The complete sets of conservation laws for the electromagnetic field

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We present a compact and simple formulation of zero- and first-order conserved currents for the electromagnetic field and give the number of independent n-order currents.

New conservation laws for the electromagnetic field, discovered by Lipkin [1], had obtained an adequate mathematical and physical interpretation long ago, see e.g. [2–6]. It happens that these conservation laws are nothing but a small part of the infinite series of conserved quantities which exist for any self-adjoint linear system of differential equations; among their number are Maxwell's equations [7]. As to the physical interpretation of Lipkin's zilch tensor it can be connected with conservation of polarization of the electromagnetic field [5, 6].

The aim of the present letter is to establish certain rules in the bewildering complexity of the conservation laws and to describe complete sets of them for the electromagnetic field.

We say that an arbitrary bilinear function $j_\mu^{(m)} = f_\mu^{(m)}(D^nF, D^kF)$ is a conserved current if it satisfies the continuity equation

$$\partial^\mu j_\mu^{(m)} = 0, \quad \mu = 0, 1, 2, 3. \tag{1}$$

Here $F = F_{\mu\nu}$ is the tensor of the electromagnetic field,

$$D^n = \prod_{\lambda=0}^{n} \partial^{\mu_\lambda}, \quad \mu_\lambda = 0, 1, 2, 3, \quad m = \max(n+k).$$

It follows from (1) according to the Ostrgradskii–Gauss theorem that the following quantity is conserved in time:

$$\langle j_0^{(m)} \rangle = \int d^3x j_0^{(m)}.$$

We say conserved currents $j_\mu^{(m)}$ and $j_\mu^{\prime(m)}$ are equivalent if

$$\langle j_0^{(m)} \rangle = \langle j_0^{\prime(m)} \rangle.$$

Proposition 1. *There exist exactly 15 non-equivalent conserved currents of zero order for Maxwell's equation. All these currents can be represented in the form*

$$j_\mu^{(0)} = T_{\mu\nu} K^\nu, \tag{2}$$

where $T_{\mu\nu}$ is the traceless energy-momentum tensor of the electromagnetic field and K^ν is a Killing vector satisfying the equations

$$\partial^\nu K^\mu + \partial^\mu K^\nu - \frac{1}{2} g^{\mu\nu} \partial_\lambda K^\lambda = 0. \tag{3}$$

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Proof. This reduces to finding the general solution of the equation

$$\partial^0 \langle j_0^{(0)} \rangle = \partial^0 \int d^3x j_0^{(0)}(F, F) = 0, \tag{4}$$

where $j_0^{(0)}(F, F)$ is a bilinear combination of components of the tensor of the electromagnetic field. It is not difficult to find such a solution, decomposing $j_0^{(0)}$ by the complete set of symmetric matrices of the dimension 6×6

$$\begin{aligned}
& j_0^{(0)} = \varphi^T Q \varphi, \quad \varphi = \text{column}(F_{01}, F_{02}, F_{03}, F_{23}, F_{31}, F_{12}), \\
& Q = (\sigma_0 A_0^{ab} + \sigma_1 A_1^{ab} + \sigma_3 A_3^{ab}) Z_{ab} + \delta_2 S_a K^a, \\
& Z_{ab} = 2\delta_{ab} + S_a S_b + S_b S_a, \quad a, b = 1, 2, 3, \\
& S_a = \begin{pmatrix} S_a & \hat{0} \\ \hat{0} & S_a \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} I & \hat{0} \\ \hat{0} & I \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} \hat{0} & I \\ I & \hat{0} \end{pmatrix}, \\
& \sigma_2 = \begin{pmatrix} \hat{0} & -I \\ I & \hat{0} \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} I & \hat{0} \\ \hat{0} & -I \end{pmatrix}, \\
& S_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

where $\hat{0}$ and I are the zero and unit matrices of dimension 3×3, A_λ^{ab}, K^a are unknown functions of x_μ. In fact substituting (5) into (4) and using the Maxwell equations

$$\partial_\mu F^{\mu\nu} = 0, \quad \partial^\mu \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} = 0$$

we come to the relations $A_1^{ab} = A_3^{ab} = 0$, $A_2^{ab} = -\delta^{ab} K^0$ and to the equations (3) for K^0 and K^a.

Thus we have found all non-equivalent $j_0^{(0)}$ satisfying (4). The corresponding expressions for $j_\mu^{(0)}$ with $\mu \neq 0$ can be obtained by Lorentz transformations.

Formula (2) gives an elegant formulation of the classical conservation laws of Bessel–Hagen [8]. We present a direct (and simple) proof that there are not another conserved bilinear combination of the electromagnetic field strengths.

In an analogous way it is possible to prove the following assertion.

Proposition 2. *There exist exactly 84 conserved currents of first order for the electromagnetic field. All these currents can be represented in the form*

$$j_\mu^{(1)} = K^{\sigma\nu} Z_{\sigma\nu,\mu} + 2\varepsilon_{\mu\nu\lambda\sigma} (\partial^\lambda K^{\rho\nu}) T^{\sigma\rho}, \tag{5}$$

where $T^{\sigma\rho}$ is the energy-momentum tensor, $Z_{\sigma\nu,\mu}$ is Lipkin's zilch tensor, $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric unit tensor, $K^{\sigma\nu}$ is a conformal Killing tensor of valence 2, satisfying the equations

$$\partial^{(\mu} K^{\sigma\nu)} = \frac{1}{3} \partial_\lambda K^{\lambda(\mu} g^{\sigma\nu)}, \quad K^{\sigma\nu} = K^{\nu\sigma}, \quad K_\mu^\mu = 0, \tag{6}$$

where symmetrization is imposed over the indices in brackets.

Using the relations

$$\begin{aligned}
& \partial^\mu Z_{\lambda\sigma,\mu} = 0, \quad Z_{\mu\nu,}{}^\nu = 0, \quad \partial_\lambda T^{\lambda\mu} = 0, \quad T^\lambda{}_\lambda = 0, \\
& \partial^\rho (\varepsilon_{\rho\lambda\nu\sigma} T^{\sigma\mu} + \varepsilon_{\rho\mu\nu\sigma} T^{\sigma\lambda}) = Z_{\lambda\nu,}{}^\mu + Z_{\mu\nu,}{}^\lambda
\end{aligned}$$

and the equations (7) we can ensure that the currents (6) really satisfy the continuity equation (1).

Thus all non-equivalent conserved currents of first order are given by formula (6). The general solution of the equation (7) is a fourth-order polynomial of x_μ depending on 84 parameters; for the explicit expression of $K^{\sigma\nu}$ see e.g. [9]. Formula (6) describes well known and also 'new' conserved currents; the latter depend on the fourth degree of x_μ.

In conclusion we note that in an analogous way it is possible to describe conserved currents for the electromagnetic field of an arbitrary order m. For $m > 1$ such currents are defined by two fundamental quantities i.e. by the conformal Killing tensor of valence $m+1$ and the Floyd–Penrose tensor of valence $R_1 + 2R_2$ where $R_1 = m-1$, $R_2 = 2$. The higher order conserved currents will be considered in a separate paper; here we present only the number of linearly independent currents of order m:

$$N_m = \frac{1}{2}(2m+5)\left[2m(m+1)(m+4)(m+5) + (m+2)^2(m+3)^2\right], \quad m > 1.$$

For the details about generalized Killing and Floyd–Penrose tensors in application to higher symmetries of Poincaré- and Galilei-invariant wave equations see the extended version of our book [10]. Non-Lie symmetries and conservation laws for Maxwell's equations are discussed in [11].

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