linear differential equations in the "separated" form (7) goes as far as the classical works by Fourier and Euler (for a modern exposition of the problem of SV, see Miller [12] and Koornwinder [13]).

2. Orthogonal separation of variables in equation (1)

It is evident that (1) admits SV in Cartesian coordinates $\omega_1 = t$, $\omega_2 = x$ under arbitrary $V = V(x)$.

Definition 3. *Equation (1) admits non-trivial SV if there exist at least one coordinate system* $\omega_1(t,x)$, $\omega_2(t,x)$ *different from the Cartesian system providing its separability.*

Next, if one makes in (1) the following transformations:

$$\begin{aligned} &t \to C_1 t, \quad x \to C_1 x, \\ &t \to t, \quad x \to x + C_2, \quad C_i \in \mathbb{R}^1 \end{aligned} \tag{10}$$

then the class of equations (1) transforms into itself and what is more

$$\begin{aligned} &V(x) \to V'(x) = C_1^2 V(C_1 x), \\ &V(x) \to V'(x) = V(x + C_2). \end{aligned} \tag{10a}$$

That is why potentials $V(x)$ and $V'(x)$, connected by one of the above relations, are considered as equivalent ones.

When separating variables in (1) one has to solve an intermediate problem of description of all inequivalent potentials such that the equation admits non-trivial SV (classification problem). The next step is to obtain a complete description of the coordinate systems providing SV in (1) with these potentials.

First, we adduce the principal results on separation of (1) into two second-order ODE and then give an outline of the proof of the corresponding theorems.

Theorem 1. *Equation (1) admits non-trivial SV in the sense of Definition 1 iff the function* $V(x)$ *is given, up to equivalence relations (10a), by one of the following formulae:*

(1) $V = mx$;

(2) $V = mx^{-2}$;

$$
\begin{array}{ll}
(3) & V = m\sin^{-2}x;\\
(4) & V = m\sinh^{-2}x;\\
(5) & V = m\cosh^{-2}x;\\
(6) & V = m\exp x;\\
(7) & V = \cos^{-2}x(m_1 + m_2\sin x);\\
(8) & V = \cosh^{-2}x(m_1 + m_2\sinh x);\\
(9) & V = \sinh^{-2}x(m_1 + m_2\cosh x);\\
(10) & V = m_1\exp x + m_2\exp 2x;\\
(11) & V = m_1 + m_2x^{-2};\\
(12) & V = m.
\end{array}
\tag{11}
$$

Here m, m_1, m_2 are arbitrary real parameters, $m_2 \neq 0$.

Note 1. Equation (1) having the potential (6) from (11) is transformed with the change of variables [7]

$$x' = \exp(x/2)\cosh t, \quad t' = \exp(x/2)\sinh t$$

into (1) with $V(x) = m$.

Note 2. Equations (1) having the potentials (3), (4), (5) from (11) are transformed into (1) with $V(x) = mx^{-2}$ by means of changes of variables [7]

$$
\begin{aligned}
&x' = \tan\xi + \tan\eta, \quad t' = \tan\xi - \tan\eta,\\
&x' = \tanh\xi + \tanh\eta, \quad t' = \tanh\xi - \tanh\eta,\\
&x' = \coth\xi + \tanh\eta, \quad t' = \coth\xi - \tanh\eta.
\end{aligned}
$$

Hereafter $\xi = \frac{1}{2}(x+t)$, $\eta = \frac{1}{2}(x-t)$ are cone variables.

By virtue of the above remarks, the validity of the assertion follows from Theorem 1.

Theorem 2. *Provided equation (1) admits non-trivial SV in the sense of Definition 1, it is locally equivalent to one of the following equations:*

$$
\begin{array}{ll}
(1) & \Box u + mxu = 0;\\
(2) & \Box u + mx^{-2}u = 0;\\
(3) & \Box u + \cos^{-2}x(m_1 + m_2\sin x)u = 0;\\
(4) & \Box u + \cosh^{-2}x(m_1 + m_2\sinh x) = 0;\\
(5) & \Box u + \sinh^{-2}x(m_1 + m_2\cosh x) = 0;\\
(6) & \Box u + \exp x(m_1 + m_2\exp x)u = 0;\\
(7) & \Box u + (m_1 + m_2x^{-2})u = 0;\\
(8) & \Box u + mu = 0.
\end{array}
\tag{12}
$$

Thus, there exist eight inequivalent types of equations of the form (1) admitting non-trivial SV.

It is well known that there are 11 coordinate systems providing separability of the Klein–Gordon–Fock equation $\Box u + mu = 0$ into two second-order ODE [6]. Besides that, in [14] it was established that the Euler–Poisson–Darboux equation (6), which is equivalent to the second equation of (12), separates in nine coordinate systems. That is why cases $V(x) = m$ and $V(x) = mx^{-2}$ are not considered here.

As is shown below, the general form of solution with separated variables of (12) is as follows:

$$u(t, x) = \varphi_1(\omega_1(t, x))\varphi_2(\omega_2(t, x)), \tag{13}$$

where $\varphi_1(\omega_1)$, $\varphi_2(\omega_2)$ are arbitrary solutions of the separated ODE

$$\ddot{\varphi}_i = (\lambda + g_i(\omega_i))\varphi_i, \quad i = 1, 2 \tag{14}$$

and explicit forms of the functions $\omega_i(t, x)$, $g_i(\omega_i)$ are given below.

Theorem 3. *Equation* $\Box u + mxu = 0$ *separates in two coordinate systems*

$$\begin{aligned}
&(1) \quad \omega_1 = t \qquad \omega_2 = x, \quad g_1 = 0, \quad g_2 = m\omega_2;\\
&(2) \quad \omega_1 = (x+t)^{1/2} + (x-t)^{1/2}, \quad \omega_2 = (x+t)^{1/2} - (x-t)^{1/2},\\
&\qquad g_1 = -\frac{1}{4}m\omega_1^4, \quad g_2 = -\frac{1}{4}m\omega_2^4.
\end{aligned} \tag{15}$$

Theorem 4. *Equation* $\Box u + \sin^{-2} x(m_1 + m_2 \cos x)u = 0$ *separates in four coordinate systems*

$$\begin{aligned}
&(1) \quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = \cos^{-2}\omega_2(m_1 + m_2 \sin\omega_2);\\
&(2) \quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \arctan\sinh(\omega_1 + \omega_2) \pm \arctan\sinh(\omega_1 - \omega_2),\\
&\qquad g_1 = (m_1 + m_2)\sinh^{-2}\omega_1, \quad g_2 = -(m_1 - m_2)\cosh^{-2}\omega_2;\\
&(3) \quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \arctan\frac{\operatorname{sn}(\omega_1+\omega_2)}{\operatorname{cn}(\omega_1+\omega_2)} \pm \arctan\frac{\operatorname{sn}(\omega_1-\omega_2)}{\operatorname{cn}(\omega_1-\omega_2)},\\
&\qquad g_1 = m_1 \operatorname{dn}^2\omega_1 \operatorname{cn}^{-2}\omega_1 \operatorname{sn}^{-2}\omega_1 + m_2[\operatorname{cn}^{-2}\omega_1 - \operatorname{dn}^2\omega_1 \operatorname{cn}^{-2}\omega_1],\\
&\qquad g_2 = m_1 k^4 \operatorname{sn}^2\omega_2 \operatorname{cn}^2\omega_2 \operatorname{dn}^{-2}\omega_2 + m_2 k^2[\operatorname{cn}^2\omega_2 \operatorname{dn}^{-2}\omega_2 - \operatorname{sn}^2\omega_2];\\
&(4) \quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \arctan\left(\frac{k}{k'}\right)^{1/2}\operatorname{cn}(\omega_1+\omega_2) \pm \arctan\left(\frac{k}{k'}\right)^{1/2}\operatorname{cn}(\omega_1-\omega_2),\\
&\qquad g_1 = m_1[\operatorname{dn}^2\omega_1 \operatorname{cn}^{-2}\omega_1 + k^2\operatorname{sn}^2\omega_1] + m_2[(k')^2\operatorname{cn}^{-2}\omega_1 + k^2\operatorname{cn}^2\omega_1],\\
&\qquad g_2 = m_1[\operatorname{dn}^2\omega_2 \operatorname{cn}^{-2}\omega_2 + k^2\operatorname{sn}^2\omega_2] + m_2[(k')^2\operatorname{cn}^{-2}\omega_2 + k^2\operatorname{cn}^2\omega_2].
\end{aligned} \tag{16}$$

In the above formulae (16) k, $k' = (1-k^2)^{1/2}$ *are the moduli of corresponding elliptic Jacobi functions, and* k *is an arbitrary constant satisfying the inequality* $0 < k < 1$.

Theorem 5. *Equation* $\Box u + \cosh^{-2} x(m_1 + m_2 \sinh x)u = 0$ *separates in four coordinate systems*

$$\begin{aligned}
&(1) \quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = \cosh^{-2}\omega_2(m_1 + m_2 \sinh\omega_2);\\
&(2) \quad \left\{\begin{matrix} t \\ x \end{matrix}\right\} = -\ln\left[\left(\frac{k'}{k}\right)^{1/2}\operatorname{cn}(\omega_1+\omega_2)\right] \pm \ln\left[\left(\frac{k'}{k}\right)^{1/2}\operatorname{cn}(\omega_1-\omega_2)\right],\\
&\qquad g_1 = m_1(k')^2\operatorname{dn}^{-2}2\omega_1 + m_2\operatorname{cn}2\omega_1\operatorname{dn}^{-2}2\omega_1,\\
&\qquad g_2 = m_1(k')^2\operatorname{dn}^{-2}2\omega_2 + m_2\operatorname{cn}2\omega_2\operatorname{dn}^{-2}2\omega_2;
\end{aligned}$$

$$
\begin{aligned}
&(3)\quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = -\ln\sinh\frac{1}{2}(\omega_1+\omega_2) \pm \ln\cosh\frac{1}{2}(\omega_1-\omega_2), \\
&\qquad g_1 = \cosh^{-2}\omega_1(m_1 - m_2\sinh\omega_1), \quad g_2 = \cosh^{-2}\omega_2(m_1 - m_2\sinh\omega_2); \\
&(4)\quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \ln\frac{\operatorname{sn}\frac{1}{2}(\omega_1+\omega_2)}{\operatorname{cn}\frac{1}{2}(\omega_1+\omega_2)} \pm \ln\,\operatorname{dn}\frac{1}{2}(\omega_1+\omega_2), \\
&\qquad g_1 = -m_1k^2\operatorname{sn}^2\omega_1 + k^2m_2\operatorname{sn}\omega_1\operatorname{cn}\omega_1, \\
&\qquad g_2 = -m_1k^2\operatorname{sn}^2\omega_2 + k^2m_2\operatorname{sn}\omega_2\operatorname{cn}\omega_2.
\end{aligned}
\tag{17}
$$

Here k, $k' = (1-k^2)^{1/2}$ are the moduli of corresponding elliptic functions, $0 \le k \le 1$.

Theorem 6. *Equation $\Box u + \sinh^{-2}x(m_1 + m_2\cosh x)u = 0$ separates in eleven coordinate systems:*

$$
\begin{aligned}
&(1)\quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = \sinh^{-2}\omega_2(m_1 + m_2\cosh\omega_2); \\
&(2)\quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = -\ln\frac{1}{2}(\omega_1+\omega_2) \pm \ln\frac{1}{2}(\omega_1-\omega_2), \\
&\qquad g_1 = (m_1 - m_2)\omega_1^{-2}, \quad g_2 = (m_1 + m_2)\omega_2^{-2}; \\
&(3)\quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = -\ln\sin\frac{1}{2}(\omega_1+\omega_2) \pm \ln\sin\frac{1}{2}(\omega_1-\omega_2), \\
&\qquad g_1 = (m_1 - m_2)\sin^{-2}\omega_1, \quad g_2 = (m_1 + m_2)\sin^{-2}\omega_2; \\
&(4)\quad \left\{ \begin{matrix} t \\ x \end{matrix} \right\} = -\ln\sinh\frac{1}{2}(\omega_1+\omega_2) \pm \ln\sinh\frac{1}{2}(\omega_1-\omega_2), \\
&\qquad g_1 = \sinh^{-2}\omega_1(m_1 + m_2)\cosh\omega_1), \quad g_2 = \sinh^{-2}\omega_2(m_1 - m_2\cosh\omega_2); \\
&(5)\quad \left\{ \begin{matrix} t \\ x \end{matrix} \right\} = -\ln\cosh\frac{1}{2}(\omega_1+\omega_2) \pm \ln\cosh\frac{1}{2}(\omega_1-\omega_2), \\
&\qquad g_1 = \sinh^{-2}\omega_1(m_1 - m_2\cosh\omega_1), \quad g_2 = \sinh^{-2}\omega_2(m_1 - m_2\cosh\omega_2); \\
&(6)\quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \ln\tanh\frac{1}{2}(\omega_1+\omega_2) \pm \ln\tanh\frac{1}{2}(\omega_1-\omega_2), \\
&\qquad g_1 = \cosh^{-2}\omega_1(m_1 - m_2), \quad g_2 = -\cosh^{-2}\omega_2(m_1 + m_2); \\
&(7)\quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \ln\tan\frac{1}{2}(\omega_1+\omega_2) \pm \ln\tan\frac{1}{2}(\omega_1-\omega_2), \\
&\qquad g_1 = \cos^{-2}\omega_1(m_1 + m_2), \quad g_2 = \cos^{-2}\omega_2(m_1 - m_2); \\
&(8)\quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \operatorname{arctanh}\operatorname{cn}(\omega_1+\omega_2) \pm \operatorname{arctanh}\operatorname{cn}(\omega_1-\omega_2), \\
&\qquad g_1 = (m_1 + m_2)\operatorname{dn}^2\omega_1\operatorname{cn}^{-2}\omega_1 + (m_1 - m_2)k^2\operatorname{sn}^2\omega_1, \\
&\qquad g_2 = (m_1 - m_2)\operatorname{dn}^2\omega_2\operatorname{cn}^{-2}\omega_2 + (m_1 + m_2)k^2\operatorname{sn}^2\omega_2; \\
&(9)\quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \operatorname{arctanh}\operatorname{dn}(\omega_1+\omega_2) \pm \operatorname{arctanh}\operatorname{dn}(\omega_1-\omega_2), \\
&\qquad g_1 = (m_1 + m_2)k^2\operatorname{cn}^2\omega_1\operatorname{dn}^{-2}\omega_1 + (m - m_2)k^2\operatorname{sn}^2\omega_1, \\
&\qquad g_2 = (m_1 - m_2)k^2\operatorname{cn}^2\omega_2\operatorname{cn}^{-2}\omega_2 + (m_1 + m_2)k^2\operatorname{sn}^2\omega_2; \\
&(10)\quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \operatorname{arctanh}\operatorname{sn}(\omega_1+\omega_2) \pm \operatorname{arctanh}\operatorname{sn}(\omega_1-\omega_2),
\end{aligned}
\tag{18}
$$

$$g_1 = (m_1 + m_2)\,\mathrm{sn}^{-2}\omega_1 + (m_1 - m_2)k^2\,\mathrm{sn}^2\omega_1,$$
$$g_2 = (m_1 + m_2)k^2\,\mathrm{cn}^2\omega_2\,\mathrm{dn}^{-2}\omega_2 + (m_1 - m_2)k^2\,\mathrm{dn}^2\omega_2\,\mathrm{cn}^{-2}\omega_2;$$
$$(11)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \pm\ln\,\mathrm{cn}\,(\omega_1 + \omega_2) \pm \ln\,\mathrm{cn}\,(\omega_1 - \omega_2),$$
$$g_1 = -m_1\,\mathrm{sn}^{-2}\omega_1 - m_2\,\mathrm{cn}\,\omega_1\,\mathrm{sn}^{-2}\omega_1,$$
$$g_2 = -m_1\,\mathrm{sn}^{-2}\omega_2 - m_2\,\mathrm{cn}\,\omega_2\,\mathrm{sn}^{-2}\omega_2.$$

Here k are the moduli of corresponding elliptic functions, $0 < k < 1$.

Theorem 7. *Equation $\Box u + \exp x(m_1 + m_2 \exp x)u = 0$ separates in six coordinate systems:*

$$
\begin{aligned}
&(1)\quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = \exp\omega_2(m_1 + m_2\exp\omega_2);\\
&(2)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = -\ln\cos(\omega_1 + \omega_2) \pm \ln\cos(\omega_1 - \omega_2),\\
&\qquad g_1 = -2m_1\cos 2\omega_1 - \frac{1}{2}m_2\cos 4\omega_1,\\
&\qquad g_2 = -2m_1\cos 2\omega_2 - \frac{1}{2}m_2\cos 4\omega_2;\\
&(3)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \ln\sinh(\omega_1 + \omega_2) \pm \ln\sinh(\omega_1 - \omega_2),\\
&\qquad g_1 = -2m_1\cosh 2\omega_1 - \frac{1}{2}m_2\cosh 4\omega_1,\\
&\qquad g_2 = -2m_1\cosh 2\omega_2 - \frac{1}{2}m_2\cosh 4\omega_2;\\
&(4)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \ln\cosh(\omega_1 + \omega_2) \pm \ln\cosh(\omega_1 - \omega_2),\\
&\qquad g_1 = -2m_1\cosh 2\omega_1 - \frac{1}{2}m_2\cosh 4\omega_1,\\
&\qquad g_2 = -2m_1\cosh 2\omega_2 - \frac{1}{2}m_2\cosh 4\omega_2;\\
&(5)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \ln\cosh(\omega_1 + \omega_2) \pm \ln\sinh(\omega_1 - \omega_2),\\
&\qquad g_1 = -2m_1\sinh 2\omega_1 - \frac{1}{2}m_2\cosh 4\omega_1,\\
&\qquad g_2 = -2m_1\sinh 2\omega_2 - \frac{1}{2}m_2\cosh 4\omega_2;\\
&(6)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \ln(\omega_1 + \omega_2) \pm \ln(\omega_1 - \omega_2),\\
&\qquad g_1 = 2m_1 + 2m_2\omega_1^2, \quad g_2 = -2m_1 + 2m_2\omega_2^2.
\end{aligned}
\tag{19}
$$

Theorem 8. *Equation $\Box u + (m_1 + m_2 x^{-2})u = 0$ separates in six coordinate systems:*

$$(1)\quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = m_1 + m_2\omega_2^{-2};$$
$$(2)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \exp(\omega_1 + \omega_2) \pm \exp(\omega_1 - \omega_2),$$
$$g_1 = 4m_1\exp 2\omega_1, \quad g_2 = m_2\cosh^{-2}\omega_2;$$

$$(3)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \sin(\omega_1+\omega_2) \pm \sin(\omega_1-\omega_2),$$

$$g_1 = 2m_1\cos 2\omega_1 + m_2\sin^{-2}\omega_1, \qquad g_2 = -2m_1\cos 2\omega_2 + m_2\cos^{-2}\omega_2;$$

$$(4)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \sinh(\omega_1+\omega_2) \pm \sinh(\omega_1-\omega_2),$$

$$g_1 = 2m_1\sinh 2\omega_1 + m_2\sinh^{-2}\omega_1,$$

$$g_2 = -2m_1\sinh 2\omega_2 - m_2\sinh^{-2}\omega_2; \tag{20}$$

$$(5)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \cosh(\omega_1+\omega_2) \pm \cosh(\omega_1-\omega_2),$$

$$g_1 = 2m_1\cosh 2\omega_1 - m_2\cosh^{-2}\omega_1, \quad g_2 = 2m_1\cosh 2\omega_2 - m_2\cosh^{-2}\omega_2;$$

$$(6)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = (\omega_1+\omega_2)^2 \pm (\omega_1-\omega_2)^2,$$

$$g_1 = -16m_1\omega_1^2 + m_2\omega_1^{-2}, \quad g_2 = -16m_1\omega_2^2 + m_2\omega_2^{-2}.$$

We now give a sketch of the proof of the above assertions. Substituting ansatz (7) into (1), expressing functions $\ddot{\varphi}_i$ via functions $\dot{\varphi}_1$, φ_i by means of equalities (8) and splitting the equation obtained with respect to independent variables $\dot{\varphi}_i$, φ_i we obtain the following system of nonlinear PDE:

$$(1)\quad Q\Box\omega_i + 2(Q_t\omega_{it} - Q_x\omega_{ix}) + QA_i(\omega_i,\lambda)(\omega_{it}^2 - \omega_{ix}^2) = 0, \quad i = 1,2; \tag{21}$$

$$(2)\quad \Box Q + Q[B_1(\omega_1,\lambda)(\omega_{1t}^2 - \omega_{1x}^2) + B_2(\omega_2,\lambda)(\omega_{2t}^2 - \omega_{2x}^2)] + QV(x) = 0; \tag{22}$$

$$(3)\quad \omega_{1t}\omega_{2t} - \omega_{1x}\omega_{2x} = 0. \tag{23}$$

Here $\Box = \partial_t^2 - \partial_x^2$.

Thus, to separate variables in the linear PDE (1) one has to construct the general solution of the system of nonlinear equations (21)–(23). The same assertion holds true for any general linear differential equation, i.e. the problem of SV is an essentially nonlinear one.

It is not difficult to become convinced of the fact that, from (23), it follows that

$$(\omega_{1t}^2 - \omega_{1x}^2)(\omega_{2t}^2 - \omega_{2x}^2) \neq 0. \tag{24}$$

Differentiating (21) with respect to λ and using (24) we obtain

$$A_{1\lambda} = A_{2\lambda} = 0,$$

whence $B_{1\lambda}B_{2\lambda} \neq 0$. Differentiating (22) with respect to λ, we have

$$B_{1\lambda}(\omega_{1t}^2 - \omega_{1x}^2) + B_{2\lambda}(\omega_{2t}^2 - \omega_{2x}^2) = 0$$

or

$$\frac{B_{1\lambda}}{B_{2\lambda}} = -\frac{\omega_{2t}^2 - \omega_{2x}^2}{\omega_{1t}^2 - \omega_{1x}^2}.$$

Differentiation of the above equality with respect to λ yields

$$B_{1\lambda\lambda}B_{2\lambda} - B_{1\lambda}B_{2\lambda\lambda} = 0$$

or

$$\frac{B_{1\lambda\lambda}}{B_{1\lambda}} = \frac{B_{2\lambda\lambda}}{B_{2\lambda}}.$$

Since functions $B_1 = B_1(\omega_1)$, $B_2 = B_2(\omega_2)$ are independent, there exists a function $\varkappa(\lambda)$ such that

$$B_{i\lambda\lambda} = \varkappa(\lambda) B_{i\lambda}, \quad i = 1, 2.$$

Integrating the above differential equation with respect to λ we obtain

$$B_i(\omega_i) = \Lambda(\lambda) f_i(\omega_i) + g_i(\omega_i), \quad i = 1, 2,$$

where f_i, g_i are arbitrary smooth functions.

On redefining the parameter $\lambda \to \Lambda(\lambda)$, we have

$$B_i(\omega_i) = \lambda f_i(\omega_i) + g_i(\omega_i). \tag{25}$$

Substitution of (25) into (22) with subsequent splitting with respect to λ yields the following equations:

$$\Box Q + Q[g_1(\omega_{1t}^2 - \omega_{1x}^2) + g_2(\omega_{1t}^2 - \omega_{1x}^2)] + V(x)Q = 0, \tag{26}$$

$$f_1(\omega_{1t}^2 - \omega_{1x}^2) + f_2(\omega_{2t}^2 - \omega_{2x}^2) = 0. \tag{27}$$

Thus, system (21)–(23) is equivalent to the system of equations (21), (23), (26), (27). Before integrating, we make a remark: it is evident that the structure of ansatz (7) is not altered by transformation

$$Q \to Q' = Q h_1(\omega_1) h_2(\omega_2), \quad \omega_i \to \omega_i' = R_i(\omega_i), \quad i = 1, 2, \tag{28}$$

where h_i, R_i are smooth-enough functions. This is why solutions of the system under study connected by relations (28) are considered to be equivalent.

Choosing the functions h_i, R_i in a proper way, we can put in (21) and (27)

$$f_1 = f_2 = 1, \quad A_1 = A_2 = 0.$$

Consequently, functions ω_1, ω_2 satisfy equations of the form

$$\omega_{1t}\omega_{2t} - \omega_{1x}\omega_{2x} = 0, \quad \omega_{1t}^2 - \omega_{1x}^2 + \omega_{2t}^2 - \omega_{2x}^2 = 0,$$

whence

$$(\omega_1 \pm \omega_2)_t^2 - (\omega_1 \pm \omega_2)_x^2 = 0.$$

Integrating the above equations, we obtain

$$\omega_1 = F(\xi) + G(\eta), \quad \omega_2 = F(\xi) - G(\eta), \tag{29}$$

where $F, G \subset C^2(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions, $\xi = (x+t)/2$, $\eta = (x-t)/2$.

Substitution of (29) into (21) with $A_1 = A_2 = 0$ yields the following equations:

$$(\ln Q)_t = 0, \quad (\ln Q)_x = 0,$$

whence $Q = 1$.

Finally, substituting the results obtained into (26), we have

$$V(x) = [g_1(F+G) - g_2(F-G)]\frac{dF}{d\xi}\frac{dG}{d\eta}. \tag{30}$$

Thus, the problem of integrating an over-determined system of nonlinear differential equations (21)–(23) is reduced to integration of the functional-differential equation (30).

Let us summarize the results obtained. The general form of solution of (1) with separated variables is as follows

$$u = \varphi(F(\xi) + G(\eta))\varphi_2(F(\xi) - G(\eta)) \tag{31}$$

where φ_i are arbitrary solutions of (14), functions $F(\xi)$, $G(\eta)$, $g_1(F+G)$, $g_2(F-G)$ being determined by (30).

To integrate Eq. (31) we make the hodograph transformation

$$\xi = P(F), \quad \eta = R(G), \tag{32}$$

where $\dot{P} \not\equiv 0$, $\dot{R} \not\equiv 0$.

After making the transformation (32), we obtain

$$g_1(F+G) - g_2(F-G) = \dot{P}(F)\dot{R}(G)V(P+R). \tag{33}$$

Evidently, equation (33) is equivalent to the following equation:

$$(\partial_F^2 - \partial_G^2)[\dot{P}(F)\dot{R}(G)V(P+R)] = 0$$

or

$$(\dddot{P}\dot{P}^{-1} - \dddot{R}\dot{R}^{-1})V + 3(\ddot{P} - \ddot{R})\dot{V} + (\dot{P}^2 - \dot{R}^2)\ddot{V} = 0. \tag{34}$$

Thus, to integrate (30) it is enough to construct all functions $P(F)$, $R(G)$, $V(P+R)$ satisfying (34) and to substitute them into (33).

In [8] we have proved the following assertion:

Lemma. *The general solution of (34) determined up to transformation (10) is given by one of the following formulae:*

(1) $V = V(x)$ *is an arbitrary function*, $\dot{P} = \alpha$, $\dot{R} = \alpha$;

(2) $V = mx$, $\dot{P}^2 = \alpha P + \beta$, $\dot{R}^2 = \alpha R + \gamma$;

(3) $V = mx^{-2}$, $P = Q_1(F)$, $R = Q_2(G)$,

$$\begin{aligned} \dot{Q}_1^2 &= \alpha Q_1^4 + \beta Q_1^3 + \gamma Q_1^2 + \delta Q_1 + \rho, \\ \dot{Q}_2^2 &= \alpha Q_2^4 - \beta Q_2^3 + \gamma Q_2^2 - \delta Q_2 + \rho; \end{aligned} \tag{35}$$

(4) $V = m\sinh^{-2}x$, $P = \operatorname{arctanh} Q_1(F)$, $R = \tan Q_2(G)$

and Q_1, Q_2 *are determined by (35);*

(5) $V = m\sinh^{-2}x$, $P = \operatorname{arctanh} Q_1(F)$, $R = \operatorname{arctanh} Q_2(G)$

and Q_1, Q_2 *are determined by (35);*

(6) $V = m\cosh^{-2}x$, $P = \operatorname{arccoth} Q_1(F)$, $R = \operatorname{arctanh} Q_2(G)$

and Q_1, Q_2 are determined by (35);

(7) $V = m\exp x$,
$$\dot{P}^2 = \alpha\exp 2P + \beta\exp P + \gamma, \quad \dot{R}^2 = \alpha\exp 2R + \delta\exp R + \rho;$$
(8) $V = \cos^{-2}x(m_1 + m_2\sin x)$,
$$\dot{P}^2 = \alpha\sin 2P + \beta\cos 2P + \gamma, \quad \dot{R}^2 = \alpha\sin 2R + \beta\cos 2R + \gamma;$$
(9) $V = \cosh^{-2}x(m_1 + m_2\sinh x)$,
$$\dot{P}^2 = \alpha\sinh 2P + \beta\cosh 2P + \gamma, \quad \dot{R}^2 = \alpha\sinh 2R - \beta\cosh 2R + \gamma;$$
(10) $V = \sinh^{-2}x(m_1 + m_2\cosh x)$,
$$\dot{P}^2 = \alpha\sinh 2P + \beta\cosh 2P + \gamma, \quad \dot{R}^2 = -\alpha\sinh 2R + \beta\cosh 2R + \gamma;$$
(11) $V = (m_1 + m_2\exp x)\exp x$,
$$\ddot{P} = -\dot{P}^2 + \beta, \quad \ddot{R} = -\dot{R}^2 + \beta;$$
(12) $V = m_1 + m_2x^{-2}$,
$$\dot{P}^2 = \alpha P^2 + \beta P + \gamma, \quad \dot{R}^2 = \alpha R^2 - \beta R + \gamma,$$
(13) $V = m$,
$$\dot{P}^2 = \alpha P^2 + \beta P + \gamma, \quad \dot{R}^2 = \alpha R^2 + \delta R + \rho.$$

Here α, β, γ, δ, ρ, m_1, m_2, m are arbitrary real parameters; $x = \xi + \eta = P + R$.

Theorems 1 and 2 are direct consequences of the above Lemma. To prove Theorems 3–8 one has to integrate the ODE for $P(F)$, $R(G)$ and substitute the expressions obtained into formulae (32)

$$\frac{1}{2}(x+t) = P(F) \equiv P((\omega_1+\omega_2)/2), \quad \frac{1}{2}(x-t) = R(G) \equiv R((\omega_1-\omega_2)/2)$$

and into (33).

Thus, the problem of separation of the wave equation (1) into two second-order differential equations is completely solved.

Since all coordinate systems ω_1, ω_2 satisfy equation (23), we have orthogonal separation of variables. To obtain non-orthogonal coordinate systems providing separability of (1) one has to carry out SV following Definition 2.

3. Non-orthogonal separation of variables in equation (1)

Utilizing the SV procedure in (1) determined by Definition 2, we come to the following assertions (corresponding computations are omitted).

Theorem 9. *Equation (1) admits SV in the sense of Definition 2 iff it is locally-equivalent to one of the following equations:*

(1) $\Box u + mu = 0$;

(2) $\Box u + mx^{-2}u = 0$,

where m is an arbitrary real constant.

Theorem 10. *Equation $\Box u + mu = 0$ separates in two coordinate systems*

$$\begin{aligned}
&(1)\quad \omega_1 = \xi, \quad \omega_2 = \xi + \eta,\\
&\qquad \dot\varphi_1 = -\lambda\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\varphi_2;\\
&(2)\quad \omega_1 = \xi, \quad \omega_2 = \ln\xi + \ln\eta,\\
&\qquad \dot\varphi_1 = -\lambda\omega_1^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\exp(\omega_2)\varphi_2.
\end{aligned}$$

Theorem 11. *Equation $\Box u + mx^{-2}u = 0$ separates in eight coordinate systems*

$$\begin{aligned}
&(1)\quad \omega_1 = \xi, \quad \omega_2 = \xi + \eta,\\
&\qquad \dot\varphi_1 = -\lambda\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\omega_2^{-2}\varphi_2;\\
&(2)\quad \omega_1 = \xi, \quad \omega_2 = \arctan\xi + \arctan\eta,\\
&\qquad \dot\varphi_1 = -\lambda(1+\omega_1^2)\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\sin^{-2}\omega_2\varphi_2;\\
&(3)\quad \omega_1 = \xi, \quad \omega_2 = \operatorname{arctanh}\xi + \operatorname{arctanh}\eta,\\
&\qquad \dot\varphi_1 = -\lambda(1-\omega_1^2)^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\sinh^{-2}\omega_2\varphi_2;\\
&(4)\quad \omega_1 = \xi, \quad \omega_2 = \operatorname{arccoth}\xi + \operatorname{arccoth}, \eta,\\
&\qquad \dot\varphi_1 = \lambda(1-\omega_1^2)^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\sinh^{-2}\omega_2\varphi_2;\\
&(5)\quad \omega_1 = \xi, \quad \omega_2 = \operatorname{arctanh}\xi + \operatorname{arctanh}\eta,\\
&\qquad \dot\varphi_1 = -\lambda(1-\omega_1^2)^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 - m\cosh^{-2}\omega_2\varphi_2,\\
&(6)\quad \omega_1 = \xi, \quad \omega_2 = \operatorname{arccoth}\xi + \operatorname{arccoth}\eta,\\
&\qquad \dot\varphi_1 = \lambda(1-\omega_1^2)^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 - m\cosh^{-2}\omega_2\varphi_2;\\
&(7)\quad \omega_1 = \xi, \quad \omega_2 = \frac{1}{2}(\ln\xi - \ln\eta),\\
&\qquad \dot\varphi_1 = -\lambda(2\omega_1)^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 - m\cosh^{-2}\omega_2\varphi_2;\\
&(8)\quad \omega_1 = \xi, \quad \omega_2 = \xi^{-1} + \eta^{-1},\\
&\qquad \dot\varphi_1 = \lambda\omega_1^{-2}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\omega_2^{-2}\varphi_2.
\end{aligned}$$

In the above formulae λ is a separation constant, $\xi = \frac{1}{2}(x+t)$, $\eta = \frac{1}{2}(x-t)$.

As a direct check shows, the above coordinate systems do not satisfy (23). Consequently, they are non-orthogonal.

4. Conclusion

Let us say a few words about the intrinsic characterization of SV in (1). It is well known that the solution of the second-order linear PDE with separated variables is a joint eigenfunction of mutually-commuting symmetry operators of the equation under study (for more detail, see [13, 14]). Below, we construct the second-order symmetry operator of (1) such that solution with separated variables is its eigenfunction and parameter λ is an eigenvalue.

Making in (1) the change of variables (29), we obtain

$$u_{\omega_1\omega_1} - u_{\omega_2\omega_2} = V(\xi+\eta)[\dot F(\xi)\dot G(\eta)]^{-1}u.$$

Provided (1) admits SV, by virtue of (33) there exist functions $g_1(F+G)$, $g_2(F-G)$ such that

$$V(\xi+\eta)[\dot F(\xi)\dot G(\eta)]^{-1} = g_1(F+G) - g_2(F-G).$$

Since $F+G=\omega_1, F-G=\omega_2$, equation (36) takes the form

$$u_{\omega_1\omega_1} - u_{\omega_2\omega_2} = [g_1(\omega_1) - g_2(\omega_2)]u$$

or

$$Xu = 0, \quad X = \partial^2_{\omega_1} - \partial^2_{\omega_2} - g_1(\omega_1) + g_2(\omega_2).$$

Clearly, the operators $Q_i = \partial^2_{\omega_i} - g_i(\omega_i)$, $i=1,2$ commute with the operator X, i.e. they are symmetry operators of (1) and, what is more, the relations

$$Q_i u = Q_i \varphi_1(\omega_1)\varphi_2(\omega_2) = \lambda\varphi_1(\omega_1)\varphi_2(\omega_2) = \lambda u, \quad i=1,2$$

hold.

It should be noted that V.N. Shapovalov carried out classification of potentials $V(x)$ such that (1) admitted a non-trivial second-order symmetry operator [15] but he lost cases (4) and (9) from Theorem 1.

It was shown by Osborne and Stuart [16] that the method of SV could be applied to nonlinear PDE. In [8] we suggested a regular approach to SV in nonlinear partial differential equations. In future publications we intend to apply this approach to separate variables in the nonlinear wave equation $u_{tt} - u_{xx} = F(u)$.

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