Conditional invariance and exact solutions of gas dynamics equations

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Изучена условная инвариантность системы уравнений газовой динамики, а также получены некоторые ее точные решения.

Let us consider the system of gas dynamics equations

$$
\begin{aligned}
&\frac{\partial \vec{u}}{\partial x_0} + (\vec{u}\vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla}p,\\
&\frac{\partial \rho}{\partial x_0} + \operatorname{div}(\rho\vec{u}) = 0,\\
&p = F(\rho),
\end{aligned}
\tag{1}
$$

where $\vec{u} = \vec{u}(x) = \{u^1(x), u^2(x), \dots, u^n(x)\}$ is speed of gas diffusion, $\rho = \rho(x)$ is density of a gas, $p = p(x)$ is pressure of a gas, $x = (x_0, \vec{x}) = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{1+n}$.

L.V. Ovsyannikov in [1] investigated Lie symmetry of gas dynamics equations. We have to take notice of Lie symmetry of one-dimensional case. In this case, system (1) takes the form

$$
S_1 \equiv u_0 + uu_1 + f(\rho)\rho_1 = 0, \quad S_2 \equiv \rho_0 + u\rho_1 + \rho u_1 = 0,
\tag{2}
$$

where $u_\mu = \frac{\partial u}{\partial x_\mu}$, $\rho_\mu = \frac{\partial \rho}{\partial x_\mu}$, $\mu = 0, 1$, $\dot{F} = \rho f$.

In [2] it is proved that if

$$
f = \lambda\rho^{\gamma-2}, \quad \gamma = \frac{2N+1}{2N-1}, \quad N = 0, 1, 2, \dots, \quad \lambda = \text{const}
$$

then equations (2) are invariant under infinite-dimensional Lie algebra, which cannot be obtained from the results of [1].

In this paper we study conditional invariance (see [2]) of the system of gas dynamics equations.

Theorem. *The system of equations (2), with corresponding $f(\rho)$, is Q-conditionally invariant under operators Q_i, $i = 1, \dots, 8$, which are listed in Table.*

Proof. We give the proof of the theorem by considering one of the operators from Table, the other cases are analogous.

In accordance with definition (see [2]), system (2) is Q-conditionally invariant under the operator

$$
Q = \partial_0 + A(x_0, x_1, u, \rho)\partial_1 + B(x_0, x_1, u, \rho)\partial_u + C(x_0, x_1, u, \rho)\partial_\rho;
$$

if

$$
\begin{aligned}
&\tilde{Q}[S_1] = \theta_1 S_1 + \theta_2 S_2 + \theta_3(Qu) + \theta_4(Q\rho),\\
&\tilde{Q}[S_2] = \theta_5 S_1 + \theta_6 S_2 + \theta_7(Qu) + \theta_8(Q\rho),
\end{aligned}
$$

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$f = \frac{\dot F(\rho)}{\rho}$	Q	Ansatz	Reduced ODE system
$\lambda_1 p^{-3} + \lambda_2 p^{-1}$	$Q_1 = \partial_0 + u\partial_1 + {}$ $+\lambda_3[(\lambda_1\rho^{-2} + \lambda_2)\partial_u + \sqrt{\lambda_1}\partial_\rho]$	$\begin{cases} \rho = e^{\varkappa}, \\ u = \varphi^1(\omega) - \sqrt{\lambda_1}e^{-\varkappa} + \lambda_2\lambda_3 x_0, \\ \varkappa = \varphi^2(\omega) + \lambda_3 x_0 \varphi^1(\omega) - \lambda_3 x_1 + \frac{\lambda_2\lambda_3^2}{2}x_0^2, \\ \omega = p - \sqrt{\lambda_1}\lambda_3 x_0 \end{cases}$	$\begin{cases} \dot\varphi^1(\omega) = 0, \\ \dot\varphi^2(\omega) = 0 \end{cases}$
$\rho^{-2}(\lambda_1 + \lambda_2\rho)^{-1}$	$Q_2 = \partial_0 + u\partial_1 + \lambda_3\partial_u -$ $-\frac{\lambda_1\rho + \lambda_2\rho^2}{\lambda_1 x_0}\partial_\rho$	$\begin{cases} \rho = -\lambda_1(\lambda_2 + \varphi^1(\omega)x_0)^{-1}, \\ u = \frac{\varphi^2(\omega)}{x_0} + \frac{1}{2}\lambda_3 x_0 + x_1 x_0^{-1}, \\ \omega = u - \lambda_3 x_0 \end{cases}$	$\begin{cases} \dot\varphi^1(\omega) = -\lambda_3\lambda_1^2, \\ \varphi^1(\omega)\dot\varphi^2(\omega) = -\lambda_2 \end{cases}$
λp^{-3}	$Q_3 = \partial_0 + u\partial_1 + {}$ $+\lambda_1\sqrt{\lambda}(\sqrt{\lambda}p^{-2}\partial_u + \partial_\rho)$	$\begin{cases} \rho = e^{\varkappa}, \\ u = \varphi^1(\omega) - \sqrt{\lambda}e^{-\varkappa}, \\ \varkappa = \varphi^2(\omega) - \lambda_1 x_1 + \lambda_1 x_0 \varphi^1(\omega), \\ \omega = \rho - \sqrt{\lambda}\lambda_1 x_0 \end{cases}$	$\begin{cases} \dot\varphi^1(\omega) = 0, \\ \dot\varphi^2(\omega) = 0 \end{cases}$
	$Q_4 = \partial_0 + u\partial_1 + \lambda_1\partial_u + \lambda_2\rho^2\partial_\rho$	$\begin{cases} \rho = (\varphi^1(\omega) - \lambda_2 x_0)^{-1}, \\ u = x_0^{-1}\varphi^2(\omega) + \frac{1}{2}\lambda_1 x_0 + x_1 x_0^{-1}, \\ \omega = u - \lambda_1 x_0. \end{cases}$	$\begin{cases} \lambda\lambda_2\dot\varphi^1(\omega) + \lambda_1 = 0, \\ \lambda_2\dot\varphi^2(\omega) - \varphi^1(\omega) = 0 \end{cases}$
	$Q_5 = \rho\partial_0 - (\rho^2 u^2 - \lambda)\partial_u$	$\begin{cases} \rho = \varphi^1(x_1), \\ u = \frac{\sqrt{-\lambda}}{\varphi^1(x_1)}\,\mathrm{tg}(\varphi^2(x_1) - \sqrt{-\lambda}x_0),\ \lambda < 0, \end{cases}$	
		$\begin{cases} \rho = \varphi^1(x_1), \\ u = \frac{\sqrt{\lambda}}{\varphi^1(x_1)}\,\mathrm{th}(\varphi^2(x_1) + \sqrt{\lambda}x_0),\ \lambda > 0, \end{cases}$	$\begin{cases} \dot\varphi^1(x_1) = -\lambda_1(\varphi^1(x_1))^2, \\ \dot\varphi^2(x_1) = 0 \end{cases}$
$\forall\ f(\rho)$	$Q_6 = f\partial_1 + \lambda\partial_\rho$	$\begin{cases} \int f d\rho - \lambda x_1 = \varphi^2(x_0), \\ u = \varphi^1(x_0) \end{cases}$	$\begin{cases} \dot\varphi^1(x_0) = -\lambda, \\ \dot\varphi^2(x_0) + \lambda\varphi^1(x_0) = 0 \end{cases}$
	$Q_7 = -\partial_0 + \lambda u\rho\partial_u + \lambda\rho^2\partial_\rho$	$\begin{cases} \rho = (\varphi^1(x_1) + \lambda x_0)^{-1}, \\ u = \varphi^2(x_1)(\varphi^1(x_1) + \lambda x_0)^{-1} \end{cases}$	$\begin{cases} \dot\varphi^1(x_1) = 0, \\ \dot\varphi^2(x_1) = 0 \end{cases}$
	$Q_8 = \partial_0 + \lambda\partial_u + \lambda u f^{-1}\partial_\rho$	$\begin{cases} \frac{1}{2}u^2 - \int f d\rho = \varphi^1(x_1), \\ u = \varphi^2(x_1) + \lambda x_0 \end{cases}$	$\begin{cases} \dot\varphi^1(x_1) = \lambda, \\ \dot\varphi^2(x_1) = 0 \end{cases}$

λ, λ_i, $i = \overline{1,3}$ are arbitrary constants.

where $\tilde{Q}$ is the prolongation of Q; θ_i are some functions, $i = 1, 8$; $Qu = u_0 + Au_1 - B$; $Q\rho = \rho_0 + A\rho_1 - C$.

Let us consider operator

$$Q_4 = \partial_0 + u\partial_1 + \lambda_1\partial u + \lambda_2\rho^2\partial_\rho, \quad \lambda_i = \text{const}, \qquad i = 1, 2.$$

We will show that system (2) with $f(\rho) = \lambda\rho^{-3}$, $\lambda = \text{const}$ is Q-conditionally invariant under operator Q_4. For this particular case we have

$$S_1 = u_0 + uu_1 + \lambda\rho^{-3}\rho_1, \tag{3}$$

$$S_2 = \rho_0 + u\rho_1 + \rho u_1, \tag{4}$$

$$Q_4 u = u_0 + uu_1 - \lambda_1, \quad Q_4\rho = \rho_0 + u\rho_1 - \lambda_2\rho^2.$$

Acting by the prolongation of Q_4 on (3), (4) and then getting together terms in a proper manner we obtain the following

$$\begin{aligned} &\tilde{Q}_4[S_1] = -\lambda\rho^{-4}\rho_1 S_2 + \lambda\rho_1\rho^{-4}(Q_4\rho) - u_1(Q_4 u), \\ &\tilde{Q}_4[S_2] = (2\lambda_2\rho - u_1)S_2 - \rho_1(Q_4 u) + u_1(Q_4\rho). \end{aligned} \tag{5}$$

It follows from (5) that the system (2) with $f = \lambda\rho^{-3}$ is Q-conditionally invariant under the operator Q_4. The theorem is proved.

All obtained operators of conditional invariance of the system (2) are used for constructing of ansätze which reduce equations (2) to the systems of ordinary differential equations (ODE). The final results are listed in the Table. Having integrated the reduced equations and substituting obtained values of φ^1, φ^2, into the corresponding ansatz we get the following solutions of system (2) with a proper value of $f(\rho)$:

$$\rho = \exp\left\{\lambda_5 + \lambda_3\lambda_4 x_0 - \lambda_3 x_1 + \frac{\lambda_2\lambda_3^2}{2}x_0^2\right\},$$

$$u = -\sqrt{\lambda_1}\exp\left\{-\lambda_5 - \lambda_3\lambda_4 x_0 + \lambda_3 x_1 - \frac{\lambda_2\lambda_3^2}{2}x_0^2\right\} + \lambda_2\lambda_3 x_0 + \lambda_4;$$

$$\rho = -\lambda_1[\lambda_2 - \lambda_3\lambda_1^2(\omega + \lambda_4)x_0]^{-1},$$

$$u = x_0^{-1}\left[\frac{\lambda_2}{\lambda_3\lambda_1^2}(\ln(\omega + \lambda_4) + \lambda_5)\right]^2 + \frac{1}{2}\lambda_3 x_0 + x_1 x_0^{-1}, \quad \omega = u - \lambda_3 x_0;$$

$$\rho = \exp\{\lambda_3 - \lambda_1 x_1 + \lambda_1\lambda_2 x_0\},$$

$$u = \lambda_2 - \sqrt{\lambda}\exp\{-\lambda_3 + \lambda_1 x_1 - \lambda_1\lambda_2 x_0\};$$

$$\rho = (-\lambda_1\lambda^{-1}\lambda_2^{-1}\omega + \lambda_3 - \lambda_2 x_0)^{-1},$$

$$u = x_0^{-1}\left(-\frac{\lambda_1}{2\lambda\lambda_2^2}\omega^2 + \frac{\lambda_3}{\lambda_2}\omega + \lambda_4\right) + \frac{1}{2}\lambda_1 x_0 + \frac{x_1}{x_0}, \quad \omega = u - \lambda_1 x_0;$$

$$\rho = \frac{1}{\lambda_1 x_1},$$

$$u = \begin{cases} -\lambda_1\sqrt{-\lambda}x_1 \operatorname{tg}\lambda_1\sqrt{-\lambda}x_0, & \lambda < 0, \\ \lambda_1\sqrt{\lambda_1}x_1 \operatorname{th}\lambda_1\sqrt{\lambda}x_0, & \lambda < 0; \end{cases}$$

$$\int f d\rho = \frac{1}{2}\lambda_1^2 x_0^2 - \lambda_1 x_1,$$
$$u = \lambda_1 x_0;$$
$$\rho = (\lambda x_0 + \lambda_1)^{-1},$$
$$u = (\lambda x_1 + \lambda_2)(\lambda_1 + \lambda x_0)^{-1};$$
$$\frac{1}{2}u^2 + \int f d\rho = \lambda x_1 + \lambda_2,$$
$$u = \lambda_1 + \lambda x_0,$$

where λ_i, $i = \overline{1,5}$ are arbitrary constants.

The results of the theorem can be generalized for n-dimensional case. After that the counterparts of Q, have the form

$$\hat{Q}_1 = \partial_0 + \vec{u}\,\vec{\nabla} + (\lambda_1\rho^{-1} + \lambda_2)\vec{\alpha}\partial_{\vec{u}} + \sqrt{\lambda_1}\partial_\rho,$$
$$\hat{Q}_2 = \partial_0 + \vec{u}\,\vec{\nabla} + \vec{\alpha}\partial_{\vec{u}} - \frac{\lambda_1\rho + \lambda_2\rho^2}{\lambda_1 x_0}\partial_\rho,$$
$$\hat{Q}_3 = \partial_0 + \vec{u}\,\vec{\nabla} + \lambda\rho^{-1}\vec{\alpha}\partial_{\vec{u}} + \sqrt{\lambda}\partial_\rho,$$
$$\hat{Q}_4 = \partial_0 + \vec{u}\,\vec{\nabla} + \vec{\alpha}\partial_{\vec{u}} + \lambda_2\rho^2\partial_\rho,$$
$$\hat{Q}_5 = \rho\partial_0 - (\rho^2\vec{u}^2 - \lambda)\vec{\alpha}\partial_{\vec{u}},$$
$$\hat{\vec{Q}}_6 = f(\rho)\vec{\nabla} + \vec{\alpha}\partial_\rho,$$
$$\hat{Q}_7 = -\partial_0 + \lambda\rho\vec{u}\partial_{\vec{u}} + \lambda\rho^2\partial_\rho,$$
$$\hat{Q}_8 = \partial_0 + \vec{\alpha}\partial_{\vec{u}} + \vec{\alpha}\vec{u}f^{-1}(\rho)\partial_\rho,$$

where $\vec{\alpha}$ is arbitrary constant unit vector. By analogy with one-dimensional case, using operators $\hat{Q}_i$, $i = \overline{1,8}$ we can construct ansätze which reduce system (1) to systems with lesser number of variables.

Let us show some examples. The ansätze for system (1) that were constructed by means of operators $\hat{Q}_7$, $\hat{Q}_8$, $\hat{Q}_5$, respectively, are of the form

$a)$ $$\rho = (\varphi^0(\vec{x}) - \lambda x_0)^{-1},$$
$$u^a = \varphi^a(\vec{x})(\varphi^0(\vec{x}) - \lambda x_0)^{-1}, \quad a = \overline{1,n};$$

$b)$ $$\int f d\rho = \varphi^0(\vec{x}) + \alpha_a\varphi^a(\vec{x})x_0 + \frac{1}{2}\vec{\alpha}^{\,2}x_0^2,$$
$$u^a = \varphi^a(\vec{x}) + \alpha_a x_0, \quad a = \overline{1,n};$$

$c)$ $$\rho = \sqrt{\lambda}\varphi^0(\omega), \quad \omega = \{\omega_1, \omega_2, \omega_3\} = \{\vec{a}\,\vec{x}, \vec{b}\,\vec{x}, \vec{c}\,\vec{x}\},$$
$$u = \begin{cases} \vec{a}\varphi^1 + \vec{b}\varphi^2 + \dfrac{\vec{c}}{\sqrt{\lambda}}(x_0\varphi^0 + \varphi^3)^{-1}, & \delta = 0, \\ \vec{a}\varphi^1 + \vec{b}\varphi^2 + \vec{c}\,\mathrm{th}\,\sqrt{\lambda}(x_0\varphi^0 + \varphi^3), & \delta = 1, \\ \vec{a}\varphi^1 + \vec{b}\varphi^2 + \vec{c}\,\mathrm{tg}\,\sqrt{\lambda}(-x_0\varphi^0 + \varphi^3), & \delta = -1, \end{cases}$$

where $\vec{x} = (x_1, x_2, x_3)$; $\delta \equiv (\varphi^1)^2 + (\varphi^2)^2 - (\varphi^0)^{-2}$; $\vec{a}$, $\vec{b}$, $\vec{c}$ are arbitrary orthonormal vectors.

These ansätze reduce (1) to the following systems

$$
\begin{aligned}
&a)\quad \varphi^0 = \text{const},\\
&\qquad (\vec\varphi\vec\nabla + \lambda)\vec\varphi = 0,\\
&\qquad \operatorname{div}\vec\varphi + \lambda = 0;\\
&b)\quad \vec\alpha + \vec\nabla\varphi^0 + (\vec\varphi\vec\nabla)\vec\varphi = 0,\\
&\qquad \operatorname{div}\vec\varphi = 0,\\
&\qquad \varphi^a_b + \varphi^b_a = 0, \quad a, b = \overline{1,n}, \quad a \neq b;\\
&c)\quad \text{for } \delta = 0; 1; -1 \text{ respectively:}\\
&\qquad \varphi^0_3 = \varphi^s_3 = \varphi^3_3 = 0, \quad s = 1, 2,\\
&\qquad \varphi^s\varphi^3_s = -\varphi^0,\\
&\qquad \varphi^s\varphi^\sigma\varphi^\sigma_s = 0, \quad s, \sigma = 1, 2,\\
&\qquad \varphi^2_1 - \varphi^1_2 = 0, \quad \varphi^1_1 + \varphi^2_2 = 0.
\end{aligned}
$$

Having defined the potential $v = v(\omega_1, \omega_2)$, $\varphi^s = \frac{\partial v}{\partial \omega_s}$, $s = 1, 2$, we can rewrite the system c) in the following form

$$
\begin{aligned}
&\varphi^0_3 = \varphi^3_3 = \varphi^s_3 = 0, \quad s = 1, 2,\\
&v_s\varphi^3 = -\varphi^0,\\
&-(\varphi^0)^{-2} + v_s v_s = \begin{cases} 0,\\ 1,\\ -1,\end{cases}\\
&\Delta v = 0,\\
&v_s v_\sigma v_s\sigma = 0.
\end{aligned}
$$

Having got a solution of the system

$$\Delta v = 0, \quad v_s v_\sigma v_s\sigma = 0, \tag{6}$$

we can write down the solution of system (6) and, using corresponding ansatz, to construct a solution for system (1). For example we can consider the particular solution of system (7)

$$v = \psi_1.$$

It leads to the solution of system (6) $v = \omega_1$, $\varphi^3 = \omega_1 + \Phi(\psi_2)$, $\varphi^0 = 1$, $\varphi^1 = 1$, $\varphi^2 = 0$.

Using the corresponding ansatz we obtain a solution of system (1) which depends on arbitrary function Φ

$$\rho = \sqrt{\lambda}, \quad \vec u = \vec a + \frac{\vec c}{\sqrt{\lambda}}(x_0 + \omega_1 + \Phi(\omega_2))^{-1},$$

where $\omega_1 = \vec a\,\vec x$, $\omega_2 = \vec b\,\vec x$, $\vec a$, $\vec b$, $\vec c$ are arbitrary orthonormal vectors.

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