# On nonlinear representation of the conformal algebra $AC(2,2)$

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Одержано вичерпний опис нееквівалентних представлень алгебри Пуанкаре $AP(2,2)$ та конформної алгебри $AC(2,2)$ у класі диференціальних операторів першого порядку. Встановлено, що існують лише два нееквівалентних представлення алгебри $AP(2,2)$ одне з яких є нелінійним. Це представлення допускає розширення до представлення повної конформної алгебри $AC(2,2)$. Розглянуто деякі узагальнення.

The central problem to be solved in the framework of the classical Lie approach to the partial differential equation (PDE) study

$$F(x,u,\underset{1}{u},\underset{2}{u},\ldots)=0 \tag{1}$$

is the construction of its maximal symmetry group. But the inverse problem of symmetry analysis of PDE-description of equations invariant under given transformation group is not of less importance. For example, relativistic field theory motion equations have to satisfy the Lorentz–Poincaré–Einstein relativity principle. It means that considered equations must int under the Poincaré group $P(1,3)$. Consequently, to study relativistically-invariant equations one has to study representations of the group $P(1,3)$ (see e.g. [1]).

There exists vast literature on the representations of the generalized Poincaré groups $P(n,m)$, $n,m\in\mathbb{N}$ but only a few papers are devoted to nonlinear representations [2, 3].

In the present paper we adduce results on description of unequivalent representations of the generalized Poincaré group $P(2,2)$ and its extention — conformal group $C(2,2)$ acting as transformation groups in the space $V=M(2,2)\times\mathbb{R}^1$, where $M(2,2)$ is the Minkowski space with the metric tensor

$$g_{\alpha\beta}=\begin{cases}1, & \alpha=\beta=1,2;\\ -1, & \alpha=\beta=3,4;\\ 0, & \alpha\neq\beta.\end{cases}$$

Lie algebra of the above conformal transformation group (called conformal algebra $AC(2,2)$) has the basis elements of the form

$$Q=\sum_{a=1}^{4}\xi_a(x,u)\frac{\partial}{\partial x_a}+\eta(x,u)\frac{\partial}{\partial u} \tag{2}$$

that satisfy the following commutational relations:

$$\begin{aligned}&[P_\alpha,P_\beta]=0,\quad [P_\alpha,J_{\beta\gamma}]=g_{\alpha\beta}P_\gamma-g_{\alpha\gamma}P_\beta,\\ &[J_{\alpha\beta},J_{\gamma\beta}]=g_{\alpha\delta}J_{\beta\gamma}+g_{\beta\gamma}J_{\alpha\delta}-g_{\alpha\gamma}J_{\beta\delta}-g_{\beta\delta}J_{\alpha\gamma},\\ &[D,J_{\alpha\beta}]=0,\quad [P_\alpha,D]=P_\alpha,\quad [K_\alpha,J_{\beta\gamma}]=g_{\alpha\beta}K_\gamma-g_{\alpha\gamma}K_\beta,\end{aligned} \tag{3}$$

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$$[P_\alpha, K_\beta] = 2(g_{\alpha\beta}D - J_{\alpha\beta}), \quad [D, K_\alpha] = K_\alpha, \quad [K_\alpha, K_\beta] = 0.$$

Here $\alpha, \beta, \gamma, \delta = \overline{1,4}$.

Let us note that operators $P_\alpha$, $J_{\beta\gamma}$ form generalized Poincaré algebra $AP(2,2)$ which is a subalgebra of the conformal algebra.

**Definition 1.** Set of operators $P_\alpha$, $J_{\beta\gamma}$, $D$, $K_\alpha$ of the form (2) satisfying the commutational relations (3) is called a representation of the conformal algebra $AC(2,2)$.

**Definition 2.** Representation of the algebra $AC(2,2)$ is called linear if coefficients of its basis operators (2) satisfy the conditions

$$\xi_\alpha = \xi_\alpha(x), \quad \eta = a(x)u. \tag{4}$$

If conditions (4) are not satisfied, representation is called nonlinear.

It is well-known that commutational relations are not altered by the change of variables

$$x'_\alpha = f_a(x,u), \quad u' = g(x,u). \tag{5}$$

That is why two representations $\{P_\alpha, J_{\beta\gamma}, D, K_\alpha\}$ and $\{P'_\alpha, J'_{\beta\gamma}, D', K'_\alpha\}$, are called equivalent provided they are connected by the relations (5).

**Theorem 1.** *There exist only two unequivalent representations of the Poincaré algebra $AP(2,2)$:*

$$1. \quad P_\alpha = \partial_\alpha, \quad J_{\beta\gamma} = g_{\beta\delta}x_\delta\partial_\gamma - g_{\gamma\delta}x_\delta\partial_\beta, \tag{6}$$

$$\begin{aligned}
2. \quad & P_\alpha = \partial_\alpha, \quad J_{12} = -x_2\partial_1 + x_1\partial_2 + \partial_u, \\
& J_{13} = x_3\partial_1 + x_1\partial_3 + \cos u\partial_u, \\
& J_{14} = x_4\partial_1 + x_1\partial_4 - \varepsilon\sin u\partial_u, \\
& J_{23} = x_3\partial_2 + x_1\partial_3 + \sin u\partial_u, \\
& J_{24} = x_4\partial_2 + x_2\partial_4 + \varepsilon\cos u\partial_u, \\
& J_{34} = x_4\partial_3 - x_3\partial_4 + \varepsilon\partial_u, \quad \varepsilon = \pm 1.
\end{aligned} \tag{7}$$

*Here $\partial_\alpha = \partial/\partial x_\alpha$, $\partial_u = \partial/\partial u$; $\alpha, \beta, \gamma, \delta = \overline{1,4}$, the summation over the repeated indices from 1 to 4 is understood.*

Because of the lack of the space we adduce only a sketch of the proof.

Since operators $P_\alpha$, $\alpha = \overline{1,4}$ commute, there exists a change of variables (5) reducing these to the form $P_\alpha \to P_\alpha$, $\alpha = \overline{1,4}$ [4]. From the commutational relations $[P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta}P_\gamma - g_{\alpha\gamma}P_\beta$ it follows that operators $J_{\beta\gamma}$ are of the form $J_{\beta\gamma} = g_{\beta\delta}x_\delta\partial_\gamma - g_{\gamma\delta}x_\delta\partial_\beta + \xi_{\beta\gamma\delta}(u)\partial_\delta + \eta_{\beta\gamma}(u)\partial_u$, where $\xi_{\beta\gamma\delta}$, $\eta_{\beta\gamma}$ are some smooth functions, $\beta, \gamma, \delta = \overline{1,4}$.

Substituting the obtained result into the third equality from (3) we get a system of nonlinear ordinary differential equations. On solving it we arrive at the formulae (6), (7).

Thus, there exists up to the equivalence relation (5) only one nonlinear representation of the algebra $AP(2,2)$. Applying the Lie method one can prove that the only first-order PDE admitting algebra (7) is the eikonal equation

$$u_{x_1}^2 + u_{x_2}^2 - u_{x_3}^2 - u_{x_4}^2 = 0.$$

Using results of subalgebraic analysis of the algebra $AP(2,2)$ obtained in [5], one can construct broad classes of exact solutions of the nonlinear PDE (8) by symmetry reduction procedure.

**Theorem 2.** *There exist only three unequivalent representations of the conformal algebra $AC(2,2)$:*

$$\begin{aligned} &1.\quad P_\alpha,\quad J_{\beta\gamma}\ \text{ are of the form (6)},\\ &\phantom{1.\quad} D = x_\alpha\partial_\alpha + \varphi(u)\partial_u,\\ &\phantom{1.\quad} K_\alpha = 2g_{\alpha\beta}x_\beta D - (g_{\beta\gamma}x_\beta x_\gamma)\partial_\alpha, \end{aligned} \tag{8}$$

$$\begin{aligned} &2.\quad P_\alpha,\quad J_{\beta\gamma}\ \text{ are of the form (6)},\\ &\phantom{2.\quad} D = x_\alpha\partial_\alpha + u\partial_u,\\ &\phantom{2.\quad} K_\alpha = 2g_{\alpha\beta}x_\beta D - (g_{\beta\gamma}x_\beta x_\gamma)\partial_\alpha \pm u^2\partial_\alpha, \end{aligned} \tag{9}$$

$$\begin{aligned} &3.\quad P_\alpha,\quad J_{\beta\gamma}\ \text{ are of the form (7)},\quad D = x_\alpha\partial_\alpha,\\ &\phantom{3.\quad} K_1 = 2x_1D - (g_{\beta\gamma}x_\beta x_\gamma)\partial_1 + 2(x_2 + x_3\cos u - \varepsilon x_4\sin u)\partial_u,\\ &\phantom{3.\quad} K_2 = 2x_2D - (g_{\beta\gamma}x_\beta x_\gamma)\partial_2 + 2(-x_1 + x_3\sin u + \varepsilon x_4\cos u)\partial_u,\\ &\phantom{3.\quad} K_3 = -2x_3D - (g_{\beta\gamma}x_\beta x_\gamma)\partial_3 + 2(\varepsilon x_4 - x_1\cos u - x_2\sin u)\partial_u,\\ &\phantom{3.\quad} K_4 = -2x_4D - (g_{\beta\gamma}x_\beta x_\gamma)\partial_4 + 2(-\varepsilon x_3 + \varepsilon x_1\sin u - \varepsilon x_2\cos u)\partial_u. \end{aligned} \tag{10}$$

Representation of the form (9) is realized on the set of solutions of the nonlinear wave equation

$$g_{\alpha\beta}u_{x\alpha}x_\beta = \lambda u^3, \quad \lambda \in \mathbb{R}^1$$

under $\varphi(u) = -\frac{3}{2}u$.

As shown in [6] the system of nonlinear PDE

$$g_{\alpha\beta}u_{x_\alpha x_\beta} = \pm 3u^{-3}, \quad g_{\alpha\beta}u_{x_\alpha x_\beta} = \pm 1$$

is invariant under the conformal algebra having basis operators (10).

A detailed study of the second-order PDE admitting conformal the algebra with basis operators (7), (11) will be the topic of our future papers.

In conclusion, we adduce some generalizations of the above assertions.

**Theorem 3.** *An arbitrary representation of the generalized Poincaré $AP(n,m)$ with $\max\{n,m\} \geq 3$ in the class of the operators (2) is equivalent to the standard representation*

$$P_\alpha = \partial_\alpha, \quad J_{\beta\gamma} = \tilde g_{\beta\delta}x_\delta\partial_\gamma - \tilde g_{\gamma\delta}x_\delta\partial_\beta, \tag{11}$$

*where $\tilde g_{\alpha\beta}$ is the metric tensor of the pseudo-Euclidean space $M(n,m)$, $\alpha,\beta,\gamma,\delta = 1,2,3,\ldots,n+m$.*

Consequently, only the algebras $AP(1,1)$ [2], $AP(1,2)$, $AP(2,1)$ [3] and $AP(2,2)$ have the nonlinear representation.

**Theorem 4.** *An arbitrary representation of the conformal group $C(n,m)$ with $\max\{n,m\} \geq 3$ is equivalent either to (9) or to (10) (where one must replace tensor $g_{\alpha\beta}$ by $\tilde g_{\alpha\beta}$).*

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