On the new approach to variable separation in the wave equation with potential

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Пропонується конструктивний підхід до розв'язання проблем розділення змінних для двомірного хвильового рівняння $u_{tt} - u_{xx} = V(x)u$. У рамках цього підходу описані усі потенціали, що допускають, розділення змінних і вказані відповідні системи координат.

A problem of variable separation (VS) in the wave equation

$$u_{x_0x_0} - u_{x_1x_1} + V(x_1)u = 0 \qquad (1)$$

as considered in [1–3], consists of two problems. The first one is to describe all functions $V(x_1)$ providing VS in (1) in, at least, two inequivalent coordinate systems. The second one is to describe all coordinate-systems such that equation (1) admits VS for a given potential $V(x_1)$. Surprisingly enough, the both problems are not completely solved yet.

Our approach to the problem of VS in the wave equation (1) is based on the idea of its reduction to two ordinary differential equations

$$\ddot{\varphi}_i = A_i(\omega_i,\lambda)\varphi_i + B_i(\omega_i,\lambda)\varphi_i, \quad i = 1,2 \qquad (2)$$

with the use of ansatz of special structure [4–6]

$$u = A(x_0,x_1)\varphi_1(\omega_1(x_0,x_1))\varphi_2(\omega_2(x_0,x_1)). \qquad (3)$$

In the formulas (2), (3) A_1, A_2, B_1, B_2, A, ω_1, ω_2 are sufficiently smooth real functions, $\lambda \in \mathbb{R}^1$ is some parameter, no summation over i is carried out.

The formulas of the form (3) can be found in the classical works Euler, d'Alembert, Batemen and by some other contemporary mathematicians (see, for example, the review by Koornwinder [7]).

Definition. *We say, that equation (1) admits VS in the coordinates ω_1, ω_2 if substitution of the ansatz (3) into (1) with subsequent exclusion of the second derivatives, $\ddot{\varphi}_1$, $\ddot{\varphi}_2$ according to formulas (2) turns it into zero identically with respect to the variables $\dot{\varphi}_1$, $\dot{\varphi}_2$, φ_1, φ_2, λ.*

Substituting ansatz (3) into differential equation (1), expressing functions $\ddot{\varphi}_i$ in terms of $\dot{\varphi}_i$, φ_i, $i = 1,2$ and splitting the obtained expression with respect to the independent variables $\dot{\varphi}_1\dot{\varphi}_2$, $\dot{\varphi}_1\varphi_2$, $\varphi_1\dot{\varphi}_2$, $\varphi_1\varphi_2$ we get the following system of nonlinear partial differential equations:

$$\begin{aligned}
&1)\quad A\Box\omega_1 + 2A_{x_\mu}\omega_{1x^\mu} + AA_1\omega_{1x_\mu}\omega_{1x^\mu} = 0,\\
&2)\quad A\Box\omega_2 + 2A_{x_\mu}\omega_{2x^\mu} + AA_2\omega_{2x_\mu}\omega_{2x^\mu} = 0,\\
&3)\quad \Box A + A(B_1\omega_{1x_\mu}\omega_{1x^\mu} + B_2\omega_{2x_\mu}\omega_{2x^\mu}) + AV(x_1) = 0,\\
&4)\quad \omega_{1x_\mu}\omega_{2x^\mu} = 0.
\end{aligned} \qquad (4)$$

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Hereafter, the summation over the repeated Greek indices is under-stood in the Minkovski space $M(1,1)$ with a metric tensor $g_{\mu\nu} = \mathrm{diag}\,(1,-1)$.

Thus to describe all potentials $V(x_1)$ and coordinate systems ω_1, ω_2 providing VS in (1) one has to solve nonlinear system (4). At first glance such an approach seems to have poor prospects: to solve linear equation (1) it is necessary to integrate rather complicated system of nonlinear partial differential equations (4). But system (4) is overdetermined one. This fact has enabled us to construct its general solution in explicit form. Let us emphasize that the same is true when reducing nonlinear wave equation to the ordinary differential equation [5, 6] .

It is not difficult to show that from the forth equation of system (4) it follows that

$$(\omega_{1x_\mu}\omega_{1x^\mu})(\omega_{2x_\mu}\omega_{2x^\mu}) \neq 0. \tag{5}$$

Differentiating equations 1), 2) from (4) and using (5) we have

$$A_{1\lambda} = A_{2\lambda} = 0.$$

Consequently, the relation $B_{1\lambda}B_{2\lambda} \neq 0$ holds. Differentiating equation (3) with respect to λ, we get

$$B_{1\lambda}\omega_{1x_\mu}\omega_{1x^\mu} + B_{2\lambda}\omega_{2x_\mu}\omega_{2x^\mu} = 0$$

or

$$\frac{B_{1\lambda}}{B_{2\lambda}} = -\frac{\omega_{2x_\mu}\omega_{2x^\mu}}{\omega_{1x_\mu}\omega_{1x^\mu}}.$$

Differentiating the above equality with respect to λ, we obtain

$$\frac{B_{1\lambda\lambda}}{B_{1\lambda}} = \frac{B_{2\lambda\lambda}}{B_{2\lambda}}. \tag{6}$$

Since function B_i depends on the variable ω_i and the functions ω_1, ω_2 are independent, it follows from (6) that

$$B_{i\lambda\lambda} = \varkappa(\lambda)B_{i\lambda}, \quad i = 1,2.$$

Integration of the above ordinary differential equations yields

$$B_i = \Lambda(\lambda)f_i(\omega_i) + g_i(\omega_i), \quad i = 1,2.$$

After redefining the parameter λ, we have

$$B_i = \lambda f_i(\omega_i) + g_i(\omega_i), \quad i = 1,2. \tag{7}$$

Substituting (7) into equation (3) and splitting the obtained equality with respect to λ, we come to the following partial differential equations:

$$\begin{aligned} &3a) \quad \Box A + A(g_1\omega_{1x_\mu}\omega_{1x^\mu} + g_2\omega_{2x_\mu}\omega_{2x^\mu}) + V(x_1)A = 0,\\ &3b) \quad f_1\omega_{1x_\mu}\omega_{1x^\mu} + f_2\omega_{2x_\mu}\omega_{2x^\mu} = 0. \end{aligned} \tag{8}$$

Before integrating overdetermined system of nonlinear equations (4), (8), make an important remark. It is evident, that the ansatz structure does not change with the transformation of the form

$$A \to Ah_1(\omega_1)h_2(\omega_2), \quad \omega_i \to \Phi_1(\omega_i), \quad i = 1,2. \tag{9}$$

That is why, solutions of system (4), (8) connected by relations (9) are considered as equivalent.

Making the change (9) in equations 1), 2), 3b) by the appropriate, choice of functions h_i, Φ_i one can obtain $f_1 = f_2 = 1$, $A_1 = A_2 = 0$. Consequently, functions ω_1, ω_2 satisfy the equations

$$\omega_{1x_\mu}\omega_{2x^\mu} = 0, \quad \omega_{1x_\mu}\omega_{1x^\mu} + \omega_{2x_\mu}\omega_{2x^\mu} = 0.$$

whence

$$(\omega_1 \pm \omega_2)x_\mu(\omega_1 \pm \omega_2)x_\mu = 0.$$

Integrating the above equations we get

$$\omega_1 = f(\xi) + g(\eta), \quad \omega_2 = f(\xi) - g(\eta), \tag{10}$$

where f, g are arbitrary smooth functions, $\xi = \frac{1}{2}(x_1 + x_0)$, $\eta = \frac{1}{2}(x_1 - x_0)$.

Substitution of the formulas (10) into equations 1), 2) from (4) yields the following equations for a function $A(x_0, x_1)$

$$(\ln A)_{x_0} = 0, \quad (\ln A)_{x_1} = 0,$$

whence $A = 1$.

At last, substituting the obtained results into the equation 3b) from (8) we come to a conclusion that the problem of integration of system (4), (8) is reduced to solution of the functional-differential equation

$$V(x_1) = [g_1(f+g) - g_2(f-g)]\frac{df}{d\xi}\frac{dg}{d\eta}. \tag{11}$$

And what is more, solution with separated variables (3) reads

$$u = \varphi_1(f(\xi) + g(\eta))\varphi_2(f(\xi) - g(\eta)). \tag{12}$$

To integrate (11) it is convenient to make the hodograph transformation

$$\xi = P(f), \quad \eta = R(g) \tag{13}$$

with $\dot{P} \not\equiv 0$, $\dot{R} \not\equiv 0$, equation (11) taking the form

$$g_1(f+g) - g_2(f-g) = \dot{P}(f)\dot{R}(g)\dot{R}(g)V(P+R). \tag{14}$$

Evidently, equality (14) is equivalent to the following relation:

$$(\partial_f^2 - \partial_g^2)[\dot{P}(f)\dot{R}(g)V(P+R)] = 0$$

or

$$(\dddot{P}\dot{R} - \dddot{R}\dot{P})V + 3\dot{P}\dot{R}(\ddot{P} - \ddot{R})\dot{V} + \dot{P}\dot{R}(\dot{P}^2 - \ddot{R}^2)\ddot{V} = 0. \tag{15}$$

Without going into details of integration of equation (15) we give the final results.

Theorem 1. *The general solution of (15) is given by one of the following formulas:*

1. $V = V(x_1)$ *is an arbitrary function*, $\dot{R} = \dot{P} = \alpha$;
2. $V = m(x_1 + C)\dot{p}^2 = \alpha P + \beta$, $\dot{R}^2 = \alpha R + \gamma$;

$$\begin{aligned} &3. \quad V = m(x_1 + C)^{-2}, \quad P = F(f), \quad R = G(g), \\ &\quad \dot{F}^2 = \alpha F^4 + \beta F^3 + \gamma F^2 + \delta F + \rho, \\ &\quad \dot{G}^2 = \alpha G^4 - \beta G^3 + \gamma G^2 - \delta G + \rho; \end{aligned} \tag{16}$$

4. $V = m \sin^{-2}(x_1 + C), \quad P = \operatorname{arctg} F(f), \quad R = \operatorname{arctg} G(g),$

where F, G are determined by (16);

5. $V = m \operatorname{sh}^{-2}(x_1 + C), \quad P = \operatorname{arth} F(f), \quad R = \operatorname{arth} G(g),$

where F, G are determined by (16);

6. $V = m \operatorname{ch}^{-2}(x_1 + C), \quad P = \operatorname{arctg} F(f), \quad R = \operatorname{arth} G(g),$

where F, G are determined by (16);

7. $V = m \exp(-\alpha x_1), \quad \dot{P}^2 = \alpha e^{2P} + \beta e^P + \gamma, \quad \dot{R}^2 = \alpha e^{2R} + \delta e^R + \rho;$
8. $V = \cos^{-2}(x_1 + C)[m_1 + m_2 \sin(x_1 + C)],$
   $\dot{P}^2 = \alpha^2 \sin 2P + \beta^2, \quad \dot{R}^2 = \alpha^2 \sin 2R + \beta^2;$
9. $V = \operatorname{ch}^{-2}(x_1 + C)[m_1 + m_2 \operatorname{sh}(x_1 + C)],$
   $\dot{P}^2 = \alpha \operatorname{sh} 2P + \beta \operatorname{ch} 2P - \gamma^2, \quad \dot{R}^2 = \alpha \operatorname{sh} 2R - \beta \operatorname{ch} 2R - \gamma^2;$
10. $V = \operatorname{sh}^{-2}(x_1 + C)[m_1 + m_2 \operatorname{ch}(x_1 + C)],$
    $\dot{P}^2 = \alpha \operatorname{sh} 2P + \beta \operatorname{ch} 2P - \gamma^2, \quad \dot{R}^2 = -\alpha \operatorname{sh} 2R + \beta \operatorname{ch} 2R - \gamma^2;$
11. $V = m_1 \exp C_1 x_1 + m_2 \exp 2C x_1, \quad \ddot{P} = \alpha \dot{P}^2 + \beta, \quad \ddot{R} = \alpha \dot{R}^2 + \beta;$
12. $V = m_1 + m_2(x_1 + C)^{-2}, \quad \dot{P}^2 = \alpha P^2 + \beta P + \gamma, \quad \dot{R}^2 = \alpha R^2 - \beta R + \gamma;$
13. $V = m, \quad \dot{P}^2 = \alpha P^2 + \beta_1 P + \gamma_1, \quad \dot{R}^2 = \alpha R^2 + \beta_2 R + \gamma_2.$

Here α, β_i, γ_i, σ, ρ, m, m_1, m_2, C are arbitrary real constants.

Thus, Theorem 1 gives the complete solution of the problem of VS in wave equation (1).

Note 1. Equation (1) with potentials $V = m \sin^{-2} x_1$, $V = m \operatorname{ch}^2 x_1$, $V = m \operatorname{sh}^{-2} x_1$ is reduced to equation (1) with the potential $V = m x_1^{-2}$ by the changes of variables

$$\begin{aligned} &\frac{1}{2}(y_1 \pm y_0) = \operatorname{arctg} \frac{1}{2}(x_1 \pm x_0), \\ &\frac{1}{2}(y_1 \pm y_0) = \operatorname{arcth} \frac{1}{2}(x_1 \pm x_0), \\ &\frac{1}{2}(y_1 + y_0) = \operatorname{arctg} \frac{1}{2}(x_1 + x_0), \\ &\frac{1}{2}(y_1 - y_0) = \operatorname{arcth} \frac{1}{2}(x_1 - x_0). \end{aligned}$$

Note 2. Equation (1) with the potential $V = m \exp C x_1$ is reduced to the Klein–Gordon–Fock equation $\Box u + m u = 0$ with the change of variables

$$\frac{1}{2}(y_1 \pm y_0) = C^{-1} \exp \frac{C}{2}(x_1 \pm x_0).$$

It is evident that the equation (1) admits VS in Cartesian coordinates $\omega_1 = x_0$, $\omega_2 = x_1$ under arbitrary function $V(x_1)$. That is why the most interesting potentials are such that there exist new coordinate systems providing VS. From the Theorem 1 and Notes 1, 2 it follows that equations (1) admitting VS in, at least, two inequivalent coordinate systems, are locally equivalent to one of the following wave equations

$$\begin{array}{ll} 1. & \Box u + mu = 0, \\ 2. & \Box u + mx_1 u = 0, \\ 3. & \Box u + mx_1^{-2} u = 0, \\ 4. & \Box u + (m_1 + m_2 x_1^{-2})u = 0, \\ 5. & \Box u + (m_1 + m_2 \sin x_1)\cos^{-2} x_1 u = 0, \\ 6. & \Box u + (m_1 + m_2 \operatorname{sh} x_1)\operatorname{ch}^{-2} x_1 u = 0, \\ 7. & \Box u + (m_1 + m_2 \operatorname{ch} x_1)\operatorname{sh}^{-2} x_1 u = 0, \\ 8. & \Box u + (m_1 + m_2 e^{x_1})e^{x_1} u = 0. \end{array} \tag{17}$$

A detailed analysis of the coordinate systems providing VS in equation (17) will be carried out in our future work.

In conclusion, we note that the equation (1) is intimately connected with the wave equation

$$v_{tt} - C^2(x)v_{xx} = 0. \tag{18}$$

This connection is given by the formula

$$v(t,x) = \sqrt{C(x)}u\left(t, \int \frac{dx}{C(x)}\right). \tag{19}$$

Applying the Theorem 1 and the formula (19) it is not difficult to carry out VS in partial differential equation (18).

Besides, Lorentz-invariant wave equation

$$u_{y_0y_0} - u_{y_1y_1} + U(y_0^2 - y_1^2)u = 0 \tag{20}$$

can also be reduced to the form (1), where $U(\tau) = \frac{1}{4\tau}V(\ln\tau)$ by the change of variables

$$y_0 = e^{x_1/2}\operatorname{ch} x_0, \quad y_1 = e^{x_1/2}\operatorname{sh} x_0.$$

That is why, one can at once, point out all potentials $U = U(\tau)$, $\tau = x_0^2 - x_1^2$ providing VS in the wave equation (20):

$$\begin{array}{l} U = m\tau^{-1}\ln\tau, \quad U = m\tau^{-1}(\ln\tau)^{-2}, \quad U = m_1\tau^{-1} + m_2\tau^{-1}(\ln\tau)^{-2}, \\ U = m\tau^{-1}, \quad U = \tau^{-1}(m_1 + m_2\sin\ln\tau)(\cos\ln\tau)^{-2}, \\ U = \tau^{-1}(m_1 + m_2\operatorname{sh}\ln\tau)(\operatorname{ch}\ln\tau)^{-2}, \\ U = \tau^{-1}(m_1 + m_2\operatorname{ch}\ln\tau)(\operatorname{sh}\ln\tau)^{-2}, \quad U = m_1 + m_2\tau. \end{array}$$

1. Miller W., Symmetry and separation of variables, London, Addison Wesley, 1977, 280 p.
2. Шаповалов В.Н., Симметрия и разделение переменных в линейных дифференциальных уравнениях второго порядка, *Дифференциальные уравнения*, 1980, **16**, № 10, 1864–1874.
3. Багров В.Г., Гитман Д.М., Тернов Д.М., О точных, решениях релятивистских волновых уравнениях, Новосибирск, Наука, 1982, 143 с.
4. Фущич В.И., Штелень В.М., Серов Н.И., Симметрийный анализ и точные решения нелинейных уравнений математической физики, Киев, Наук. думка, 1989, 336 с.
5. Fushchych W.I., Zhdanov R.Z., On some new exact solutions of the nonlinear d'Alembert–Hamilton system, *Phys. Lett. A*, 1989, **141**, № 3–4, 113–115.
6. Fushchych W.I., Zhdanov R.Z., Yegorchenko I.A., On the reduction of the nonlinear multi-dimensional wave equations and compatibility of the d'Alembert–Hamilton system, *J. Math. Anal. Appl.*, 1991, **161**, № 2, 352–360.
7. Koornwinder T.H., A priecise definition of separation of variables, *Lecture Notes in Math.*, 1980, **810**, 240–263.