New conditionally invariant solutions for non-linear d'Alembert equation

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We describe all ansatzes of a specific form that reduce the non-linear d'Alembert equation. In this way we obtain some new solutions of the equation with a polynomial non-linearity.

1. Introduction. Let us consider a non-linear d'Alembert equation of the form

$$\Box u = \lambda u^k, \tag{1}$$

where $u = u(x_0, x_1, x_2, x_3)$ is a real function; $k \neq 1$, λ are parameters,

$$\Box u \equiv \frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2}.$$

Equation (1) is invariant under the Poincaré algebra $AP(1,3) \Supset D$ with the following basis operators:

$$\begin{gathered}\partial_0, \quad \partial_a, \quad J_{0a} = x_0\partial_a + x_a\partial_0, \quad J_{ab} = x_a\partial_b - x_b\partial_a, \\ D = x_0\partial_0 + x_a\partial_a + \frac{2}{1-k}u\partial_u,\end{gathered} \tag{2}$$

when k is arbitrary, $k \neq 1$. Here $a, b = 1, 2, 3$, and we imply summation over the repeated indices from 1 to 3. We shall not consider here the special case $k = 3$ when equation (1) is invariant under the conformal algebra.

All similarity solutions for equation (1) are adduced in [1, 2]. The similarity ansatzes corresponding to three-dimensional subalgebras of the algebra (2) have the form

$$u = f(x)\varphi(\omega), \tag{3}$$

where $f(x)$ is some function, $\omega = \omega(x)$ is a new invariant variable.

In this paper we try to search for a wider class of solutions than similar ones by means of the ansatz (3). Some ansatzes of this form were described in [3]. An example of such ansatz is

$$u = (x^2)^{-1/2}\varphi(\alpha x), \tag{4}$$

where $x^2 = x_0^2 - x_a x_a$, $\alpha_0^2 - \alpha_a\alpha_a = 0$.

The substitution (3) reduces equation (1) to an ordinary differential equation of the functions f and ω satisfy the following set of equations:

$$\begin{gathered}\Box f = f^k S(\omega), \\ 2f_\mu\omega_\mu + f\Box(\omega) = f^k T(\omega), \quad \omega_\mu\omega_\mu = R(\omega) f^{k-1}.\end{gathered} \tag{5}$$

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Here $f_\mu \equiv \frac{\partial f}{\partial x_\mu}$, the summation over the repeated Greek indices is as follows: $f_\mu \omega_\mu \equiv f_0\omega_0 - f_a\omega_a$, $a = 1,2,3$; S, T, R are some functions; T and R do not vanish simultaneously.

Further we shall consider the system (5) for the case $\omega_\mu\omega_\mu = 0$.

2. New ansatzes for the d'Alembert equation (1). We succeeded to find all solutions of the system (5) for $\omega = \alpha x$, $\alpha^2 = 0$. In this case the system (5) reduces to the equations

$$\Box f = f^k S(\alpha x), \qquad 2f_\mu\alpha_\mu = f^k T(\alpha x).$$

Its solutions have the following form:

$$f = [h(\omega, \beta x, \gamma x) + \delta x)]^{\frac{1}{1-k}}, \tag{6}$$

where the parameters α_μ, β_μ, γ_μ, δ_μ satisfy the relations $\alpha\beta = \alpha\gamma = \delta^2 = \beta\gamma = 0$, $\alpha\delta = -\beta^2 = -\gamma^2 = 1$.

$$h = \frac{1}{2}\frac{(\beta x)^2(\omega + B_1) + 2B_3^2(\beta x)(\gamma x) + (\gamma x)^2(\omega + B_2)}{(\omega + B_1)(\omega + B_2) - B_3^2}, \tag{7}$$

$$h = \frac{(\beta x)^2}{2\omega + B_1}, \tag{8}$$

$$h = B_1\beta x + B_2 + \frac{B_1^2}{2}\omega. \tag{9}$$

Here B_1, B_2, B_3 are some constants. If $B_1 = B_2$, $B_3 = 0$ we get an ansatz that is equivalent to (4).

3. Operators of conditional symmetry for equation (1). The notion of conditional symmetry had been defined in [2, 4–6]. This approach enabled to construct wide classes of exact solutions for nonlinear partial differential equations of mathematical physics (see [2, 4–6, 8]). In this paper we do not search specially for operators of conditional symmetry but for ansatzes of the form (3) explicitly.

The following statement describes the operators of conditional invariance corresponding to ansatzes of the form (3) with $\omega = \alpha x$, $\alpha^2 = 0$, f being of the form (6), (7).

Theorem 1. *Equation (1) with the additional conditions*

$$\begin{aligned} &L_1 = f\beta_\mu u_\mu - \beta_\mu f_\mu u = 0, \\ &L_2 = f\gamma_\mu u_\mu - \gamma_\mu f_\mu u = 0, \\ &L_3 = 2\delta_\mu u_\mu(1-k) - f^{k-1}u = 0 \end{aligned} \tag{10}$$

is invariant under operators:

$$\begin{aligned} &Q_1 = r(x)(f\beta_\mu\partial x_\mu - \beta_\mu f_\mu u\partial u) = 0, \\ &Q_2 = r(x)(f\gamma_\mu\partial x_\mu - \gamma_\mu f_\mu u\partial u = 0, \\ &Q_3 = r(x)(2\delta_\mu\partial x_\mu - \tfrac{1}{1-k}f^{k-1}u\partial u = 0, \end{aligned} \tag{11}$$

where $r(x)$ is an arbitrary non-zero function, f satisfies the equations

$$f_\mu\delta_\mu = \frac{1}{1-k}f^k, \qquad \Box f = f^k S(\omega), \tag{12}$$

where S is some function.

The above theorem can be proved by means of the Lie algorithm (see e.g. [7]).

Note 1. The same ansatzes may also be obtained from the Lie symmetry operators.

4. Exact solutions of equation (1). The ansatz (3) with $\omega = \alpha x$, $\alpha^2 = 0$, f of the form (6), (7) reduces equation (1) to the following ordinary differential equation:

$$\varphi' \frac{2}{1-k} + S(\omega)\varphi = \lambda \varphi^k, \tag{13}$$

$S(\omega)$ being of the form

$$S(\omega) = -\frac{1}{1-k} \frac{\omega + B_1 + B_2}{(\omega + B_1)(\omega + B_2) - B_3^2},$$

Equation (13) for arbitrary constants B_1, B_2, B_3, $k \neq 1$ can be solved in quadratures:

$$\varphi = \sqrt{\theta} \left[\frac{\lambda(1-k)^2}{2} \int \theta(\omega)^{\frac{k-1}{2}} d\omega \right]^{\frac{1}{1-k}}, \tag{14}$$

where $\theta = (\omega + B_1)(\omega + B_2) - B_3^2$.

Substituting (14) into (3) with f of the form (6), (7), we can obtain a class of solutions for the non-linear d'Alembert equation (1).

5. Compatibility and solutions of the system (5) with $\boldsymbol{\omega_\mu \omega_\mu = 0}$. In this case $R(\omega) = 0$, so $T(\omega)$ must not vanish. We can take $T(\omega) = \frac{2}{1-k}$ and obtain the system

$$f_\mu \omega_\mu + \frac{1}{2} f \Box \omega = \frac{1}{1-k} f^k, \qquad \Box f = f^k S(\omega). \tag{15}$$

If $\Box \omega = 0$, then from the first equation of (15)

$$f = \left[h(\omega, \theta^1, \theta^2) + \theta^3) \right]^{\frac{1}{1-k}}, \tag{16}$$

where θ^1, θ^2, θ^3 are functions on x,

$$\begin{aligned} &\theta^1_\mu \theta^1_\mu = \theta^2_\mu \theta^2_\mu = -1, \\ &\theta^1_\mu \omega_\mu = \theta^2_\mu \omega_\mu = \theta^3_\mu \theta^3_\mu = \theta^1_\mu \theta^2_\mu = 0, \\ &\theta^1_\mu \theta^3_\mu = 1. \end{aligned} \tag{17}$$

With the substitution (16) the second equation (15) reduces to the form

$$\Phi_{\theta^1\theta^1} + \Phi_{\theta^2\theta^2} = \hat{S}(\omega), \qquad 2\Phi_\omega - \Phi^2_{\theta^1} - \Phi^2_{\theta^2} = 0. \tag{18}$$

The compatibility and solutions of the system of Laplace and Hamilton–Jacobi equations were considered in detail in [8]. The system (18) is compatible iff

$$\hat{S}(\omega) = \frac{\rho'}{\rho}, \quad \text{where} \quad \rho''' = 0.$$

If we take for the solutions of the system (17)

$$\theta^1 = \beta x, \quad \theta^2 = \gamma x, \quad \theta^3 = \delta x,$$

where β_μ, γ_μ, δ_μ are parameters satisfying (6), we shall get the solutions (6), (7)–(9) of the system (15).

Note 2. A system similar to (15) arose in [8] while searching for ansatzes of the form $u = \exp(if(x))\varphi(\omega)$ for a nonlinear Schrödinger equation $2iu_t + u_{aa} - uF(|u|) = 0$. It is known [9] that complex n-dimensional non-linear d'Alembert equation can be reduced by similarity methods to $(n-1)$-dimensional Schrödinger equation.

Note 3. The ansatz (3), (6), (7) can be used to get solutions also for complex non-linear d'Alembert equation, the function φ being complex-valued.

For the equation

$$\Box u = \lambda u(uu^*)^{\frac{k-1}{2}},$$

we get the reduced equation

$$2\varphi' - \frac{\rho'}{\rho}\varphi = \lambda(1-k)(\varphi\varphi^*)^{\frac{k-1}{2}},$$

where $\rho = (\omega + B_1)(\omega + B_2) - B_3^2$.

From the reduced equation we can find φ:

$$\varphi = \sqrt{\rho}\left[\frac{\lambda(1-k)^2}{2}\int \rho^{\frac{k-1}{2}} d\omega\right]^{\frac{1}{1-k}} \exp i\sigma,$$

where σ is an arbitrary constant.

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