Generation of solutions for nonlinear equations via the Euler–Amperé transformation

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За допомогою контактного перетворення Ейлера–Ампера одержано формули розмноження розв'язків. Побудовано класи ДРЧП другого порядку, які інваріантні відносно цього перетворення.

The invariance of DE under a nonlocal transformation of variables allows us to generate its solutions from the known ones. The reducing of a nonlinear PDE to a linear equation makes it possible to construct for it the formula of a nonlinear superposition of solutions. In the present paper the solutions generating formulae are obtained via the Euler–Amperé contact transformation. Classes of Euler–Amperé invariant PDEs are constructed. The efficiency of the obtained formulae is illustrated in several of examples.

1. Nonlocal invariance and the solutions generating formula. Let us consider the Euler–Amperé transformation in the space $\mathbb{R}(1,n-1)$ of n independent variables [1, 2]:

$$
\begin{aligned}
&u = y_a v_a - v, \quad x_0 = y_0, \quad x_a = v_a, \\
&v_\mu = \partial_\mu v = \frac{\partial v}{\partial y_\mu}, \quad a,b = \overline{1,n-1}, \quad \delta \equiv \det(v_{ab}) \neq 0, \quad \mu,\nu = \overline{1,n-1}.
\end{aligned} \tag{1}
$$

The first and second order derivatives are changing as

$$
\begin{aligned}
&u_0 = -v_0, \quad u_a = y_a, \\
&v_{00} = -\det{}^{-1}(v_{ab})\det(v_{\mu\nu}), \quad u_{0a} = -\det{}^{-1}(v_{ab}) v_{0b} a_{ba}(v_{cd}), \\
&u_{ab} = -\det{}^{-1}(v_{cd}) a_{ab}(v_{cd}) \quad (a,b,c,d = \overline{1,n-1}).
\end{aligned} \tag{2}
$$

Hereafter the summation over repeated Greek indices is understood in the space $\mathbb{R}(1,n-1)$ with the metric $g_{\mu\nu} = \mathrm{diag}\,(1,-1,\dots,-1)$ and over repeated Latin indices it is understood in the space $\mathbb{R}(0,n-1)$ with the metric $g_{ab} = \mathrm{diag}\,(1,1,\dots,1)$, $u_{\mu\nu} = \partial_{\mu\nu} u = \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}$, $\det(u_{ab}) = a_{00}(u_{ab})$. $a_{\lambda\sigma}(u_{\mu\nu})$, $a_{ab}(u_{cd})$ are the cofactors to the elements $u_{\lambda\sigma}$ and u_{ab} respectively, $\lambda,\sigma = \overline{0,n-1}$.

Following expressions are absolute differential invariants of order ≤ 2 with respect to (1) due to its involutivity:

$$
\begin{aligned}
&f^0(x_0), \quad f^1(x_a, u_a), \quad f^2(x_0, -u_0), \quad f^3(u, x_a u_a - u), \\
&f^4(u_{00}, -\det{}^{-1}(u_{ab})\det(u_{\mu\nu})), \quad f^5(u_{0a}, -\det{}^{-1}(u_{ab}) u_{0b} a_{ba}(u_{cd})), \\
&f^6(u_{ab}, -\det{}^{-1}(u_{cd}) a_{ab}(u_{cd})).
\end{aligned} \tag{3}
$$

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Here f^0 is an arbitrary smooth function, and f^k, $k = 1,6$ are arbitrary smooth and symmetric on arguments functions:

$$f^k(x,z) = f^k(z,x).$$

Let us construct by means of the expressions (3) the absolutely invariant under transformation (1) second order PDE

$$F(\{f^\sigma\}) = 0 \quad (\sigma = \overline{0,6}). \tag{4}$$

$F(\cdot)$ is an arbitrary smooth function. Such equations are contained in the class (4):

$$u_0 - u_a u_a + x^2 = 0, \quad x^2 = x_a x_a; \tag{5.1}$$

$$\lambda u_0 - \Delta u - \det{}^{-1}(u_{cd})\,\mathrm{Slid}\,(u_{cd}) = 0; \tag{5.2}$$

$$u_{00} - \det{}^{-1}(u_{cd})\det(u_{\mu\nu}) = 0; \tag{5.3}$$

$$\lambda u_0 - \det{}^m(u_{cd}) + (-1)^{m\cdot n}\det{}^{-m\cdot n}(u_{cd})\det[a_{ab}(u_{cd})] = 0; \tag{5.4}$$

$$\lambda u_0^{2h} + \varphi(x_c)u_a u_a + \varphi(u_c)x^2 = 0; \tag{5.5}$$

$$\lambda u_0^{2h} + \varphi(u_c)u_a u_a + \varphi(x_c)x^2 = 0; \tag{5.6}$$

$$\lambda u_0 - \varphi(x_c,u)\Delta - \det{}^{-1}(u_{cd})\varphi(u_c, x_a u_a - u)\cdot\mathrm{Slid}\,(u_{cd}) = 0. \tag{5.7}$$

One can continue this list of equations (5) in the obvious manner. Δ is the Laplacian,

$$\mathrm{Slid}\,(u_{cd}) \stackrel{\mathrm{def}}{=} g_{ab}a_{ab}(u_{cd}),$$

$\varphi(x,z)$ is an arbitrary smooth function, λ is an arbitrary parameter, m, h are real numbers.

Let $\overset{(1)}{u}(x_0,x)$ be a known partial solution of Eq. (4). For constructing new solution $\overset{(2)}{u}(x_0,x)$ of this Eq. (4) we rewrite the formula (1) in parametric form, replacing x_a for parameters τ^a, $a = \overline{1,n-1}$. Substitute $\overset{(1)}{u}(x_0,\tau)$, $\overset{(1)}{u}{}_a(x_0,\tau)$ to (1). So, as a result, we obtain the formula

$$\begin{aligned} &\overset{(2)}{u}(x_0,x) = \tau^a\,\overset{(1)}{u}{}_a(x_0,\tau) - \overset{(1)}{u}(x_0,\tau) = \tau^a x_a - \overset{(1)}{u}(x_0,\tau),\\ &x_a = \overset{(1)}{u}{}_a(x_0,\tau), \quad a = \overline{1,n-1}. \end{aligned} \tag{6}$$

Here $x = (x_1, x_2, \ldots, x_{n-1})$, $\tau = (\tau^1, \tau^2, \ldots, \tau^{n-1})$. The formula (6) allows us to construct efficiently the new solutions of nonlinear equations (5) by resolving the last system (6) with respect to parameters τ.

Example 1. Let us consider the equation

$$u_0 u_{11} - u_{11}^2 + 1 = 0, \tag{7}$$

which is invariant under the transformation (1). The function

$$\overset{(1)}{u}(x_0,x_1) = \varphi(\omega), \quad \omega = \alpha_0 x_0 + \alpha_1 x_1$$

is a solution of Eq. (7), when φ satisfies the first order ODE

$$\begin{gathered} q + 2(k\dot\varphi)^{-1}\sqrt{(k\dot\varphi)^2+1} + 4k\omega + c_1 = 0, \\ q \equiv \ln\left|k\dot\varphi - \sqrt{(k\dot\varphi)^2+1}\right| - \ln\left|k\dot\varphi + \sqrt{(k\dot\varphi)^2+1}\right|. \end{gathered} \tag{8}$$

Here $k = \frac{1}{2}\alpha_0$, $\alpha_1 = 1$, $\alpha_1 = 1$, α_0, c_1 are arbitrary constants. In this case the generating of new solutions is realized according to the formulae

$$\overset{(2)}{u}(x_0, x_1) = \tau\cdot\varphi(\omega) - \dot\varphi(\omega), \quad x_1 = \dot\varphi(\omega), \quad \omega = 2kx_0 + \tau. \tag{9}$$

From the second equation of the system (9) we get

$$\tau = [\dot\varphi]^{-1}(x_1) - 2kx_0.$$

Here $[\dot\varphi]^{-1}(x)$ is the inverse function to $\dot\varphi(x)$. Note, that

$$[\dot\varphi]^{-1}(x_1) = \omega, \quad \dot\varphi(\omega) = \dot\varphi(\omega) = \dot\varphi([\dot\varphi]^{-1}(x_1)) = x_1.$$

Then from the first equation of the system (9) we obtain

$$\overset{(2)}{u}(x_0, x_1) = \tau\cdot x_1 + \varphi(2kx_0 + \tau). \tag{10}$$

Here $\varphi(2kx_0 + \tau)$ is a solution of ODE (8) of argument $2kx_0 + \tau$. Due to equality $x_1 = \overset{(1)}{u}(x_0, \tau) = \dot\varphi(2kx_0 + \tau)$ we get τ from the correlation (8)

$$\begin{gathered} \tau = -\left\{g^* + 2(kx_1)^{-1}\sqrt{(kx_1)^2+1} + 2kx_0 + (4k)^{-1}\cdot c_1\right\}, \\ g^* \equiv \ln\left|kx_1 - \sqrt{(kx_1)^2+1}\right| - \ln\left|kx_1 + \sqrt{(kx_1)^2+1}\right|. \end{gathered} \tag{11}$$

Thus, the solution $u(x_0, x_1)$ is determined by the parametric system of equations

$$\begin{aligned} \overset{(2)}{u}(x_0, x_1) = \varphi\left(-q^* - 2(kx_1)^{-1}\sqrt{(kx_1)^2+1} - (4k)^{-1}\cdot c_1\right) - \\ - x_1\left\{g^* + 2(kx_1)^{-1}\sqrt{(kx_1)^2+1} + 2kx_0 + (4k)^{-1}\cdot c_1\right\}, \end{aligned} \tag{12}$$

$$q + 2(k\dot\varphi)^{-1}\sqrt{(k\dot\varphi)^2+1} + 4k\omega + c_1 = 0, \quad \varphi = \varphi(\omega). \tag{13}$$

Example 2. The equation

$$(u_0 - \Delta_{(2)}u)(u_{11}u_{22} - u_{12}^2) - \Delta_{(2)}u = 0 \tag{14}$$

is (1)-invariant, when the condition $u_{11}u_{22} - u_{12}^2 \neq 0$ is satisfied. The partial solution of Eq. (13) is

$$\overset{(1)}{u} = \ln r^2, \quad r^2 = x_1^2 + x_2^2.$$

Let us replace x_a, $a = 1, 2$ in $\overset{(1)}{u}$ for parameters τ^a

$$\overset{(1)}{u} = \ln\rho^2, \quad \rho^2 = (\tau^1)^2 + (\tau^2)^2.$$

and substitute this result into to the formula (6). We obtain

$$\overset{(2)}{u}(x_0, x_1, x_2) = 2 - \ln \rho^2, \quad x_a = 2\tau^a \rho^{-2}, \quad a = 1, 2. \tag{15}$$

Let us express τ^1, τ^2 through x_1, x_2 from the last two conditions of the system (15)

$$\tau^a = 2x_a r^{-2}, \quad a = 1, 2. \tag{16}$$

Substituting τ from (16) into the first equation of the system (15), we get the solution $\overset{(2)}{u}$:

$$\overset{(2)}{u} = 2(1 - \ln 2) + \ln r^2.$$

2. Nonlocal linearization and nonlinear superposition formula. Let us apply the transformation (1) to a general second order linear PDE

$$b^{\mu\nu}(y_0, y)v_{\mu\nu} + b^{\mu}(y_0, y)v_{\mu} + b(y_0, y)v + c(y_0, y) = 0. \tag{17}$$

$y = (y_1, y_2, \dots, y_{n-1})$, $b^{\mu\nu} = b^{\nu\mu}$, b^{μ}, b, c are arbitrary smooth functions of y_0, y. As a result we get the nonlinear equation

$$\begin{aligned} &\{b^{00}(x_0, \underset{1}{u}) \det(u_{\mu\nu}) - 2b^{0a}(x_0, \underset{1}{u})u_{0b}a_{ba}(u_{cd}) + \\ &\quad + b^{ab}(x_0, \underset{1}{u})a_{ab}(u_{cd})\} \det{}^{-1}(u_{cd}) + b^0(x_0, \underset{1}{u}) \cdot u_0 + \\ &\quad + b^a(x_0, \underset{1}{u})x_a - b(x_0, \underset{1}{u})[x_a u_a - u] - c(x_0, \underset{1}{u}) = 0. \end{aligned} \tag{18}$$

Here $\underset{1}{u} = (u_1, u_2, \dots, u_{n-1})$, $a, b = \overline{1, n-1}$. Eq. (18) possesses the solutions superposition property, which arises from the superposition of solutions of the linear equation (17)

$$\overset{(3)}{v}(y_0, y) = \overset{(1)}{v}(y_0, y) + \overset{(2)}{v}(y_0, y).$$

Let $\overset{(k)}{u}$, $k = 1, 2$ be known solutions of Eq. (18) and $\overset{(3)}{u}(x_0, x)$ be a new solution of the same equation. Let us express $\overset{(3)}{u}$ through $\overset{(1)}{u}$ and $\overset{(2)}{u}$. Making use of Euler–Amperé transformation (1), we get

$$\begin{aligned} &\overset{(3)}{u}(x_0, x) = y_a \overset{(3)}{v}_a - \overset{(3)}{v} = y_a(\overset{(1)}{v}_a + \overset{(2)}{v}_a) - \overset{(1)}{v} - \overset{(2)}{v}, \\ &x_a = \overset{(3)}{v}_a = \overset{(1)}{v}_a + \overset{(2)}{v}_a, \quad x_0 = y_0. \end{aligned} \tag{19}$$

One can express $\overset{(1)}{v}$ and $\overset{(2)}{v}$ via $\overset{(1)}{u}$ and $\overset{(2)}{u}$ accordingly, where x are replaced for parameters $\tau = (\tau^1, \tau^2, \dots, \tau^{n-1})$ in the first and $\theta = (\theta^1, \theta^2, \dots, \theta^{n-1})$ in the second ones:

$$\begin{aligned} &\overset{(k)}{v} = \overset{(k)}{\tau}{}^a \overset{(k)}{u}{}_a - \overset{(k)}{u}, \quad k = 1, 2, \quad y_0 = x_0 = \overset{(k)}{\tau}{}^0, \quad \overset{(1)}{\tau} \equiv \tau, \\ &y_a = \overset{(k)}{u}{}_a(\overset{(k)}{\tau}{}^0, \overset{(k)}{\tau}), \quad \overset{(2)}{\tau} \equiv \theta, \quad \overset{(k)}{v}{}_a = \overset{(k)}{\tau}{}^a. \end{aligned} \tag{20}$$

Substituting the relations (20) into (19) we obtain the solutions superposition formula for Eq. (18)

$$\overset{(3)}{u}(x_0,x) = \overset{(1)}{u}(x_0,\tau) + \overset{(2)}{u}(x_0,x-\tau), \quad \overset{(1)}{u}_a(x_0,\tau) = \overset{(2)}{u}_a(x_0,\theta), \quad \theta = x-\tau. \tag{21}$$

Here the second equation of the system (19) is used essentially $x_a = \tau^a + \theta^a$ for eliminating parameters θ^a in the formula (21) and the designation $\overset{(2)}{u}_a(x_0,\theta) \equiv \overset{(2)}{u}_{\theta^a}(x_0,\theta)$ is adopted as well.

Example 3. Let us use as initial partial solutions

$$\overset{(1)}{u}(x_0,x_1) = x_0 - \frac{1}{2}x_1^2, \quad \overset{(2)}{u}(x_0,x_1) = k[1+x_1-x_0]^{\frac{3}{2}}, \quad k = -\frac{2}{3}\sqrt{2}$$

of the Euler–Amperé-linearizable equation

$$u_0u_{11} + 1 = 0, \quad u_{11} \neq 0. \tag{22}$$

Replacing the argument x_1 for parameter τ in $\overset{(1)}{u}$ and for parameter $\theta = x_1 - \tau$ in $\overset{(2)}{u}$ and making use of the formula (21), we obtain a new solution of Eq. (22)

$$\overset{(3)}{u}(x_0,x_1) = x_0 - h - \frac{2}{3}\sqrt{2}h^{\frac{3}{2}}, \quad h \equiv 2 + x_1 - x_0 \pm \sqrt{2(x_1-x_0)+3}. \tag{23}$$

Example 4. The nonlinear heat conduction equation

$$\begin{aligned} &u_0\det(u_{ab}) + \Delta_{(2)}u = 0, \\ &\Delta_{(2)} \equiv \partial_1^2 + \partial_2^2, \quad \det(u_{ab}) \neq 0 \end{aligned} \tag{24}$$

admits the linearization under the transformation (1) to the equation

$$v_0 - \Delta_{(2)}v = 0.$$

This Eq. (24) possesses the partial solution in parametric form

$$\begin{aligned} &\overset{(1)}{u}(x_0,x_1,x_2) = \theta x_1^{-1}r^2 + 2x_0x_1\theta^{-1}, \quad r^2 = x_1^2 + x_2^2, \\ &x_1(8\pi x_0^2) = \pm\theta\exp\{-\theta^2r^2(8x_0x_1^2)^{-1}\}. \end{aligned} \tag{25}$$

Let the second solution of Eq. (24) take the form

$$\overset{(2)}{u}(x_0,x_1,x_2) = x_1^2 - x_2^2. \tag{26}$$

Making use of the formula (21) we obtain the new solution $\overset{(3)}{u}$:

$$\begin{aligned} \overset{(3)}{u}(x_0,x_1,x_2) &= \theta(x_1-\theta)^{-2}\left(x_1 - \frac{1}{2}\theta\right)(r^2+\theta^2-2x_1\theta) + \\ &\quad + 2x_0\theta^{-1}\left(x_1-\frac{1}{2}\theta\right) + \frac{1}{4}\theta^2 - x_2^2(x_1-\theta)^{-2}\left(x_1-\frac{1}{2}\theta\right)^2, \\ 8\pi x_0^2\left(x_1-\frac{1}{2}\theta\right) &= \pm\theta\exp\left\{-\theta^2\frac{(r^2+\theta^2-2x_1\theta)}{4x_0(x_1-\theta)^2}\right\}, \end{aligned} \tag{27}$$

θ is the parameter to be eliminated.

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