A new conformal-invariant non-linear spinor equation

W. FUSHCHYCH, W. SHTELEN, P. BASARAB-HORWATH

We propose a new model for a spinor particle, based on a non-linear Dirac equation. We invoke group invariance and use symmetry reduction in order to obtain a multi-parameter family of exact solutions of the proposed equation.

1. Introduction

Since the discovery of the electron, many people have proposed and discussed the hypothesis that the mass of the electron is generated by an electromagnetic field, which the electron produces itself, so that the electron can be thought of as localized electromagnetic energy. In other words, this means that the electron is described by a non-linear dynamical system (see, for instance [1, 2] for these ideas). We propose a realization of this old and interesting physical idea in the framework of the classical theory of spinor fields. For the electron, we propose the following Lorentz-invariant spinor equation

$$(i\gamma\partial - m(u, v, \bar{\Psi}\Psi, j_\mu j^\mu))\Psi = 0, \tag{1.1}$$

where

$$\gamma\partial = \gamma^\mu\partial_\mu, \quad \mu = 0, 1, 2, 3$$

and the γ^μ are the Dirac matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}, \quad a = 1, 2, 3,$$

where the σ^a are the 2×2 Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$u = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu}, \quad v = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu},$$

where $F^{\mu\nu}$ is an antisymmetric tensor and

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad \partial_\mu \tilde{F}^{\mu\nu} = 0$$

with

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$$

and $\varepsilon_{\mu\nu\alpha\beta}$ is the antisymmetric Kronecker symbol.

Preprint LiTH-MAT-R-93-05, Department of Mathematics, Linköping University, Sweden, 8 p.

The electromagnetic field which the electron itself produces satisfies Maxwell's equations:

$$\partial_\nu F^{\mu\nu} = j\nu, \quad \text{with} \quad j\nu = e\bar{\Psi}\gamma^\nu\Psi, \tag{1.2}$$

where e is the charge of the electron.

We can interpret (1.1) as follows: the mass, m, of an electron is generated by the electromagnetic field $F^{\mu\nu}$, and its own spinor field Ψ. In the usual Dirac equation, m is a parameter which does not depend on the electromagnetic and spinor fields. Equation (1.1), in contrast to the standard Dirac equation, is a complicated non-linear equation, and as a result one has the following problem: how does one find at least some non-trivial solutions of such an equation?

For the case of m depending only on the spinor field, some classes of exact solutions of (1.1) have been found [5, 6, 10, 11]. In order to construct solutions of (1.1), (1.2), we first examine the symmetries of this system, and then we give some families of exact solutions. The system (1.1), (1.2) is non-linear even for $m = \text{const}$, and can be thought of as a first modification of the Dirac equation in our approach.

2. Symmetries

In the spinor equation (1.1), (1.2), we shall consider the fields $F^{\mu\nu}$, $\bar{\Psi}$, Ψ as independent, and we shall look for symmetry operators of that system in the form

$$X = \xi_\mu \frac{\partial}{\partial x^\mu} + \eta_\mu^{(1)} \frac{\partial}{\partial \Psi_\mu} + \eta_\mu^{(2)} \frac{\partial}{\bar{\Psi}_\mu} + \eta_{\mu\nu}^{(3)} \frac{\partial}{\partial F^{\mu\nu}},$$

where the coefficients are functions of x, Ψ, $\bar{\Psi}$, $F^{\mu\nu}$. In finding these symmetry operators, we use the method of Lie [4, 8, 9]. Indeed, after a painstaking calculation, we obtain the following:

Theorem 1. *The maximal point symmetry algebra of the system of (1.1), (1.2), with* $m = \text{const}$*, has as basis the following vector fields:*

$$\partial_\mu = \partial/\partial x^\mu, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu + (\sigma_{\mu\nu}\Psi)^\rho \frac{\partial}{\partial \Psi^\rho} + F^{\mu\rho}\frac{\partial}{\partial F^{\nu\rho}} - F^{\nu\rho}\frac{\partial}{\partial F^{\mu\rho}}, \tag{2.1}$$

$$D = \Psi^\mu \frac{\partial}{\partial \Psi^\mu} + F^{\mu\nu}\frac{\partial}{\partial F^{\mu\nu}}, \tag{2.2}$$

$$P = P^{\mu\nu}\frac{\partial}{\partial F^{\mu\nu}}, \tag{2.3}$$

where $\partial_\mu P^{\mu\nu} = 0$*,* $\partial_\mu \tilde{P}^{\mu\nu} = 0$ *and*

$$\sigma_{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu].$$

Remark 1. The operator D generates scale transformations in the space of the field variables Ψ^m, $F^{\mu\nu}$, not in Minkowski space $\mathbb{R}(1,3)$. The operators $\langle \partial_\mu, J_{\mu\nu}, D\rangle$, generate the extended Poincaré algebra [4].

If we assume dependence of the mass on the Lorentz-invariant quantities u, v, defined in (1.1), (1.2), we retain invariance under the Poincaré group, but not always under the extended Poincaré group. In fact, we have the following result:

Theorem 2. *The system (1.1), (1.2), where m is a function of the invariants u, v defined in (2.1), (2.2), is invariant under the algebra generated by (1.3), (1.4) if and only if*

$$m = \begin{cases} m\left(\dfrac{v}{u}\right), & u \neq 0, \\ m = \text{const}, & u = 0 \end{cases}$$

Remark 2. Theorem 2 implies that there exists a wide class of non-linear systems of the form (1.1), (1.2), which are invariant with respect to the extended Poincaré algebra. This is so when we assume that the mass depends only on the electromagnetic field.

3. Conformally invariant equations

In this paragraph, we shall describe equations of the form (1.1), (1.2), which are invariant under the conformal group, under the assumption that the mass has the following dependence on the fields:

$$m = \lambda_1 F_1(u, v) + \lambda_2 (\bar{\Psi}\Psi)^k. \tag{3.1}$$

The conformal group, $C(1,3)$ is well-known (see for instance [4], [5]). It consists of the Poincaré group together with the following non-linear transformations:

$$x'_\mu = \frac{x_\mu - c_\mu x^2}{\sigma}, \tag{3.2}$$

$$\Psi'(x') = \sigma(1 - (\gamma c)(\gamma x))\Psi(x), \tag{3.3}$$

$$\begin{aligned} F'_{\mu\nu}(x') = \sigma^2 F_{\mu\nu} + 2\sigma\{ & x^\beta[(2(cx) - 1)(c_\mu F_{\beta\nu} - c_\nu F_{\beta\mu}) - \\ & - c^2(x_\mu F_{\beta\nu} - x_\nu F_{\beta\mu})] + c^\alpha[x_\alpha F_{\alpha\nu} - x_\nu F_{\alpha\mu} - \\ & - x^2(c_\mu F_{\alpha\nu} - c_\nu F_{\alpha\mu})] + 2(c_\mu x_\nu - c_\nu x_\mu) F_{\alpha\beta} c^\alpha x^\beta\}, \end{aligned} \tag{3.4}$$

$$x'_\mu = e^\theta x_\mu, \tag{3.5}$$

$$\Psi'(x') = e^{-\frac{3}{2}\theta}\Psi(x), \tag{3.6}$$

$$F'_{\mu\nu}(x') = e^{-2\theta} F_{\mu\nu}, \tag{3.7}$$

where the primes denote transformed quantities, θ and c_μ are arbitrary real constants, $cx = c_\mu x^\mu$, $c^2 = c_\mu c^\mu$, $x^2 = x_\mu x^\mu$.

Applying Lie's method for calculating symmetry operators, one can prove the following result:

Theorem 3. *The system of equations (1.1), (1.2), with mass given by (3.1), is invariant under the conformal group if and only if $k = \frac{1}{3}$ and*

$$F_1(u, v) = \begin{cases} u^{\frac{1}{4}} F\left(\dfrac{v}{u}\right), & u \neq 0, \\ v^{\frac{1}{4}}, & u = 0, \end{cases} \tag{3.8}$$

where F is an arbitrary, smooth function.

One can easily verify that (1.1), (1.2), with mass defined by (3.1), is indeed invariant under the scale transformations (2.4)–(2.6). Substituting these into the equations yields

$$[i\gamma\partial - \lambda_1 F_1(e^{-4\theta}u, e^{-4\theta}v) - \lambda_2 e^{\theta(1-3k)}(\bar\Psi\Psi)^k]\Psi = 0,$$
$$\partial_\nu F^{\mu\nu} = e\bar\Psi\gamma^\mu\Psi, \quad \partial_\nu \tilde F^{\mu\nu} = 0.$$

The condition of invariance then gives

$$e^\theta F_1(e^{-4\theta}u, e^{-4\theta}v) = F_1(u,v), \quad \theta(1-3k) = 0$$

which immediately implies $k = \frac{1}{3}$, and, differentiating with respect to θ, that F_1 satisfies the equation

$$4u\frac{\partial F_1}{\partial u} + 4v\frac{\partial F_1}{\partial v} = F_1.$$

The general solution of this equation is easily shown to be that given by (3.8). Conformal invariance follows by using the transformations

$$\bar\Psi\Psi \mapsto \sigma^3\bar\Psi\Psi, \quad u \mapsto \sigma^4 u, \quad v \mapsto \sigma^4 v.$$

Remark 3. Requiring conformal invariance narrows quite considerably the class of admissible systems (1.1), (1.2). Fixing the function $F\left(\frac{u}{v}\right)$, we obtain different conformally-invariant equations for a spinor particle.

4. Exact solutions

We shall construct a class of exact solutions for the simplest conformally-invariant system (1.1), (1.2), namely for the case $F = 1$, so that our system becomes

$$\begin{aligned} &\left(i\gamma\partial - \lambda_1 u^{\frac14} - \lambda_2(\bar\Psi\Psi)^{\frac13}\right)\Psi = 0, \\ &\partial_\nu F^{\mu\nu} = e\bar\Psi\gamma^\mu\Psi, \quad \partial_\nu\tilde F^{\mu\nu} = 0. \end{aligned} \tag{4.1}$$

We shall look for solutions of this system by the method of reduction [4], that is we reduce the system of partial differential equations to systems of ordinary differential equations. For these, we use the following ansatzes [4, 5, 6, 7, 10, 11]:

$$\Psi(x) = \varphi(\omega), \quad F^{\mu\nu}(x) = f^{\mu\nu}(\omega), \tag{4.2}$$

where $\varphi(\omega)$ is a four-component vector, $f^{\mu\nu}(\omega)$ an antisymmetric tensor, $\omega = \beta x$, with β a constant vector satisfying $\beta^2 = 1$. Substituting (4.2) into (4.1), we obtain the reduced system of ordinary differential equations

$$\begin{aligned} &i(\gamma\beta)\dot\varphi - \left(\lambda_1 z^{\frac14} + \lambda_2(\bar\varphi\varphi)^{\frac13}\right) = 0, \\ &\beta_\nu \dot f^{\mu\nu} = e\bar\varphi\gamma^\mu\varphi, \quad \beta_\nu f^{\mu\nu} = 0 \end{aligned} \tag{4.3}$$

with $z = -\frac12 f_{\mu\nu}f^{\mu\nu}$ and the dot denotes differentiation with respect to the argument ω. Since $f^{\mu\nu}$ is anisymmetric, it follows that $\beta_\mu\beta_\nu\dot f^{\mu\nu} = 0$, so that the second equation in (4.3) yields $\bar\varphi(\gamma\beta)\varphi = 0$. Using the relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

and the fact that β is chosen so that $\beta^2 = 1$, it is easy to show that $(\gamma\beta)(\gamma\beta) = 1$. Multiplying the first equation of (4.3) on the left by $\bar{\varphi}(\gamma\beta)$ we then obtain

$$\bar{\varphi}\dot{\varphi} = 0.$$

We therefore find that φ satisfies

$$\bar{\varphi}\varphi = \text{const}, \quad \bar{\varphi}(\gamma\beta)\varphi = 0. \tag{4.4}$$

These equations imply that we should look for solutions φ in the form

$$\varphi = \exp(i(\gamma\beta)g(\omega))\chi, \tag{4.5}$$

where $g(\omega)$ is a function we must find and χ is a constant vector which satisfies $\bar{\chi}(\gamma\beta)\chi = 0$. Since $(\gamma\beta)^2 = 1$, it follows that

$$\varphi = [\cos(g(\omega)) - i(\gamma\beta)\sin(g(\omega))]\chi, \tag{4.6}$$

$$\bar{\varphi}\gamma_\mu\varphi = \alpha^\mu\cos(2g(\omega)) + c^\mu\sin(2g(\omega)), \tag{4.7}$$

$$\alpha^\mu = \bar{\chi}\gamma^\mu\chi, \quad c^\mu = \frac{i}{2}\bar{\chi}[(\gamma\beta), \gamma^\mu]\chi. \tag{4.8}$$

Clearly, $\alpha\beta = 0$. Equation (4.3) together with (4.6), (4.7), (4.8), can be written as

$$\begin{aligned} &\dot{g} = \lambda_1 z^{\frac{1}{4}} + \lambda_2(\bar{\chi}\chi)^{\frac{1}{3}}, \\ &\beta_\nu\dot{f}^{\mu\nu} = e(\alpha^\mu\cos(2g) + c^\mu\sin(2g)), \\ &\beta_\nu\dot{\tilde{f}}^{\mu\nu} = 0. \end{aligned} \tag{4.9}$$

We now seek solutions of (4.9) of the form

$$\begin{aligned} &g(\omega) = \kappa\omega, \\ &f^{\mu\nu} = \varepsilon[(\alpha^\mu\beta^\nu - \alpha^\nu\beta^\mu)\sin(2\kappa\omega) - (c^\mu\beta^\nu - c^\nu\beta^\mu)\cos(2\kappa\omega)], \end{aligned} \tag{4.10}$$

where κ, ε are constants. Without loss of generality, we assume $\alpha^2 = c^2 = -1$, since we have $\beta^2 = 1$, $\alpha\beta = \beta c = \alpha c = 0$. With these conventions, (4.9) and (4.10) give

$$\kappa = \lambda_1\sqrt{\varepsilon} + \lambda_2(\bar{\chi}\chi)^{\frac{1}{3}}, \quad e = 2\varepsilon\kappa. \tag{4.11}$$

Let us now consider solutions of (4.11). The first case is when $\lambda_1 \neq 0$, $\lambda_2 = 0$. Then

$$\varepsilon = \left(\frac{e}{2\lambda_1}\right)^{\frac{2}{3}}, \quad \kappa = \left(\frac{e\lambda_1^2}{2}\right)^{\frac{1}{3}}. \tag{4.12}$$

The second case is $\lambda_1 = 0$, $\lambda_2 \neq 0$, which gives

$$\kappa = \lambda_2(\bar{\chi}\chi)^{\frac{1}{3}}, \quad \varepsilon = \frac{e}{2\lambda_2(\bar{\chi}\chi)^{\frac{1}{3}}}. \tag{4.13}$$

Finally, when $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ equation (4.12) becomes the cubic equation

$$y^3 + py + q = 0, \quad \varepsilon = \frac{e}{2\kappa}, \tag{4.14}$$

where

$$y = \sqrt{\kappa} = \sqrt[3]{-\frac{q}{2} + \sqrt{Q}} + \sqrt[3]{-\frac{q}{2} - \sqrt{Q}}, \quad Q = \frac{e\lambda_1^2}{8} - \frac{\lambda_2^2(\bar{\chi}\chi)}{27}.$$

In this way we obtain exact solutions of the system (4.1), (4.2) in the following form

$$\psi(x) = \exp(-i\kappa(\gamma\beta)\omega)\chi, \quad \omega = \beta x, \tag{4.15}$$

$$F^{\mu\nu} = \frac{e}{2\kappa}[(\alpha^\mu\beta^\nu - \alpha^\nu\beta^\mu)\sin(2\kappa\omega) - (c^\mu\beta^\nu - c^\nu\beta^\mu)\cos(2\kappa\omega)], \tag{4.16}$$

$$\alpha^\mu = \bar{\chi}\gamma^\mu\chi, \quad c^\mu = \frac{i}{2}\bar{\chi}[(\gamma\beta), \gamma^\mu]\chi,$$

$$\beta^2 = 1, \quad \alpha^2 = c^2 = -1, \quad \alpha\beta = \alpha c = \beta c = 0.$$

For conformally invariant solutions of (4.1) we exploit the ansatzes [6, 7]

$$\begin{aligned} &\psi(x) = \frac{\gamma x}{(x^2)^2}\varphi(\omega), \quad \omega = \frac{\beta x}{x^2}, \quad \beta^2 = 1, \\ &F^{\mu\nu} = \frac{f^{\mu\nu}(\omega)}{x^2} - \frac{2x_\rho[x^\mu f^{\rho\nu}(\omega) - x^\nu f^{\rho\nu}(\omega)]}{(x^2)^3}. \end{aligned} \tag{4.17}$$

Combining (4.1) and (4.17) yields the system of ordinary differential equations

$$\begin{aligned} &-i(\gamma\beta)\dot{\varphi} = \lambda_1 z^{\frac{1}{4}} + \lambda_2(\bar{\varphi}\varphi)^{\frac{1}{3}}, \\ &\beta_\nu \dot{f}^{\mu\nu} = -e\bar{\varphi}\gamma^\mu\varphi, \\ &\beta_\nu \dot{\tilde{f}}^{\mu\nu} = 0 \end{aligned} \tag{4.18}$$

with $z = -\frac{1}{2}f_{\mu\nu}f^{\mu\nu}$, which is formally similar to (4.3). Using this fact, we can write down the following solutions of (4.1), (4.17):

$$\Psi(x) = \frac{\gamma x}{(x^2)^2}\exp(i\kappa(\gamma\beta)\omega)\chi, \quad \omega = \frac{\beta x}{x^2}, \tag{4.19}$$

$$\begin{aligned} F^{\mu\nu} = \frac{e}{2\kappa(x^2)^2}\Big\{ &\Big[\beta^\mu\alpha^\nu - \beta^\mu\alpha^\nu) + 2(\alpha^\mu x^\nu - \alpha^\nu x^\mu)\omega + \\ &+ 2\frac{\alpha x}{x^2}(x^\mu\beta^\nu - x^\nu\beta^\mu)\Big]\sin(2\kappa\omega) + \Big[(c^\mu\beta^\nu - c^\nu\beta^\mu) + \\ &+ 2\omega(x^\mu c^\nu - x^\nu c^\mu) - 2\frac{cx}{x^2}(x^\mu\beta^\nu - x^\nu\beta^\mu)\Big]\cos(2\kappa\omega)\Big\}, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} &\alpha^\mu = \bar{\chi}\gamma^\mu\chi, \quad c^\mu = \frac{i}{2}\bar{\chi}[(\gamma\beta), \gamma^\mu]\chi, \\ &\beta^2 = 1, \quad \alpha^2 = c^2 = -1, \quad \alpha\beta = \alpha c = \beta c = 0. \end{aligned}$$

The solutions found show that the system (1.1), (1.2) is consistent, at least in certain cases of the mass function. Furthermore, we can calculate the mass corresponding to these solutions:

$$m = \lambda_1 u^{\frac{1}{4}} + \lambda_2(\bar{\psi}\psi)^{\frac{1}{3}} = \frac{\kappa}{x^2}.$$

5. Conclusion

We have shown that there exists a consistent non-linear dynamical model for a classical spinor particle, in which the mass is generated by an electromagnetic field and a spinor field, which the particle itself creates. The proposed model (3.1) is conformally-invariant, as is the class of solutions we obtain, For these solutions, we have also found an explicit form for the Lorentz-invariant mass. The question of quantizing the model (3.1) will be taken up in future papers.

1. Yukawa H. (ed.), Nonlinear field theory, *Progress of Theoretical Physics, Supplements*, 1959, V.9, 1–128.
2. Heisenberg W., Introduction to the unified theory of elementary particles. Collected works, Series B, Berlin, Springer-Verlag, 1984.
3. Bedrij O., Fushchych W., On the electromagnetic structure of elementary particle masses, *Proceedings of the Ukrainian Academy of Sciences*, 1991, № 2, 38–40.
4. Fushchych W., Shtelen W., Serov M., Symmetry and exact solutions of equationsof nonlinear mathematical physics, Dordrecht, Kluwer, 1993.
5. Fushchych W., Nikitin A., Symmetries of Maxwell's equations, Dordrecht, D. Reidel, 1987.
6. Fushchych W., Shtelen W., On some exact solutions of the nonlinear Dirac equation, *J. Phys. A*, 1983, **16**, 271–277.
7. Fushchych W., Shtelen W., On some exact solutions of the nonlinear equations of quantum electrodynamics, *Phys. Lett. B*, 1983, V.128, 215–217.
8. Olver P., Applications of Lie groups to differential equations, New York, Springer, 1986.
9. Ovsjannikov L.V., Group analysis of differential equations, New York, Academic Press, 1982.
10. Fushchych W., Zhdanov R., Symmetry and exact solutions of non-linear spinor equations, *Physics Reports*, 1989, V.172, 123–174.
11. Fushchych W., Zhdanov R., Non-linear spinor equations: symmetry and exact solutions, Kiev, Naukova Dumka, 1992 (in Russian).