On non-local symmetries of nonlinear heat equation

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Для нелинейного уравнения теплопроводности приведены нелокальные формулы размножения и суперпозиции его решений.

1. Introduction. L.V. Ovsiannikov [1] gave the group classification of nonlinear one-dimensional heat equation

$$u_0 = \partial_1(F(u)u_1), \tag{1}$$

where $u = u(x)$, $x = (x_0, x_1)$, $u_\mu = \partial_\mu u$, $\partial_\mu = \frac{\partial}{\partial x_\mu}$, $\mu = 0, 1$; $F(u)$ is arbitrary differentiable function. These results can be formulated as follows:

Theorem 1. *The widest algebra of invariance of equation (1) with $F(u) \neq \mathrm{const}$ in class of S. Lie operators is given by the following basis elements*

$$a)\ \ \partial_0 = \frac{\partial}{\partial x_0},\ \ \partial_1 = \frac{\partial}{\partial x_1},\ \ D_1 = 2x_0\partial_0 + x_1\partial_1, \tag{2}$$

if $F(u)$ is arbitrary differentiable function;

$$b)\ \ \partial_0,\ \ \partial_1,\ \ D_1,\ \ D_2 = x_1\partial_1 + \frac{2}{k}u\partial_u, \tag{3}$$

if $F(u) = \lambda u^k$, λ, k are arbitrary constants, not equal to zero;

$$c)\ \ \partial_0,\ \ \partial_1,\ \ D_1,\ \ D_3 = x_1\partial_1 + 2\partial_u, \tag{4}$$

if $F(u) = \lambda \exp u$;

$$d)\ \ \partial_0,\ \ \partial_1,\ \ D_1,\ \ D_4 = x_1\partial_1 - \frac{3}{2}u\partial_u,\ \ \Pi = x_1^2\partial_1 - 3x_1u\partial_u, \tag{5}$$

if $F(u) = \lambda u^{-\frac{4}{3}}$.

It is well known (see for example [3]) that the sequence of transformations

$$u(x_0, x_1) = \frac{\partial v(x_0, x_1)}{\partial x_1}, \tag{6}$$

$$x_0 = t, \quad x_1 = w(t, x), \quad v = x, \tag{7}$$

$$\frac{\partial w(t, x)}{\partial x} = z(t, x) \tag{8}$$

do not take out of the equations class (1), i.e. if the sequence of transformations (6), (7), (8) is carried out then equation (1) goes to the form

$$z_t = \partial_x(F^*(z)z_x), \tag{9}$$

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where

$$F^*(z) = z^{-2}F(z^{-1}). \tag{10}$$

In this paper transformations (6)–(8) are used to construct nonlocal ansätzes, which reduce equation (1) to ordinary differential equations (ODE). The generating and superposition formulas for solutions of equation (1) are given for corresponding nonlinearities $F(u)$.

2. The equation $u_0 = \partial_1(u^{-2}u_1)$. In the case when $F(u) = u^{-2}$ the equation (1) takes the form

$$u_0 = \partial_1(u^{-2}u_1). \tag{11}$$

It follows from (10) that equation (11) can be reduced to the linear heat equation by means of transformations (6)–(8):

$$z_t = z_{xx}. \tag{12}$$

As it was established by S. Lie, the widest algebra of invariance of equation (12) consists of the operators:

$$\begin{gathered}\partial_t = \frac{\partial}{\partial t}, \quad \partial_x = \frac{\partial}{\partial_x}, \quad G = t\partial_x - \frac{1}{2}xz\partial_z, \quad I = z\partial_z, \\ D = 2t\partial_t + x\partial_x, \quad P = t\left(t\partial_t + x\partial_x - \frac{1}{2}z\partial_z\right) - \frac{x^2}{4}z\partial_z.\end{gathered} \tag{13}$$

The symmetry of equation (11) is given by only four operators (3), whereas the symmetry of equation (12) is given by six operators (13). It means that nonlinear equation (11) has some non-Lie symmetry which cannot be obtained by Lie's method. Let us use this fact to construct nonlocal ansätzes for nonlinear equation (11), i.e. having used operators G, P (13) we will construct non-local ansätzes for equation (11) by means of transformations (6)–(8). Below we will show only those ansätzes, which cannot be obtained from Lie symmetry of equation (11)

$$\begin{gathered}u(x_0, x_1) = \frac{1}{x_0x_1 + x_1h(\omega)}, \quad \omega = \tau + x_0^2, \\ \exp\left(x_0\tau + \frac{2}{3}x_0^3\right)\varphi(\omega) = x_1;\end{gathered} \tag{14}$$

$$\begin{gathered}u(x_0, x_1) = \frac{2(x_0^2+1)}{x_1\left[2(x_0^2+1)^{1/2}h(\omega) - x_0\tau\right]}, \quad \omega = \tau(x_0^2+1)^{-\frac{1}{2}}, \\ \exp\left\{\lambda \operatorname{arctg} x_0 - \frac{x_0\tau^2}{4(x_0^2+1)}\right\}\varphi(\omega) = x_1(x_0^2+1)^{1/4}.\end{gathered} \tag{15}$$

In formulas (14), (15) $\tau = \tau(x_0, x_1)$ is functional parameter, functions $\varphi(\omega)$ and $h(\omega)$ are connected by the relation

$$h(\omega) = \frac{\dot\varphi(\omega)}{\varphi(\omega)}.$$

Ansätzes (14), (15) reduce equation (11) to Riccati equations for unknown function h:

$$\dot h + h^2 = \omega, \tag{16}$$

$$\dot h + h^2 = -\frac{\omega^2}{4} + \lambda, \tag{17}$$

respectively. Equations (16), (17) being written down for function $\varphi(\omega)$, have the form

$$\ddot\varphi - \omega\varphi = 0, \quad \ddot\varphi + \left(\frac{\omega^2}{4} - \lambda\right)\varphi = 0.$$

The solutions of these equations can be expressed only in terms of special functions. As it follows from transformations (6)–(7) the relation between the solutions of equations (11) and (12) is given by following formula

$$u(x_0, x_1) = \left[\frac{\partial z(x_0,\tau)}{\partial\tau}\right]^{-1}, \tag{18}$$

where $\tau = \tau(x_0, x_1)$ is functional parameter, which can be obtained from the relation

$$z(x_0, \tau) = x_1. \tag{19}$$

3. Non-Lie generating of equation solutions. Let us illustrate the process of finding new solutions by means of formulas (18), (19). The function

$$z(t,x) = t + \frac{x^2}{2},$$

is a solution of heat equation (12). For this solution, in accordance with (19), we have $\tau = \sqrt{2(x_1 - x_0)}$. Having substituted this value of parameter τ into (18), we obtain the solution of equation (11):

$$u(x_0, x_1) = [2(x_1 - x_0)]^{-\frac{1}{2}}.$$

Linear equation (12) has a remarkable property: any operator of invariance algebra of this equation maps it's solution into another solution, i.e. the following generating formula takes place

$$\overset{2}{z}(t,x) = Q\,\overset{1}{z}(t,x), \tag{20}$$

where $\overset{1}{z}$, $\overset{2}{z}$ are solutions of equation (12), Q is an operator that belongs to algebra (13).

Let us use formula (20) and the relation between the solutions of equations (11) and (12) to construct generating solutions formula for equation (11). If we, for example, choose operator ∂_x instead of Q in (20), then we get one of the formulas which describe the generating solutions of nonlinear equation (11)

$$\overset{2}{u}(x_0, x_1) = -[\overset{1}{u}(x_0,\tau)]^3 \left[\frac{\partial\,\overset{1}{u}(x_0,\tau)}{\partial\tau}\right]^{-1}, \tag{21}$$

where $\overset{1}{u}(x_0, x_1)$ and $\overset{2}{u}(x_0, x_1)$ are solutions of equations (11) while function $\tau = \tau(x_0, x_1)$ is determined by the equation

$$\overset{1}{u}(x_0, \tau) = x_1^{-1}. \tag{22}$$

So equation (11) solutions of the form

$$\overset{1}{u}(x_0,x_1)=x_0^{\frac12}x_1^{-1}\left(-\ln x_0^{\frac12}x_1\right)^{-\frac12}$$

are multiplied into parametrical solutions:

$$\overset{2}{u}(x_0,x_1)=x_0^{\frac32}\tau\left(\ln\tau-\frac12\right)^{-1},\quad \ln\tau=x_0^2x_1^2\tau^2$$

by means of formulas (21), (22).

In the case $Q=\partial_t$, it follows from (18)–(20) that

$$\overset{2}{u}(x_0,x_1)=\frac{[\overset{1}{u}(x_0,\tau)]^5}{2[\overset{1}{u}_\tau(x_0,\tau)]^2-[\overset{1}{u}(x_0,\tau)]^2\,\overset{1}{u}_0(x_0,\tau)}, \tag{23}$$

where $\tau=\tau(x_0,x_1)$ is defined by the condition

$$\overset{1}{u}_\tau(x_0,\tau)+x_1[\overset{1}{u}(x_0,\tau)]^3=0. \tag{24}$$

Note. If we choose anyone of operators (13) in the capacity of Q in formula (20) then the generating solutions formula for equation (11) is constructed analogously. The synthesis of Galilei local transformations:

$$t'=t,\quad x'=x+2at,\quad z'=z\exp\{-ax-a^2t\} \tag{25}$$

and non-local relation (18), (19) leads to the new generating solutions formula of equation (11)

$$\overset{2}{u}(x_0,x_1)=\frac{\overset{1}{u}(x_0,\tau)}{-ax_1\,\overset{1}{u}(x_0,\tau)+x_1\tau^{-1}}, \tag{26}$$

where a is arbitrary real parameter and $\tau=\tau(x_0,x_1)$ is functional parameter which is a solution of the following equations

$$\tau_1=\frac{1}{-ax_1\,\overset{1}{u}(x_0,\tau)+x_1\tau^{-1}},\quad [\overset{1}{u}(x_0,\tau)]^2\tau_0=\tau_1^{-2}\tau_{11}+2a\,\overset{1}{u}{}^2(x_0,\tau). \tag{27}$$

It should be noted that x_0 is a parameter of first equation (27) and that is why this equation can be considered as first-order ODE with separable variables. Because of this the second of the equation (27) is only the correlating condition of obtained τ with respect for x_0. The following example show the effectivity of formulas (26)–(27). The constant solution $u^1(x_0,x_1)=1$ being generated by these formulas takes the form of following implicit solution:

$$\ln\frac{1}{(x_1u)^{-1}-a}+\frac{a}{(x_1u)^{-1}-a}=\ln x_1+a^2x_0.$$

4. Nonlinear superposition principle. Solutions of equation (12) have linear superposition principle. Using formulas (18), (19) we get nonlinear superposition principle for the solutions of equation (11). Let $\overset{1}{u}(x_0,x_1)$, $\overset{2}{u}(x_0,x_1)$ are the pair of solutions of equation (11) then third solution of this equation can be obtained by the formula

$$\frac{1}{\overset{3}{u}(x_0,x_1)}=\frac{1}{\overset{1}{u}(x_0,\overset{1}{\tau})}+\frac{1}{\overset{2}{u}(x_0,\overset{2}{\tau})}, \tag{28}$$

where $\overset{k}{\tau}=\overset{k}{\tau}(x_0,x_1)$ are functional parameters which can be obtained from the conditions

$$\begin{gathered}\overset{1}{u}(x_0,\overset{1}{\tau})d\overset{1}{\tau}=\overset{2}{u}(x_0,\overset{2}{\tau})\,d\overset{2}{\tau},\\ \overset{1}{\tau}+\overset{2}{\tau}=x_1,\quad \overset{k}{\tau}_0=\frac{\overset{k}{\tau}_{11}}{\overset{k}{\tau}{}_1^2}\overset{k}{u}{}^{-2}(x_0,\overset{k}{\tau}),\quad k=1,2.\end{gathered} \tag{29}$$

The substitution $u(x_0,x_1)=\frac{1}{U(x_0,x_1)}$ leads equation (11) and formulas (28), (29) to the following form

$$U_0=U^2U_{11}, \tag{30}$$

$$\overset{3}{U}(x_0,x_1)=\overset{1}{U}(x_0,\overset{1}{\tau})+\overset{2}{U}(x_0,\overset{2}{\tau}), \tag{31}$$

$$\begin{gathered}\frac{d\overset{1}{\tau}}{\overset{1}{U}(x_0,\overset{1}{\tau})}=\frac{d\overset{2}{\tau}}{\overset{2}{U}(x_0,\overset{2}{\tau})},\\ \overset{1}{\tau}+\overset{2}{\tau}=x_1,\quad \overset{k}{\tau}_0=\frac{\overset{k}{\tau}_{11}}{\overset{k}{\tau}{}_1^2}\overset{k}{U}{}^2(x_0,\overset{k}{\tau}),\quad k=1,2.\end{gathered} \tag{32}$$

Example. Having two simplest stationary solutions

$$\overset{1}{U}(x_0,x_1)=x_1,\quad \overset{2}{U}(x_0,x_1)=2x_1$$

of equation (30) and using formulas (31)–(32) we can obtain nonstationary solution of that equation

$$\overset{3}{U}(x_0,x_1)=\pm e^{-2x_0}\left(1-2x_1e^{2x_0}\pm\sqrt{1-2x_1e^{2x_0}}\right).$$

5. Non-Lie ansätzes for equation $u_0=\partial_1(u^{-\frac{2}{3}}u_1)$. It follows from (10), that transformations (6)–(8) map equation

$$u_0=\partial_1(u^{-\frac{2}{3}}u_1) \tag{33}$$

into the equation

$$z_t=\partial_x(z^{-\frac{4}{3}}z_x). \tag{34}$$

Symmetry of equation (34) in class of Lie transformations is wider then that of equation (33) (see Theorem 1). By analogy with section 1 let us use Lie symmetry of equation (34) for a construction of non-local ansätzes which reduce equation (33).

Let us concisely adduce the results of our analysis. Ansätzes for function z:

$$
\begin{aligned}
&1)\quad z = x^{-3}\varphi(\omega), \quad \omega = t,\\
&2)\quad z = x^{-3}\varphi(\omega), \quad \omega = at + \frac{1}{x},\\
&3)\quad z = t^{\frac{3}{4}}x^{-3}\varphi(\omega), \quad \omega = a\ln t + \frac{1}{x},\\
&4)\quad z = (x^2+1)^{-\frac{3}{2}}\varphi(\omega), \quad \omega = t + \lambda \operatorname{arctg} x,\\
&5)\quad z = (x^2-1)^{-\frac{3}{2}}\varphi(\omega), \quad \omega = t + \lambda \operatorname{arctg} x,\\
&6)\quad z = t^{\frac{3}{4}}(x^2+1)^{-\frac{3}{2}}\varphi(\omega), \quad \omega = \ln t + \lambda \operatorname{arctg} x,\\
&7)\quad z = t^{\frac{3}{4}}(x^2-1)^{-\frac{3}{2}}\varphi(\omega), \quad \omega = \ln t + \lambda \operatorname{arctg} x.
\end{aligned}
\tag{35}
$$

Ansätzes for function u:

$$
\begin{aligned}
&1)\quad u = [\varphi^1(x_0)x_1^2 + \varphi^2(x_0)]^{-\frac{2}{3}};\\
&2)\quad [x_1 + \varphi^1(x_0)][\dot\varphi^2(x_0)]^{\frac{3}{4}} = -\tau\dot\varphi^3(\omega) + \varphi^3(\omega),\\
&\qquad \omega = \varphi^2(x_0) + \tau, \quad -\frac{\tau_1}{\tau} = u;\\
&3)\quad [x_1 + \varphi^1(x_0)][\dot\varphi^1(x_0)]^{\frac{3}{4}} = \int [\dot\varphi^3(\tau)]^{\frac{3}{2}}\varphi^4(\omega)d\tau,\\
&\qquad \omega = \varphi^2(x_0) + \varphi^3(\tau), \quad \tau_1 = u.
\end{aligned}
\tag{36}
$$

Reduced equations which where obtained by the substitution of ansätzes (36) into the equation (33):

$$
\begin{aligned}
1)\quad &\dot\varphi^1 + 4(\varphi^1)^2 = 0,\\
&\dot\varphi^2 - 2\varphi^1\varphi^2 = 0;
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
2)\quad &\dot\varphi^1 = \lambda_1(\dot\varphi^2)^{\frac{1}{4}}, \quad \ddot\varphi^2 = \lambda_2(\dot\varphi^2)^2,\\
&3(\ddot\varphi^3)^{-\frac{1}{3}} + \lambda_3\dot\varphi^3 + \frac{3}{4}\lambda_2\varphi^3 - \lambda_1 = 0;
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
3)\quad &\dot\varphi^1 = 0, \quad \ddot\varphi^2 = \lambda_2(\dot\varphi^2)^2,\\
&2\,\dddot\varphi^3\dot\varphi^3 - 3(\ddot\varphi^3)^2 = 2\lambda_1(\dot\varphi^3)^4,\\
&(\varphi^4)^{-\frac{4}{3}}\ddot\varphi^4 - \frac{4}{3}(\varphi^4)^{-\frac{7}{3}}(\dot\varphi^4)^2 + 3\lambda_1(\varphi^4)^{-\frac{1}{3}} + \frac{3}{4}\lambda_2\varphi^4 - \dot\varphi^4 = 0,
\end{aligned}
\tag{39}
$$

where λ_1, λ^2, λ^3 are arbitrary constants. In particular, having integrated the system of equations (38) with $\lambda_2 = 0$, we obtain the parametrical solutions of equation (33) of the form:

$$
\begin{aligned}
&u^{\frac{1}{3}} = \frac{-c_1\left(\frac{5}{4}c_1^3x_1 + c_2x_0\right)}{\tau(\tau - 4c_3x_0)},\\
&(\tau + c_3x_0)(\tau - 4c_3x_0)^4 = \left(\frac{5}{4}c_1^3x_1 + c_2x_0\right)^4,
\end{aligned}
\tag{40}
$$

where c_1, c_2, c_3 are arbitrary constants of integration.

6. The invariance of equation (1) under the transformations (6)–(8). For the invariance of equation (1) under non-local transformations (6)–(8) the following condition must be satisfied

$$z^{-2}F(z^{-1}) = F(z). \tag{41}$$

The solution of equation (41) can be written down in the form

$$F(z) = z^{-1} f(\ln z), \tag{42}$$

where f is arbitrary differentiable even function. So transformations (6)–(8) are non-local invariance transformations of equation

$$u_0 = \partial_1 \left[\frac{f(\ln u)}{u} u_1 \right], \quad (f(-\alpha) = f(\alpha)). \tag{43}$$

Using this fact, we construct generating formula for solutions of equation (43):

$$\overset{2}{u}(x_0, x_1) = \frac{1}{\overset{1}{u}(x_0, \tau)}, \tag{44}$$

where $\tau = \tau(x_0, x_1)$ is the functional parameter which is a solution of the equations

$$\tau_1 = \frac{1}{\overset{1}{u}(x_0, \tau)}, \quad \tau_0 = f(\ln \tau_1) \frac{\tau_{11}}{\tau_1}. \tag{45}$$

Example. Let us consider the solution

$$\overset{1}{u}(x_0, x_1) = \frac{x_0}{1 + \cos x_1} \tag{46}$$

of the equation

$$u_0 = \partial_1 \left(\frac{u_1}{u} \right). \tag{47}$$

By means of formulas (44), (45) we construct new solution

$$\overset{2}{u}(x_0, x_1) = \frac{2x_0}{x_0^2 + x_1^2} \tag{48}$$

of the equation (47). It should be noted that the solutions (46) and (48) have essentially different properties (boundaryness, periodicity, the behavior at zero and at the infinity and so on). If we will apply Lie transformations to manifold of the solutions of equation (47), then the majority of those properties of the solutions will be conserved.

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