Anti-reduction of the nonlinear wave equation

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Ми запропонували конструктивний метод зведення рівняння з частинними похідними до декількох рівнянь з меншим числом незалежних змінних. Застосувавши цей підхід до багатовимірного нелінійного хвильового рівняння, ми побудували низку принципово нових анзаців, які редукують його до двох звичайних диференціальних рівнянь.

The wide class of solutions of the multi-dimensional wave equation

$$\Box u \equiv u_{x_0x_0} - \Delta_3 u = F(u) \tag{1}$$

can be obtained by means of the following ansatz [1–3]:

$$u = \varphi(\omega), \tag{2}$$

where φ is an arbitrary smooth function and $\omega = \omega(x)$ is the absolute invariant of some three-dimensional subgroup of the Poincaré group $P(1,3)$. As a result, one gets ordinary differential equation (ODE) for a function $\varphi(\omega)$. That is why, the term "reduction" is used: a number of dependent and independent variables is decreased.

On the other hand, there are examples of ansatzes reducing one nonlinear partial differential equation (PDE) to two or even to three equations [4]. Such procedure leads to an increase of the number of dependent variables and is called an "anti-reduction" [4].

In the present paper we suggest a regular approach to the anti-reduction of the nonlinear differential equation (1).

Consider the ansatz

$$u(x) = f(x, \varphi_1(\omega_1), \varphi_2(\omega_2), \ldots, \varphi_N(\omega_N)) \tag{3}$$

and the following ordinary differential equations:

$$\ddot{\varphi}_i = R_i(\omega_i, \varphi_i, \dot{\varphi}_i), \quad i = \overline{1, N}, \tag{4}$$

where f, R_i are smooth enough functions, $\omega_i = \omega_i(x) \in C^2(\mathbb{R}^n, \mathbb{R}^1)$, $i = \overline{1, N}$. If substitution of (3) into Eq. (1) with subsequent exclusion of the second derivatives $\ddot{\varphi}^i$, $i = \overline{1, N}$ according to (4) yields an identity with respect to variables $\dot{\varphi}^i$, φ_i, $i = \overline{1, N}$ then we say that the anti-reduction of nonlinear PDE (1) to N ODE takes place.

In fact, the above definition contains an algorithm of the anti-reduction. We are going to realize it, provided $N = 2$.

Theorem. *The equation (1) with a logarithmic nonlinearity*

$$\Box u = \lambda u \ln u, \quad \lambda \in \mathbb{R}^1 \tag{5}$$

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is the only nonlinear wave equation belonging to the class of PDE (1) that admits anti-reduction to two second-order ODE and that is more the ansatz (2) has the form

$$u(x) = a(x)\varphi_1(\omega_1)\varphi_2(\omega_2), \tag{6}$$

where $a(x)$, $\omega_1(x)$, $\omega_2(x)$ *are smooth functions satisfying the system of PDE*

$$\begin{aligned} &1)\quad \omega_{1x_\mu}\omega_{2x_\mu} = 0,\\ &2)\quad a\Box\omega_i + 2a_{x\mu}\omega_{ix_\mu} = 0, \quad i=\overline{1,2},\\ &3)\quad \omega_{ix_\mu}\omega_{ix_\mu} = Q_i(\omega_i), \quad i=\overline{1,2},\\ &4)\quad \Box a = \lambda \ln a. \end{aligned} \tag{7}$$

Here Q_i *are arbitrary smooth functions,* $h_{x_\mu}g_{x_\mu} = h_{x_0}g_{x_0} - \sum\limits_{a=1}^{3} h_{x_a}g_{x_a}$.

Omitting intermidiate computations, we adduce main steps of the proof.

Substituting (3) with $N = 2$ into Eq. (1), we get

$$\begin{aligned} &f_{x_\mu x_\mu} + \sum_{i=1}^{2}\{f_{\varphi_i}(\ddot\varphi_i\omega_{ix_\mu}\omega_{ix_\mu} + \dot\varphi_i\Box\omega_i) + f_{\varphi_i\varphi_i}\dot\varphi_i^2\omega_{ix_\mu}\omega_{ix_\mu} + 2f_{\varphi_i x_\mu}\omega_{ix_\mu}\varphi_i\} + \\ &\qquad + 2f_{\varphi_1\varphi_2}\dot\varphi_1\dot\varphi_2\omega_{1x_\mu}\omega_{2x_\mu} = F(f(x,\varphi_1,\varphi_2)). \end{aligned}$$

Replacing $\ddot\varphi_i$ by $R_i(\omega_i,\varphi_i,\dot\varphi_i)$ and splitting the obtained equality with respect to $\dot\varphi_1$, $\dot\varphi_2$, we have

$$\begin{aligned} &R_i = A_i(\omega_i,\varphi_i)\dot\varphi_i^2 + B_i(\omega_i,\varphi_i)\dot\varphi_i + C_i(\omega_i,\varphi_i), \quad i=\overline{1,2},\\ &\omega_{1x_\mu}\omega_{2x_\mu}f_{\varphi_1\varphi_2} = 0. \end{aligned}$$

Since the equality $f_{\varphi_1\varphi_2} = 0$ leads to the case $F_{uu} = 0$, we can put $f_{\varphi_1\varphi_2} \neq 0$ whence $\omega_{1x_\mu}\omega_{2x_\mu} = 0$.

By force of the above facts we get

$$\begin{aligned} &1)\quad f_{\varphi_i\varphi_i} + A_i f_{\varphi_i} = 0, \quad i=\overline{1,2},\\ &2)\quad f_{\varphi_i}(B_i\omega_{ix_\mu}\omega_{ix_\mu} + \Box\omega_i) + 2f_{\varphi_i x_\mu}\omega_{ix_\mu} = 0,\\ &3)\quad f_{x_\mu x_\mu} + \sum_{i=1}^{2} C_i f_{\varphi_i}\omega_{ix_\mu}\omega_{ix_\mu} = F(f),\\ &4)\quad \omega_{1x_\mu}\omega_{2x_\mu} = 0. \end{aligned} \tag{8}$$

From the first two equations of the system (8) it follows that

$$f = H_1(\omega_1,\varphi_1)H_2(\omega_2,\varphi_2)a(x) + b(x),$$

where H_i, $a(x)$, $b(x)$ are arbitrary smooth functions.

By redefining functions $\varphi_i : \varphi_i \to \tilde\varphi_i H_i(\omega_i,\varphi_i)$, $i = 1, 2$, we may choose

$$f = a(x)\varphi_1(\omega_1)\varphi_2(\omega_2) + b(x), \tag{9}$$

whence $A_1 = A_2 = 0$.

From the Eq. 2 of the system (8) by force of (9) it follows that $B_i = B_i(\omega_i)$, $i = 1, 2$. Consequently, by redefining functions ω_i

$$\omega_i \to \tilde{\omega}_i = W_i(\omega_i), \quad i = \overline{1,2},$$

we may choose $B_1 = B_2 = 0$. As a result, the system (8) is read

$$\begin{aligned}
&1) \quad \omega_{1x_\mu}\omega_{2x_\mu} = 0,\\
&2) \quad a\Box\omega_i + 2a_{x_\mu}\omega_{ix_\mu} = 0, \quad i = \overline{1,2},\\
&3) \quad (\Box a)\varphi_1\varphi_2 + \Box b + a[C_1(\omega_1,\varphi_1)\varphi_2\omega_{1x_\mu}\omega_{1x_\mu} +\\
&\qquad\qquad + C_2(\omega_2,\varphi_2)\varphi_1\omega_{2x_\mu}\omega_{2x_\mu}] = F(a\varphi_1\varphi_2 + b).
\end{aligned} \tag{10}$$

The only thing left is to split Eq. 3 from (10) with respect to variables φ_1, φ_2. Dividing Eq. 3 into $\varphi_1\varphi_2$ and differentiating it with respect to variables $\varphi_1\varphi_2$ we get $\{(\varphi_1\varphi_2)^{-1}[F(a\varphi_1\varphi_2 + b) - \Box b]\}_{\varphi_1\varphi_2} = 0$, whence

$$a^2x^2\frac{d^2F}{d\omega^2} - ax\frac{dF}{d\omega} + F = \Box b, \quad x = \varphi_1\varphi_2, \quad \omega = ax + b. \tag{11}$$

Differentiation of (11) with respect to x yields

$$ax\frac{d^3F}{d\omega^3} + \frac{d^2F}{d\omega^2} = 0.$$

Since we are interested in a nonlinear case, the inequality $\ddot{F} \neq 0$ holds. Hence, it follows that

$$\dddot{F}(\ddot{F})^{-1} = -(ax)^{-1}$$

or

$$\ddot{F}(\dddot{F})^{-1} = -\omega + b.$$

Differentiating the above equality with respect to ω we obtain nonlinear ODE for $F(\omega)$: $\ddot{F}\,\ddddot{F} - 2(\dddot{F})^2 = 0$, which general solution reads $F(\omega) = \alpha_1^{-2}(\alpha_1\omega + \alpha_2)\ln(\alpha_1\omega + \alpha_2) + \alpha_3\omega + \alpha_4$ and what is more $b = -\alpha_2\alpha_1^{-1}$ (without loss of generality we may put $b = \alpha_2 = 0$).

In the above formulae α_1, α_2, α_3, α_4 are arbitrary real constants, $\alpha_1 \neq 0$.

Substitution of the expression for F

$$F = \lambda_1\omega\ln\omega + \lambda_1\omega + \lambda_3 \tag{12}$$

into Eq. 3 from the system (10) yields

$$\begin{aligned}
&\omega_{ix_\mu}\omega_{ix_\mu} = Q_i(\omega_i), \quad i = \overline{1,2},\\
&C_i = \lambda_1 Q_i^{-1}(\omega_i)\varphi_i\ln\varphi_i, \quad i = \overline{1,2},\\
&\Box a = \lambda_1 a\ln a + \lambda_2 a, \quad \lambda_3 = 0.
\end{aligned}$$

Since in Eq. (12) $\lambda_1 \neq 0$, we can rescale the function $\omega \to k\omega$ in such a way that $F(\omega)$ takes the form $F = \lambda_1\omega\ln\omega$. The theorem is proved.

Note. A classical example of the anti-reduction of mathematical physics equations is the procedure of separation of variables. But the method of separation of variables can be effectively applied to linear second-order PDEs only, whereas the anti-reduction procedure is evidently applicable to nonlinear differential equations.

Thus each solution of the system (7) after being substituted into ansatz (6) reduces the nonlinear PDE (5) to two second-order ODEs

$$Q_i(\omega_i)\dot{\varphi}_i = \lambda\varphi_i \ln\varphi_i, \quad i = \overline{1,2}.$$

Let us write down some particular solutions of Eqs. (7) under $a = 1$.

1. $\omega_1 = \ln(x_0^2 - x_3^2), \quad \omega_2 = \ln(x_1^2 + x_2^2);$
2. $\omega_1 = \ln(x_0^2 - x_3^2), \quad \omega_2 = x_1;$
3. $\omega_1 = x_0, \quad \omega_2 = \ln(x_1^2 + x_2^2);$
4. $\omega_1 = \ln(x_1^2 + x_2^2), \quad \omega_2 = x_3;$
5. $\omega_1 = x_0, \quad \omega_2 = x_1;$
6. $\omega_1 = (x_0^2 - x_1^2 - x_2^2)^{-1/2}, \quad \omega = x_3;$
7. $\omega_1 = x_0, \quad \omega_2 = (x_1^2 + x_2^2 + x_3^2)^{-1/2};$
8. $\omega_1 = x_1 \cos\omega_1 + x_2 \sin\omega_1 + \omega_2, \quad \omega_2 = x_1 \sin\omega_1 - x_2 \cos\omega_1 + \omega_3.$

In the above formulae ω_1, ω_2, ω_3 are arbitrary smooth functions on $x_0 + x_3$.

Let us emphasize that the above ansatzes can not be obtained within the framework of the classical Lie approach (see, e.g. [5, 6]), because the maximal symmetry group admitted by Eq. (5) is the Poincaré group $P(1,3)$ [2] and the general form of Poincaré-invariant ansatz is given by the formula (2).

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