Hodograph transformations and generating of solutions for nonlinear differential equations

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Перетворення годографа однієї скалярної функції в $\mathbb{R}(1,1)$ та $\mathbb{R}(1,3)$, а також двох скалярних функцій в $\mathbb{R}(1,1)$ використані для розмноження розв'язків нелінійних рівнянь; побудовані класи годограф-інваріантних рівнянь другого порядку.

The results of using the hodograph transformations for solution of applied problems are well-known. One can find them for example in [1, 2, 3]. We note also the paper [4], in which a number of invariants for hodograph transformation as well as hodograph-invariant equations were constructed.

1. Hodograph-invariant and -linearizable equations in $\mathbb{R}(1,1)$. Let us consider the hodograph transformation for one scalar function ($M=1$) of two independent variables $x=(x_0,x_1)$, $n=2$:

$$\begin{aligned} &u(x)=y_1, \quad x_0=y_0, \quad x_1=v(y), \\ &\delta=v_1=\partial_1 v=\frac{\partial v}{\partial y_1}\neq 0, \quad y=(y_0,y_1). \end{aligned} \tag{1}$$

Differential prolongations of the transformation (1) generate such expressions for the first and second order derivatives:

$$u_1=v_1^{-1}, \quad u_0=-v_0v_1^{-1}, \tag{2}$$

$$\begin{aligned} &u_{11}=-v_1^{-3}v_{11}, \quad u_{10}=-v_1^{-3}(v_1v_{10}-v_0v_{11}), \\ &u_{00}=-v_1^{-3}[v_0^2v_{11}-2v_0v_1v_{10}+v_1^2v_{00}]. \end{aligned} \tag{3}$$

It is clear that (1) is an involutory transformation. This allows to write a set of differential expressions of order ≤ 2, which are absolutely invariant under the transformation (1):

$$\begin{aligned} &f^0(x_0), \quad f^1(x_1,u), \quad f^2(u_1,u_1^{-1}), \quad f^3(u_0,-u_0u_1^{-1}), \quad f^4(u_{11},-u_1^{-3}u_{11}), \\ &f^5(u_{10},-u_1^{-3}(u_1u_{10}-u_0u_{11})), \quad f^6(u_{00},-u_1^{-3}[u_0^2u_{11}-2u_0u_1u_{10}+u_1^2u_{00}]). \end{aligned} \tag{4}$$

Here f^0 is an arbitrary smooth function, f^i, $i=\overline{1,6}$ are arbitrary functions symmetric on arguments, i.e. $f^i(x,z)=f^i(z,x)$. So, the second order PDE invariant under the transformation (1) has the form

$$F(\{f^\sigma\})=0, \quad \{f^\sigma\}=\{f^0,f^1,\ldots,f^6\}, \quad \sigma=\overline{0,6}, \tag{5}$$

F is an arbitrary smooth function.

Such well-known equations are contained in the class (5):

1. $u_0^2-u_1^2-1=0$ — the eikonal equation; (6)

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2. $u_{11} - u_{00}[u_0^2 u_{11} - 2u_0 u_1 u_{10} + u_1^2 u_{00}] = 0$ — the Born–Infeld equation; (7)

3. $u_{00}u_{11} - u_{10}^2 = 0$ — the Monge–Amperé equation; (8)

4. $u_0 = f(u_1)u_{11}$, $f(u_1) = f(u_1^{-1})u_1^{-2}$ — the nonlinear heat equation [5]. (9)

Particularly, such equation as

$$u_0 - u_1^{-1}u_{11} = 0 \tag{10}$$

is contained in the last class (9).

Let $\overset{(1)}{u}(x_0, x_1)$ be a known solution of Eq. (5). To construct a new solution $\overset{(2)}{u}(x_0, x_1)$ let us write the first solution replacing in it an argument x_1 for parameter τ: $\overset{(1)}{u}(x_0, \tau)$ and substitute it to the hodograph transformation formula (1). So, we obtain the solutions generating formula for Eq. (5).

$$\overset{(2)}{u}(x_0, x_1) = \tau, \quad x_1 = \overset{(1)}{u}(x_0, \tau). \tag{11}$$

Let us now describe some class of (1)-linearizable equations. Making use of formulae (1) to transform general linear second order PDE

$$b^{\mu\nu}(y)v_{\mu\nu} + b^{\mu}(y)v_{\mu} + b(y)v + c(y) = 0, \quad y = (y_0, y_1), \quad \mu, \nu = 0, 1, \tag{12}$$

we obtain

$$\begin{aligned} &b^{00}(x_0, u)u_1^{-3}(u_0^2 u_{11} - 2u_0 u_1 u_{10} + u_1^2 u_{00}) - \\ &\quad - 2b^{10}(x_0, u)u_1^{-3}(u_1 u_{10} - u_0 u_{11}) + b^{11}(x_0, u)u_1^{-3}u_{11} + \\ &\quad + b^0(x_0, u)u_1^{-1}u_0 + b^1(x_0, u)u_1^{-1} - b(x_0, u)x_1 - c(x_0, u) = 0. \end{aligned} \tag{13}$$

$b^{\mu\nu}$, b^{μ}, c are arbitrary smooth functions, $b^{10} = b^{01}$. Summation over repeated indices is understood in the space $\mathbb{R}(1,1)$ with the metric $g_{\mu\nu} = \mathrm{diag}\,(1, -1)$. The repeated use of this transformation to Eq. (12) turn us again to the Eq. (11).

For any equation of the class (12) the principle of nonlinear superposition is satisfied

$$\overset{(3)}{u}(x_0, x_1) = \overset{(1)}{u}(x_0, \tau), \quad \overset{(1)}{u}(x_0, x_1) = \overset{(2)}{u}(x_0, x_1 - \tau), \tag{14}$$

Here $\overset{(k)}{u}(x_0, x_1)$, $k = 1, 2$ are known solutions of Eq. (12), $\overset{(3)}{u}(x_0, x_1)$ is a new solution of this equation. Parameter τ must be eliminated due to second equality of the system (13). For example, such equations important for applications are contained in this class (12):

$$\begin{aligned} &u_0 - u_1^{-2}u_{11} = 0, \quad u_0 u_{11} - u_1 u_{10} = 0, \\ &u_0^2 u_{11} - 2u_0 u_1 u_{10} + u_1^2 u_{00} = 0, \quad u_0 - c(x_0, u)u_1 = 0. \end{aligned}$$

Let us consider now an example of constructing new solutions from two known ones by means of solutions superposition formula (13).

Example 1. A nonlinear heat equation

$$u_0 - u_1^{-2}u_{11} = 0$$

is reduced to the linear equation

$$v_0 - v_{11} = 0 \tag{15}$$

Therefore, the formula (13) is true for (14). The functions

$$\overset{(1)}{u} = x_1, \quad \overset{(2)}{u} = \sqrt{x_1 - 2x_0} \tag{16}$$

are both partial solutions of Eq. (14). We construct a new solution $\overset{(3)}{u}$ of this Eq. (14) via $\overset{(1)}{u}$ and $\overset{(2)}{u}$. It has the form

$$\overset{(3)}{u}(x_0, x_1) = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + x_1 - 2x_0}, \tag{17}$$

2. Hodograph-invariant and -linearizable equations in $\mathbb{R}(1,3)$. The hodograph transformation of a scalar function $u(x)$ of four independent variables $x = (x_0, x_1, x_2, x_3)$ has the form

$$v(x) = y_1, \quad x_1 = v(y), \quad x_\theta = y_\theta, \quad \theta = 0, 2, 3. \tag{18}$$

Prolongation formulae for (18) are obtained via calculations [6, 7]:

$$\begin{aligned}
&u_1 = v_1^{-1}, \quad u_\theta = -v_1^{-1} v_\theta, \quad u_{11} = -v_1^{-3} v_{11},\\
&u_{1\theta} = -v_1^{-3}(v_1 v_{1\theta} - v_\theta v_{11}), \quad v_{\theta\theta} = -v_1^{-3}(v_1^2 v_{\theta\theta} - 2v_\theta v_1 v_{1\theta} + v_\theta^2 v_{11}),\\
&u_{\theta\gamma} = -v_1^{-3}[v_1(v_1 v_{\theta\gamma} - v_\gamma v_{1\theta}) - v_\theta(v_1 v_{1\gamma} - v_\gamma v_{11})].
\end{aligned} \tag{19}$$

Here $\theta, \gamma = 0, 2, 3$, $\theta \neq \gamma$. Making use of involutivity of the transformation (18) we list for it a such set of absolute differential invariant expressions of order ≤ 2:

$$\begin{aligned}
&f^0(x_0, x_2, x_3), \quad f^1(x_1, u), \quad f^2(u_1, u_1^{-1}), \quad f^3(u_\theta, -u_1^{-1} u_\theta),\\
&f^4(u_{11}, -u_1^{-3} u_{11}), \quad f^5(u_{1\theta}, -u_1^{-3}(u_1 u_{1\theta} - u_\theta u_{11})),\\
&f^6(u_{\theta\theta}, -u_1^{-3}(u_1^2 u_{\theta\theta} - 2u_1 u_\theta u_{1\theta} + u_\theta^2 u_{11})).\\
&f^7(u_{\theta\gamma}, -u_1^{-3}[u_1(u_1 u_{\gamma\theta} - u_\gamma u_{1\theta}) - u_\theta(u_1 u_{1\gamma} - u_\gamma u_{11})]).
\end{aligned} \tag{20}$$

There is no summation over θ here, as before, f^0 is an arbitrary smooth function, f^j, $j = \overline{1,7}$ are arbitrary symmetric.

An equation invariant under transformation (18) has the form

$$F(\{f^\lambda\}) = 0 \quad (\lambda = \overline{0,7}). \tag{21}$$

The solutions generating formula has the same form as (10)

$$\overset{(2)}{u}(x_0, x_1, x_2, x_3) = \tau, \quad x_1 = \overset{(1)}{u}(x_0, \tau, x_2, x_3). \tag{22}$$

Here $\overset{(1)}{u}(x)$ is a known solution of Eq. (21), $\overset{(2)}{u}(x)$ is its new solution. The following well-known equations are contained in this class (21):

1. $u_0^2 - u_a u_a - 1 = 0$, $a = \overline{1,3}$, the eikonal equation;
2. $(1 - u_\nu u^\nu)\Box u - u^\mu u^\nu u_{\mu\nu} = 0$, $\mu, \nu = \overline{0,3}$, the Born–Infeld equation [8];
3. $\det(u_{\mu\nu}) = 0$ the Monge–Amperé equation.

Here summation over repeated indices is understood in the space $\mathbb{R}(1,3)$ with the metric $g_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$.

$$\Box u = \partial_\mu \partial^\mu u = u_{00} - u_{11} - u_{22} - u_{33}$$

is the d'Alembert operator,

$$u_a u_a = u_1^2 + u_2^2 + u_2^2 + u_3^2 = (\nabla u)^2.$$

The class of hodograph-linearizable equations in $\mathbb{R}(1,3)$ is constructed analogously as above. Making use of transformation (18) for linear equation (11), written in $\mathbb{R}(1,3)$, we get

$$\begin{aligned} & b^{11}(x_\delta,u)u_1^{-3}u_{11} + b^{\theta\theta}(x_\delta,u)u_1^{-3}(u_1^2 u_{\theta\theta} - 2u_1 u_\theta u_{10} + u_\theta^2 u_{11}) + \\ & \quad + b^{\gamma\theta}(x_\delta,u)u_1^{-3}[u_1(u_1 u_{\gamma\theta} - u_\gamma u_{10}) - u_\theta(u_1 u_{1\gamma} - u_\gamma u_{11})] + \\ & \quad + b^1(x_\delta,u)u_1^{-1}u_\theta - b(x_\delta,u)x_1 - c(x_\delta,u) = 0, \quad x_\delta = (x_0,x_2,x_3). \end{aligned} \tag{23}$$

Here $\delta,\theta = 0,2,3$ and summation over θ is understood in the space $\mathbb{R}(1,2)$ with metric $\tilde g_{\theta\gamma} = \mathrm{diag}\,(1,-1,-1)$.

Note, that multidimensional nonlinear heat equation

$$u_0 - u_1^{-2}(1 + u_2^2 + u_3^2)u_{11} - u_{22} - u_{33} + 2u_1^{-1}(u_2 u_{12} + u_3 u_{13}) = 0 \tag{24}$$

reduces due to transformation (18) to linear equation $v_0 = \Delta_{(3)} v$, where $\Delta_{(3)} \equiv \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator.

So, the solutions superposition formula for the equations (23) and (24) is

$$\overset{(3)}{u}(x_0,x_1,x_2,x_3) = \overset{(1)}{u}(x_0,\tau,x_2,x_3), \tag{25}$$

$$\overset{(1)}{u}(x_0,\tau,x_2,x_3) = \overset{(2)}{u}(x_0,x_1-\tau,x_2,x_3). \tag{26}$$

Example 2. Let partial solutions of Eq. (24)

$$\overset{(1)}{u} = x_0 - x_2 - x_3 - \ln\frac{x_1 - c_2}{c_1}, \qquad \overset{(2)}{u} = \left[\frac{9}{4}c_3^2(x_1 - c_4)^2 - x_2^2 - x_3^2\right]^{\frac{1}{2}}$$

be initial for generating a new solution $\overset{(3)}{u}$. Then this new solution of Eq. (24) is determined via (25), (26) by the equality

$$\begin{aligned} & \overset{(3)}{u}{}^2(x) + x_2^2 + x_3^2 = c_3\big[x_1 - c_2 - c_1\exp\{x_0 - x_2 - x_3 - \overset{(3)}{u}(x)\}\big]^2, \\ & c_3 = \frac{9}{4}c_3^2, \quad c_2 = c_4 + c_2. \end{aligned} \tag{27}$$

Thus, the formula (27) gives us a new solution of Eq. (24) in the implicite form.

3. Hodograph-invariant and -linearizable systems of PDE in $\mathbb{R}(1,1)$. Let us consider two functions $u^\mu(x_0,x_1)$, $\mu = 0,1$ of independent variables x_0, x_1. The hodograph transformation in this case, as is known [2], has the form

$$\begin{aligned} & u^0(x_0,x_1) = y_0, \quad u^1(x_0,x_1) = y_1, \quad x_0 = v^0(y_0,y_1), \quad x_1 = v^1(y_0,y_1), \\ & \delta = u_1^1 u_0^0 - u_0^1 u_1^0 \neq 0, \quad \delta^* = v_1^1 v_0^0 - v_0^1 v_1^0 \neq 0. \end{aligned} \tag{28}$$

The first and second order derevatives are changing as

$$u_1^1=\delta^{*-1}v_0^0,\quad u_0^1=-\delta^{*-1}v_0^1,\quad u_1^0=-\delta^{*-1}v_1^0,\quad u_0^0=\delta^{*-1}v_1^1, \tag{29}$$

$$\begin{aligned}
u_{11}^1 &= -\delta^{*-3}\cdot[(v_0^0)^2(v_0^1v_{11}^0-v_0^0v_{11}^1)+(v_1^0)^2(v_0^1v_{00}^0-v_0^0v_{00}^1)-\\
&\quad -2v_1^0v_0^0(u_0^1v_{10}^0-v_0^0v_{10}^1)],\\
u_{00}^1 &= -\delta^{*-3}\cdot[(v_0^1)^2(v_0^1v_{11}^0-v_0^0v_{11}^1+(v_1^1)^2(v_0^1v_{00}^0-v_0^0v_{00}^1)-\\
&\quad -2v_0^1v_1^1(v_0^1v_{10}^0-v_0^0v_{10}^1)],\\
u_{10}^1 &= \delta^{*-3}\cdot[v_0^0v_0^1(v_0^1v_{11}^0-v_0^0v_{11}^1)+v_1^0v_1^1(v_0^1v_{00}^0-v_0^0v_{00}^1)-\\
&\quad -(v_0^1v_{10}^0-v_1^0v_{10}^1)(v_1^1v_0^0+v_0^1v_1^0)],\\
u_{11}^0 &= -\delta^{*-3}[(v_0^0)^2(v_1^0v_{11}^1-v_1^1v_{11}^0)+(v_1^0)^2(v_1^0v_{00}^1-v_1^1v_{00}^0)-\\
&\quad -2v_1^0v_0^0(v_1^0v_{10}^1-v_1^1v_{10}^0)],\\
u_{00}^0 &= -\delta^{*-3}[(v_0^1)^2(v_1^0v_{11}^1-v_1^1v_{11}^0)+(v_1^1)^2(v_1^0v_{00}^1-v_1^1v_{00}^0)-\\
&\quad -2v_1^1v_0^1(v_1^0v_{10}^1-v_1^1v_{10}^0)],\\
u_{10}^0 &= -\delta^{*-3}[v_0^0v_0^1(v_1^0v_{11}^1-v_1^1v_{11}^0)+v_1^0v_1^1(v_1^0v_{00}^1-v_1^1v_{00}^0)-\\
&\quad -(v_1^0v_{10}^1-v_1^1v_{10}^0)(v_1^1v_0^0+v_0^1v_1^0)].
\end{aligned} \tag{30}$$

Let us now construct the absolute differential invariants with respect to (28)–(30) of order ≤ 2. Making use of involutivity of this transformation we get

$$f^1(x_\mu,u^\mu),\quad \mu=0,1,\quad f^2(u_\mu^\mu,\delta u_\nu^\nu),\quad \mu\neq\nu,\ \ \mu,\nu=0,1,$$

there is no summation over repeated indices here,

$$\begin{aligned}
&f^3(u_\nu^\mu,-\delta^{-1}u_\nu^\mu),\quad \mu\neq\nu,\ \ \mu,\nu=0,1;\\
&f^4(u_{11}^1,-\delta^{-3}[(u_0^0)^2(u_0^1v_{11}^0-u_0^0u_{11}^1)+(u_1^0)^2(u_0^1u_{00}^0-u_0^0u_{00}^1)-\\
&\quad -2u_1^0u_0^0(u_0^1u_{10}^0-u_0^0u_{10}^1)]),\\
&f^5(u_{00}^1,-\delta^{-3}\cdot[(u_0^1)^2(u_0^1u_{11}^0-u_0^0u_{11}^1)+(u_1^1)^2(u_0^1u_{00}^0-u_0^0u_{00}^1)-\\
&\quad -2u_0^1u_1^1(u_0^1v_{10}^0-u_0^0u_{10}^1)]),\\
&f^6(u_{10}^1,-\delta^{-3}\cdot[u_0^0u_0^1(u_0^1u_{11}^0-u_0^0u_{11}^1)+u_1^0u_1^1(u_0^1u_{00}^0-u_0^0u_{00}^1-\\
&\quad -(u_0^1u_{10}^0-u_0^0u_{10}^1)(u_1^1u_0^0+u_0^1u_1^0)]),\\
&f^7(u_{11}^0,-\delta^{-3}\cdot[(u_0^0)^2(u_1^0u_{11}^1-u_1^1u_{11}^0)+(u_1^0)^2(u_1^0u_{00}^1-u_1^1u_{00}^0)-\\
&\quad -2u_1^0u_0^0(u_1^0u_{10}^1-u_1^1u_{10}^0)]),\\
&f^8(u_{00}^0,-\delta^{-3}[(u_0^1)^2(u_1^0u_{11}^1-u_1^1u_{11}^0)+(u_1^1)^2(u_1^0u_{00}^1-u_1^1u_{00}^0)-\\
&\quad -2u_1^1u_0^1(u_1^0u_{10}^1-u_1^1u_{10}^0)]),\\
&f^9(u_{10}^0,-\delta^{-3}[u_0^0u_0^1(u_1^0u_{11}^1-u_1^1u_{11}^0)+u_1^0u_1^1(u_1^0u_{00}^1-u_1^1u_{00}^0)-\\
&\quad -(u_1^0u_{10}^1-u_1^1u_{10}^0)(u_1^1u_0^0+u_0^1u_1^0)]).
\end{aligned} \tag{31}$$

All functions f^k, $k=\overline{1,9}$ are arbitrary smooth and symmetric.

So, we now are able to construct the hodograph-invariant system of second order PDEs

$$F^\sigma(\{f^k\})=0,\quad k=\overline{1,9},\quad \sigma=1,2,\ldots,N. \tag{32}$$

We construct a new solution $\overset{(2)}{u} = (\overset{(2)}{u}{}^0, \overset{(2)}{u}{}^1)$ of system (32) via known solution $\overset{(1)}{u} = (\overset{(1)}{u}{}^0, \overset{(1)}{u}{}^1)$ according to the formula

$$\overset{(2)}{u}(x) = \tau, \quad x = \overset{(1)}{u}(\tau). \tag{33}$$

Here $x = (x_0, x_1)$, $\tau = (\tau^0, \tau^1)$, τ^μ are parameters to be eliminated out of system (33).

Example 3. Let us consider the simplest hodograph-invariant system of first order PDE

$$u_0^1 - u_1^0 = 0, \quad u_1^1 - u_0^0 = 0. \tag{34}$$

It is easily to verify, that pair of functions

$$\overset{(1)}{u}{}^0 = 2x_0x_1 + c, \quad \overset{(1)}{u}{}^1 - x_0^2 + x_1^2$$

is the solution of system (34). Making use of formula (33) one obtain the new solution of this system

$$\begin{aligned} \overset{(2)}{u}{}^1 &= \pm\frac{1}{\sqrt{2}}\left[x_1 \pm \sqrt{x_1^2 + (x_0 - c)^2}\right]^{\frac{1}{2}}, \\ \overset{(2)}{u}{}^0 &= \pm\frac{x_0 - c}{\sqrt{2}}\left[x_1 \pm \sqrt{x_1^2 + (x_0 - c)^2}\right]^{-\frac{1}{2}}. \end{aligned} \tag{35}$$

Let us consider the linear system of first order PDEs

$$b_\mu^{\sigma\nu}(y)v_\mu^\nu + b^{\sigma\nu}(y)v^\nu + c^\sigma(y) = 0. \tag{36}$$

Here $b_\mu^{\sigma\nu}$, $b^{\sigma\nu}$, c^σ are arbitrary smooth functions of $y = (y_0, y_1)$, summation over repeated indices is understood in the space with metric $g_{\mu\nu}^* = \operatorname{diag}(1,1)$. This system (36) under transformation (28) reduces into system of nonlinear PDEs

$$\begin{aligned} & b^{\sigma 0}(u)\delta^{-1}u_1^1 - b_1^{\sigma 0}(u)\delta^{-1}u_1^0 - b_0^{\sigma 1}(u)\delta^{-1}u_0^1 + \\ & \qquad + b_1^{\sigma 1}(u)\delta^{-1}u_0^0 + b^{\sigma 0}(u)x_0 + b^{\sigma 1}(u)x_1 + c^\sigma(u) = 0. \end{aligned} \tag{37}$$

The solutions superposition formula for the system (37) has the form

$$\begin{aligned} \overset{(3)}{u}{}^0(x_0, x_1) &= \overset{(1)}{u}{}^0(\tau^0, \tau^1), \quad \overset{(1)}{u}{}^0(\tau^0, \tau^1) = \overset{(2)}{u}{}^0(x_0 - \tau^0, x_1 - \tau^1), \\ \overset{(3)}{u}{}^1(x_0, x_1) &= \overset{(1)}{u}{}^1(\tau^0, \tau^1), \quad \overset{(1)}{u}{}^1(\tau^0, \tau^1) = \overset{(2)}{u}{}^1(x_0 - \tau^0, x_1 - \tau^1). \end{aligned} \tag{38}$$

Making use of designations $u = (u^0, u^1)$, $x = (x_0, x_1)$, $\tau = (\tau^0, \tau^1)$, one can rewrite the formula (38) in another way:

$$\overset{(3)}{u}(x) = \overset{(1)}{u}(\tau), \quad \overset{(1)}{u}(\tau) = \overset{(2)}{u}(x - \tau). \tag{38a}$$

Example 4. It is obviously, that two pairs of functions

$$\begin{aligned} \overset{(1)}{u} &= \frac{1}{2}x_0, \quad \overset{(1)}{\rho} = (2\lambda)^{-1}\sqrt{\frac{1}{4}x_0^2 - x_1}, \\ \overset{(2)}{u} &= x_0^{-1}\left[\frac{1}{2}c_1 + x_1\right], \quad \overset{(2)}{\rho} = (2\lambda x_0)^{-1}c_0 \end{aligned} \tag{39}$$

give two partial solutions of the system

$$
\begin{aligned}
&u_0 + uu_1 + 4\lambda^2 \rho\rho_1 = 0, \\
&\rho_0 + u_1\rho + u\rho_1 = 0.
\end{aligned}
\tag{40}
$$

Let us apply the formula (38) to construct a new solution $\overset{(3)}{u}$, $\overset{(3)}{\rho}$ via (39). Finally we get

$$
\begin{aligned}
&\overset{(3)}{u}{}^2(x_0, x_1) - c_2^2(x_0 - 2\overset{(3)}{u}(x_0, x_1))^{-2} - x_0\overset{(3)}{u}(x_0, x_1) + x_1 + \frac{1}{2}c_1 = 0, \\
&\overset{(3)}{\rho}(x_0, x_1) = (2\lambda)^{-1}\left[x_0\overset{(3)}{u}(x_0, x_1) - \overset{(3)}{u}{}^2(x_0, x_1) - x_1 - \frac{1}{2}c_1\right]^{\frac{1}{2}}.
\end{aligned}
$$

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