The conditional invariance and exact solutions of the nonlinear diffusion equation

W.I. FUSHCYCH, N.I. SEROV, L.A. TULUPOVA

Исследована условная инвариантность нелинейного уравнения диффузии. Операторы условной инвариантности использованы для построения анзацев, редуцирующих данное уравнение к обыкновенным дифференциальным уравнениям. Найдены некоторые точные решения исходного уравнения.

Let us consider the nonlinear diffusion equation

$$H(u)u_0 + u_{11} = F(u), \tag{1}$$

where $u = u(x) \in \mathbb{R}_1$, $x = (x_0, x_1) \in \mathbb{R}_2$, $u_0 = \frac{\partial u}{\partial x_0}$, $u_{11} = \frac{\partial^2 u}{\partial x_1^2}$, $H(u)$ and $F(u)$ are arbitrary smooth functions.

Usually the equation (1) is investigated in the equivalent form

$$u_0 + \partial_1(f(u)u_1) = g(u). \tag{2}$$

In this way, for example, in papers [1, 2] Lie invariance of this equation was investigated.

The present paper is a continuation of the works [3, 4], where the Q-conditional invariance of the equation (1) was studied when $H(u) \equiv 1$ and $H(u) = u^{-1}$, $F(u) = 0$. In this paper Q-conditional invariance of the equation (1) is studied when $H(u)$ and $F(u)$ are arbitrary functions. Using obtained operators of Q-conditional invariance exact solutions of the given equation are found.

Let

$$Q = A(x,u)\partial_0 + B(x,u)\partial_1 + C(x,u)\partial_u, \tag{3}$$

where A, B, C are smooth functions, be a differential operator of the first order, acting on the manifold $(x,u) \in \mathbb{R}_3$.

The following theorem is proved analogously, as in [5].

Theorem 1. *The equation (1) is Q-conditionally invariant under the operator (3), if the functions A, B, C satisfy the following conditions:*

Case 1. $A = 1$.

$$\begin{aligned} &B_{uu} = 0, \quad C_{uu} = 2(B_{1u} + HBB_u),\\ &3B_uF = 2(C_{1u} + HB_uC) - (HB_0 + B_{11} + 2HBB_1 + H_uBC),\\ &CF_u - (C_u - 2B_1)F = HC_0 + C_{11} + 2HCB_1 + H_uC^2; \end{aligned} \tag{4}$$

Case 2. $A = 0$, $B = 1$.

$$CF_u - C_uF = HC_0 + C_{11} + 2CC_{1u} + C^2C_{uu} + \frac{H_u}{H}C(F - C_1 - CC_u). \tag{5}$$

Доповіді АН України, 1993, № 4, С. 37–40.

In formulae (4), (5) and everywhere below a subscript means differentiation with respect to corresponding argument.

Theorem 2. *The equation (1) is Q-conditionally invariant under the operator*

$$Q = \partial_0 + u\partial_1 + C(u)\partial_u, \tag{6}$$

if it has form

$$\left(3\lambda_1 + \frac{\lambda_2}{u}\right) u_0 + u_{11} = \left(2\lambda_1 + \frac{\lambda_2}{u}\right) P_3(u), \tag{7}$$

where $P_3(u) = \lambda_1 u^3 + \lambda_2 u^2 + \lambda_3 u + \lambda_4$ *is arbitrary third-order polynomial of* u, λ_k *are arbitrary constants,* $k = \overline{1,4}$. *In this case* $C(u) = P_3(u)$.

Proof. Substituting $B = u$, $C = C(u)$ into (4), we have

$$C_{uu} = 2uH, \quad F = \frac{1}{3}(2H - uH_u)C, \quad uH_{uu} + 2H_u = 0.$$

Whence it appears that

$$H = 3\lambda_1 + \frac{\lambda_2}{u}, \quad C = P_3(u), \quad F = \left(2\lambda_1 + \frac{\lambda_2}{u}\right) P_3(u),$$

The theorem is proved.

We use the operator (6) for finding solutions of the equation (7). The ansatz obtained with the help of the operator (6) has the form

$$x_1 - \int \frac{udu}{P_3(u)} = \varphi(\omega), \quad \omega = x_0 - \int \frac{du}{P_3(u)}. \tag{8}$$

The ansatz (8) reduces the equation (7) to the ordinary differential equation (ODE)

$$\ddot{\varphi} + P_3(\dot{\varphi}) = 0. \tag{9}$$

Integration of the equation (9) depends on a form of the roots of the polynomial P_3. There are seven essentially different cases. We give one example of each case.

1) $P_3(u) = (u-1)^3, \qquad (\varphi - \omega)^2 = 2\omega,$
$$u = 1 + \frac{x_1 - x_0}{x_0 - \frac{1}{2}(x_1 - x_0)^2};$$

2) $P_3(u) = (u+1)(u-1)^2, \quad \mathrm{th}\,(\varphi - \omega) - 1 = \dfrac{1}{\varphi + \omega},$
$$\frac{\left(x_0 + x_1 + \frac{1}{u-1}\right) u - 1}{x_0 + x_1 + \frac{1}{u-1} - u} = \mathrm{th}\,(x_0 - x_1);$$

3) $P_3(u) = (u-2)(u^2-1), \quad \exp 3(\varphi - \omega) - 3\exp(\varphi + \omega) + 2 = 0,$
$$u = -\frac{\exp 3(x_1 - x_0) + 3\exp(x_1 + x_0) - 4}{\exp 3(x_1 - x_0) - 3\exp(x_1 + x_0) + 2};$$

4) $P_3(u) = (u-1)(u^2+2u+2)$,
$3\cos(2\varphi - 2\omega) + 4\sin(2\varphi - 2\omega) + 5 = 2\exp(-4\varphi - 6\omega)$,
$u = \dfrac{\exp(-3x_0 - 2x_1) + 3\sin(x_0 - x_1) - \cos(x_0 - x_1)}{\exp(-3x_0 - 2x_1) + 2\sin(x_0 - x_1) - \cos(x_0 - x_1)}$;

5) $P_3(u) = (u-1)^2, \quad \varphi = \omega + \ln\omega, \quad u = 1 + \exp(x_1 - x_0)$;

6) $P_3(u) = u^2 - 1, \quad \varphi = \ln\operatorname{ch}\omega, \quad u = \dfrac{\operatorname{ch} x_0 - \exp x_1}{\operatorname{sh} x_0}$;

7) $P_3(u) = u^2 + 1, \quad \varphi = \ln\cos\omega, \quad u = \dfrac{-\cos x_0 + \exp x_1}{\sin x_0}$.

Theorem 3. *The equation*

$$u_0 + uu_{11} = \lambda_1 u + \lambda_2, \quad (\lambda_1, \lambda_2 = \text{const}) \tag{10}$$

is Q-conditionally invariant under the operator

$$Q = \partial_0 + \frac{u}{x_1}\partial_1 + (\lambda_1 u + \lambda_2)\partial_u. \tag{11}$$

Proof. If we find a prolongation of the operator (11) and act on the equation (10), then we have

$$\tilde{Q}(u_0 + uu_{11} - \lambda_1 u - \lambda_2) = \left(\frac{2u}{x_1^2} - \frac{3u_1}{x_1} + 2\lambda_1 + \frac{\lambda_2}{u}\right) \times$$
$$\times (u_0 + uu_{11} - \lambda_1 u - \lambda_2) - \left(\frac{2u}{x_1^2} - \frac{2u_1}{x_1} + \lambda_1 + \frac{\lambda_2}{u}\right) \times$$
$$\times \left(u_0 + \frac{uu_1}{x_1} - \lambda_1 u - \lambda_2\right),$$

i.e.

$$\tilde{Q}S = \alpha S + \beta Q u,$$

The theorem is proved.

The ansatz

$$\lambda_1 v + \lambda_2 = e^{\lambda_1 x_0}\varphi(\omega), \quad \omega = u - \lambda_2 x_0 - \lambda_1 \frac{x_1^2}{2}, \tag{12}$$

obtained with the help of the operator (11) reduces the equation (10) to the following ODE

$$\ddot{\varphi} = 0. \tag{13}$$

Solving the equation (13) and using the ansatz (12), we find the solution of the equation (10):

$$\lambda_1 u + \lambda_2 = e^{\lambda_1 x_0}\left[c_1\left(u - \lambda_2 x_0 - \lambda_1 \frac{x_1^2}{2}\right) + c^2\right]. \tag{14}$$

Now we give some more results on the Q-conditional invariance of the equation (1). The results are written in the following order — an equation (1), a corresponded operator, an ansatz, a reduced equation, a solution of the equation (1).

1) $(\lambda_1 u^2+\lambda_2)u_0+u_{11}=\lambda_2 u^3,\quad Q=\lambda_2 x_1^2\partial_0+3x_1\partial_1+3u\partial_u,$

$u=x_1\psi(\omega),\quad \omega=x_0-\dfrac{\lambda_2}{6}x_1^2,\qquad \lambda_1\varphi^2\varphi+\dfrac{\lambda_2^2}{9}\ddot\varphi=\lambda_3\varphi^3,$

2) $(\lambda_1 u^2+\lambda_2)u_0+u_{11}=\lambda_3 u^3+2u,$

$Q=\lambda_2(1+\cos 2x_1)\partial_0-3\sin 2x_1\partial_1+6u\partial_u,\quad u=\operatorname{ctg}x_1\varphi(\omega),$

$\omega=\dfrac{3}{\lambda_2}x_0+\ln\sin x_1,\quad \ddot\varphi-3\dot\varphi+2\varphi+3\dfrac{\lambda_1}{\lambda_2}\varphi^2\dot\varphi=\lambda_3\varphi^3,$

3) $(\lambda_1 u^2+\lambda_2)u_0+u_{11}=\lambda_3 u^3-2u,\quad Q=\lambda_2\partial_0+3\operatorname{th}x_1\partial_1-\dfrac{3u}{\operatorname{ch}^2 x_1}\partial_u,$

$u=\operatorname{cth}x_1\varphi(\omega),\quad \omega=\dfrac{3}{\lambda_2}x_0-\ln\operatorname{sh}x_1,\quad \ddot\varphi+3\dot\varphi+2\varphi+3\dfrac{\lambda_1}{\lambda_2}\varphi^2\dot\varphi=\lambda_3\varphi^3,$

4) $e^u u_0+u_{11}=e^u,$

$a)$ $Q=x_1\partial_1-2\partial_u,\quad u=\varphi(x_0)-2\ln x_1,\quad e^\varphi\dot\varphi+2=e^\varphi,\quad u=\ln\dfrac{e^{x_0}+2}{x_1^2},$

$b)$ $Q=\partial_1+\operatorname{tg}\dfrac{x_1}{2}\partial_u,\quad u=\varphi(x_0)-2\ln\cos\dfrac{x_1}{2},\quad e^\varphi\dot\varphi+\dfrac12=e^\varphi,$

$u=\ln\dfrac{e^{x_0}+\frac12}{\cos^2\frac{x_1}{2}},$

$c)$ $Q=\partial_1+\operatorname{th}\dfrac{x_1}{2}\partial_u,\quad u=\varphi(x_0)-2\ln\operatorname{ch}\dfrac{x_1}{2},\quad e^\varphi\dot\varphi-\dfrac12=e^\varphi,$

$u=\ln\dfrac{e^{x_0}-\frac12}{\cos^2\frac{x_1}{2}},$

5) $\lambda u u_0+u_{11}=\lambda u^2,$

$a)$ $Q=\partial_0+\left(u+\dfrac{1}{\lambda x_1^2}\right)\partial_u,\quad u=\dfrac{1}{\lambda x_1^2}+e^{x_0}\varphi(x_1),$

$x_1^2\ddot\varphi-6(2\lambda-1)\varphi=0,$

$b)$ $Q=\partial_0-\left(u-\dfrac{1}{\lambda}W(x_1)\right)\partial_u,\quad u=\dfrac{1}{\lambda}W+e^{x_0}\varphi(x_1),\quad \ddot\varphi=W\varphi,$

$u=\dfrac{1}{\lambda}W(x_1)+e^{x_0}\lambda(x_1),$

where $W(x)$ is the Weierstrass function, $\lambda(x)$ is the Lame function.

1. Овсянников Л.В., Групповой анализ дифференциальных уравнений, М., Наука, 1978, 400 с.
2. Дородницын В.А., Князева И.В., Свищевский С.Р., Групповые свойства уравнени теплопроводности с источником в двумерном и трехмерном случаях, *Дифференц. уравнения*, 1983, **19**, № 7, 1215–1224.
3. Фущич В.И., Условная симметрия уравнений нелинейной математической физики, *Укр. мат. журн.*, 1991, **43**, № 11, 1456–1470.
4. Фущич В.I., Сєров М.I., Умовна інваріантність і точні розв'язки нелінійного рівняння акустики, *Доп. АН УРСР, Сер. А*, 1988, № 1, 28–32.
5. Серов Н.И., Условная инвариантность и точные решения нелинейного уравнения теплопроводности, *Укр. мат. журн.*, 1990, **42**, № 10, 1370–1376.