Conditional symmetry of equations of nonlinear mathematical physics

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1. Introduction. In this paper we present some results on conditional symmetry of nonlinear equations of mathematical and theoretical physics which were obtained in the Institute of Mathematics of Ukrainian Academy of Science. The term and the concept of "conditional symmetry of equation" or "conditional invariance" had been introduced in [1–10].

Speaking about the conditional symmetry of an equation, we mean of the symmetry of some subset of its solutions. To be a constructive one, such general definition needs, some more details. To study conditional symmetry means to give analytical description of conditions (constraints) for the set of solutions of an equation under study picking out subsets having wider (or another) symmetry properties than the whole set of solutions. Having carried out such description one can obtain solutions which cannot be obtained within the framework of the classical Vie approach (as it is known, in the Lie approach reduction, of the multi-dimensional partial differential equation (PDE) to equations with less number of independent variables is carried out by means of symmetry of the set of its solutions in a whole).

Euler, Bateman, Lie, Smirnov and Sobolev (1932) and many other classics used implicitly symmetry of subsets of solutions for linear d'Alembert and Laplace equations to construct their exact solutions. Not long ago Bluman and Cole [11] suggested the "non-classical method of solutions invariant under group" for the linear heat equation. Olver and Rosenau (1986) [12] constructed solutions of the one-dimensional nonlinear acoustics equation

$$u_{00} = uu_{11}, \quad u_{00} = \partial^2 u/\partial t^2, \quad u_{11} = \partial^2 u/\partial x^2 \tag{1}$$

which cannot be obtained by means of Lie method. Clarkson and Kruskal suggested "new method of invariant reduction of the Boussinesq equation"

$$u_{00} + \frac{1}{2}u_{11} + u_{1111} = 0. \tag{2}$$

Conclusion 1. *Using the concept of "conditional symmetry of PDE" we can obtain the above results within the framework of the unified symmetry approach.*

Conclusion 2. *The majority of linear and nonlinear equations of mathematical physics: d'Alembert, Maxwell, Schrödinger, Dirac, heat, acoustics, KdV equations possess some conditional symmetry.*

Note 1. All solutions of the Boussinesq equation (2) constructed by Clarkson and Kruskal had been obtained independently by Levi and Winternitz [14], and by Fushchych and Serov [10], using the concept of conditional symmetry.

in Symmetry Analysis of Equations of Mathematical Physics, Kyiv, Institute Mathematics, 1992, P. 7–27.

Let us consider some PDE

$$L(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u}) = 0, \tag{3}$$

where $u = u(x)$, $x \in \mathbb{R}(n+1)$, $u(x) \in \mathbb{R}$, $\underset{s}{u}$ is the set of s-th order partial derivatives of $u(x)$.

According to Lie, the equation (3) is invariant under the first-order differential operator

$$X = \xi^{\mu}(x, u)\frac{\partial}{\partial x^{\mu}} + \eta(x, u)\frac{\partial}{\partial u} \tag{4}$$

if the following condition is satisfied:

$$\underset{s}{X} L = \lambda L \quad \Leftrightarrow \quad \underset{s}{X} L\Big|_{L=0} = 0, \tag{5}$$

where $\underset{s}{X}$ is the s-th prolongation of the operator X, $\lambda = \lambda(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u})$ is some differential expression.

Let us designate by the symbol $Q = \{\underset{1}{Q}, \ldots, \underset{k}{Q}\}$ a collection of operators not belonging to the invariance algebra (IA) of the equation (3), i.e. $Q \notin$ IA.

Definition 1 [2, 5]. *We say that the equation (3) is conditionally-invariant under the operators Q if there exists some additional condition*

$$L_1(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u}) = 0 \tag{6}$$

to be compatible.

The additional condition (6) picks out some subset from the whole set of solutions of the equation (3). It appears that for many important equations of mathematical physics such subsets admit the wider symmetry than the whole set of solutions. Such subsets are to be constructed.

Let the operator Q act on the equation (3) as follows:

$$\underset{s}{Q} L = \lambda_0 L + \lambda_1 L_1 \tag{7}$$

or

$$\underset{s}{Q} L\Big|_{\substack{Lu=0\\ L_1 u = 0}} = 0,$$

where λ_0, λ_1 are some differential expressions depending on $x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u}$, $\underset{s}{Q}$ is the s-th prolongation of the operator Q. Then the invariance condition reads

$$\underset{s}{Q} L_1 = \lambda_2 L + \lambda_3 L_1, \tag{8}$$

where λ_2, λ_3 are some differential expressions.

The principal problem of our approach is to describe in explicit form equations of the form (6) which extend symmetry of the equation (6).

The principal and difficult problem can be essentially simplified if one chooses the following nonlinear first-order PDE as an additional condition (6):

$$Qu = 0, \tag{9}$$

where

$$Q = j^\mu(x^\mu, u)\partial_\mu + z(x^\mu, u)\partial_u, \quad \partial_\mu \equiv \partial/\partial x_\mu, \quad \partial_u \equiv \partial/\partial u. \tag{10}$$

In this case, the invariance condition for the system of equations (3), (9) takes the form

$$\underset{s}{Q}\, L = \lambda_0 L + \lambda_1 (Qu). \tag{11}$$

Definition 2. *We say that the equation (3) is Q-conditionally invariant if the system (3), (9) is invariant under the operator (10).*

Let us turn now to the simplest one-dimensional acoustics equation.

2. Conditional symmetry of the equation (1).

Theorem 1 [18]. *The equation (1) is Q-conditionally invariant under the operator (10) if its coefficient functions*

$$j^0 \equiv A(x), \quad j^1 \equiv B(x), \quad z \equiv h(x)u + q(x), \quad x = (x_0, x_1)$$

satisfy the following differential equations:

Case 1. $A \neq 0$, $B \neq 0$:

$$\begin{aligned}
&h = 2\left(B_1 - A_0 + \frac{B}{A}A_1\right), \quad q = 2\frac{B}{A}B_0,\\
&h_{00} + \frac{2}{A}hh_0 - \left[\frac{h}{A}A_{00} + \frac{2}{A}hA_{00} + 2\left[\frac{h}{A}\right]_1 B_0\right] = q_{11} - \frac{q}{A}A_{11} + 2\left[\frac{q}{A}\right]_1 A_1,\\
&h_{11} = \frac{h}{A}A_{11} + 2\left[\frac{h}{A}\right]_1 A_1,\\
&q_{00} + 2\frac{q}{A}q_0 - \left[\frac{q}{A}A_{00} + 2\left[\frac{q}{A}\right]_1 B_0\right] = 0,\\
&B_{11} - 2h_1 - \left[\frac{B}{A}A_{11} + 2\left[\frac{B}{A}\right]_1 A_1 + 2\frac{h}{A}A_1\right] = 0,\\
&B_{00} + 2\frac{B}{A}h_0 - \left[\frac{B}{A}A_{00} + 2\left[\frac{B}{A}\right]_1 B_0\right] = 0.
\end{aligned} \tag{12}$$

The subscripts denote the corresponding derivatives.

Case 2. $A = 0$, $B \neq 0$ *(without loosing generality one may choose* $B = 1$*):*

$$\begin{aligned}
&h_0 = 0, \quad h_{11} + 3hh_1 + h^3 = 0,\\
&q_{11} + hq_1 + \left(3h_1 + 2h^2\right) q = 0,\\
&q_{00} - qq_1 - hq^2 = 0.
\end{aligned} \tag{13}$$

Case 3. $A = 1$, $B = 0$:

$$\begin{aligned}
&h_1 = 0, \quad h_{00} + hh_0 - h^3 = q_{11}, \quad q(q_0 + hq) = 0,\\
&q_{00} + h_0 q - h^2 q - 0.
\end{aligned} \tag{14}$$

Thus a problem of study of Q- conditional symmetry of the equation (1) is reduced to search of the general solution for the equations (12)–(14). Let us emphasize that coefficient functions j, z of the operator Q unlike coefficient functions ξ, η (4) satisfy a system of nonlinear equations. This fact makes difficult to describe conditional symmetry of given equations. Nevertheless it is possible to construct their partial solutions.

We had found 12 inequivalent operators of conditional symmetry for the equation (1) [8]. Two of them have the form

$$Q_1 = x_0^2 x_1 \partial_1 + \left(u x_0^2 + 3x_1^2 + b_5 x_0^5 + b_6\right)\partial_u, \tag{15}$$

$$Q_2 = \partial_1 + [W(x_0)x_1 + f(x_0)]\partial_u, \quad W'' = W^2, \quad f'' = Wf, \tag{16}$$

W is the Weierstrass function.

The operator (15) generates the ansatz

$$U = x_1\varphi(x_0) + 3x_0^{-2}x_1 - b_5 x_0^3 + b_6 x_0^{-2}. \tag{17}$$

The ansatz (17) reduces the nonlinear equation (2) to linear differential equation (ODE)

$$x_0^2\varphi''(x_0) = 6\varphi(x_0) \tag{18}$$

operator (16) gives rise to the ansatz

$$u = \frac{1}{2}W(x_0)x_1^2 + f(x_0)x_1 + \varphi(x_0) \tag{19}$$

reducing the equation (1) to linear ODE with the Weierstrass potential

$$\varphi''(x_0) = W\varphi(x_0). \tag{20}$$

Note 2. In an analogous way we had constructed families of exact solutions for the multi-dimensional equation [8]

$$u_{00} = u\Delta u. \tag{21}$$

Conclusion 3. *Ansätzes generated by operators of conditional invariance often reduce the initial nonlinear equation to a linear one. The reduction by Lie operators, as a rule, does not change the nonlinear structure of the equation under study.*

3. Conditional invariance of the d'Alembert equation. Let us consider the nonlinear equation

$$\Box u = F_1(u), \quad u = u(x_0, x_1, x_2, x_3), \tag{22}$$

where $F_1(u)$ is an arbitrary smooth function. The equation (22) is invariant under the conformal group (that is the maximal invariance group admitted by (22)) iff $F_1 = 0$ or $F_1 = \lambda u^3$. Let us impose on the solutions of (22) the Poincaré-invariant eikonal constraint

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} = F_2(u), \tag{23}$$

where F_2 is a smooth function.

Theorem 2 [15]. *Provided $F_1 = F_2 = 0$ the equation (22) with the condition (23) is invariant under the infinite-dimensional Lie algebra with coefficients of the operator (4) having the form*

$$\xi^\mu(x,u) = c^{00}(u)x^\mu + c^{\mu\nu}(u)x^\nu + d^\mu(u), \quad \eta(x,u) = \eta(u),$$

where $c^{00}(u)$, $c^{\mu\nu}(u)$, $d^\mu(u)$, $\eta(u)$ are arbitrary smooth functions.

Consequently, the additional condition (23) ($F_2 = 0$) picks out from the whole set of solutions of the linear d'Alembert equation ($F_1 = 0$) the subset having the unique symmetry properties. Besides, an arbitrary smooth function of a solution of the system (22), (23) ($F_1 = F_2 = 0$) is its solution too.

Theorem 3. *The system (22), (23) is invariant under the conformal group $C(1,3)$ iff*

$$F_1 = 3\lambda(u+C)^{-1}, \quad F_2 = \lambda, \tag{24}$$

where λ, C are constants.

Thus, the additional eikonal constraint (23) extends the class of nonlinear wave equations admitting conformal group. It means that we can construct wide classes of exact solutions of the equation (22) using the subgroup structure of the group $C(1,3)$.

Note 3. The system (22), (23) had been completely integrated in [16].

Let us consider the Lorentz non-invariant wave equation

$$Lu \equiv \Box u + F(x, u, \underset{1}{u}), \tag{25}$$

$$F = -\left(\frac{\lambda_0}{x_0}\right)^2 \left(\frac{\partial u}{\partial x_0}\right)^2 + \sum_{a=1}^{3} \left(\frac{\lambda_a}{x_a}\right)^2 \left(\frac{\partial u}{\partial x_a}\right)^2. \tag{26}$$

The maximal invariance group admitted by the equations (25), (26) is the following two-parameter group

$$x_\mu \to x'_\mu = e^a x_\mu, \quad u \to u' = u + b,$$

where a, b are group parameters.

An additional condition of the type (6) is chosen in the form

$$J_{\mu\nu}u = 0, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad \mu,\nu = 0,1,2,3. \tag{27}$$

By direct check one can assure that the equations (25), (27) are invariant under the Lorentz group $O(1,3)$. It means that the Lorentz-invariant ansatz

$$u = \varphi(\omega), \quad \omega = x_\mu x^\mu = x_0^2 - x_1^2 - x_2^2 - x_3^2 \tag{28}$$

reduces the nonlinear wave equation (25) to the following ODE

$$\omega\frac{d^2\varphi}{d\omega^2} + 2\frac{d\varphi}{d\omega} + \lambda^2\left(\frac{d\varphi}{d\omega}\right)^2 = 0, \quad \lambda^2 = \lambda_\mu\lambda^\mu.$$

The solution of the above equation is given by the formulae

$$\varphi(\omega) = 2(-\lambda^2)^{-1/2}\tan^{-1}\left[\omega(-\lambda^2)^{-1/2}\right], \quad \lambda^2 < 0,$$

$$\varphi(\omega) = -(\lambda^2)^{-1/2}\ln\left(\frac{(\lambda^2)^{-1/2}+\omega}{(\lambda^2)^{-1/2}-\omega}\right), \quad \lambda^2 > 0,$$

$$\varphi(\omega) = C_1\omega^{-1} + C_2, \quad \lambda^2 = 0,$$

where C_1, C_2 are constants.

Thus, the condition (27) selects from the set of solutions of the Lorentz non-invariant equation (25) a subset invariant under the six-parameter Lorentz group. This essential extension of the symmetry makes it possible to construct wide classes of exact solutions of the nonlinear wave equation (25).

4. Conditional invariance of the nonlinear Schrödinger equation. Let us consider the nonlinear equation of the form

$$Su + F(|u|)u = 0, \quad S \equiv i\frac{\partial}{\partial x_0} + \lambda_1\Delta. \tag{29}$$

The equation (29) is invariant under the Galilei algebra $AG(1,3)$ having the basis elements

$$\begin{gathered} P_0 = \partial_0, \quad P_a = \partial_a, \quad J_{ab} = x_aP_b - x_bP_a, \quad a,b = \overline{1,n}, \\ G_a = x_0P_a + \frac{1}{2\lambda_1}x_aR_1, \end{gathered} \tag{30}$$

where

$$R_1 = i\left(u\frac{\partial}{\partial u} - u^*\frac{\partial}{\partial u^*}\right).$$

In the class of nonlinear equations (29) there are two well-known ones having wider symmetry algebra than the equation (29) has [17, 18]:

$$Su + \lambda_2|u|^r u = 0, \tag{31}$$

$$Su + \lambda_3|u|^{4/n}u = 0, \tag{32}$$

where λ_2, λ_3, r are arbitrary parameters, n is the number or space variables in the equation (29).

The equation (31) is invariant, under the extended Galilei algebra $AG_1(1,n) = \langle AG(1,n), D\rangle$ having the basis elements (30) and

$$D = 2x_0P_0 + x_aP_a + \frac{2}{r}R_2, \tag{33}$$

where

$$R_2 = u\frac{\partial}{\partial u} + u^*\frac{\partial}{\partial u^*}.$$

The equation (32) is invariant under the generalized Galilei algebra $AG_2(1,n) = \langle AG_1(1,n), A\rangle$ having the basis elements (30), (33) and

$$A = x_0^2P_0 + x_0x_aP_a + \frac{x^2}{4\lambda_1}R_1 - \frac{nx_0}{2}R_2, \quad x^2 = x_1^2 + \cdots + x_n^2.$$

Theorem 4 [18]. *The Schrödinger equation (29) is conditionally-invariant under the operator*

$$Q_1 = \ln(uu^{*-1})R_1 + x_a P_a - cR_2, \quad c = \text{const}, \tag{34}$$

provided

$$F(|u|) = \lambda_4|u|^{-4/r} + \lambda_5|u|^{4/r},$$

where λ_4, λ_5, r are arbitrary real parameters, and the condition

$$\lambda_1\Delta|u|^{r+4} + \lambda_6|u|^r = 0 \tag{35}$$

holds.

Thus imposing on solutions of the nonlinear equation (29) an additional constraint (35) we extend its symmetry.

Theorem 5 [18]. *The equation (32) being taken together with the the equation (35) is invariant under the algebra $AG_2(1,n)$ and the operator Q_1 (34).*

5. Conditional symmetry of nonlinear heat equations. To describe nonlinear processes of heat and mass transfer the one-dimensional equations of the form are used

$$u_0 + u_{11} = F(u), \tag{36}$$

$$u_0 + uu_{11} = 0, \tag{37}$$

where F is a smooth function.

We look for operators of conditional symmetry in the form

$$Q = A(x,u)\partial_0 + B(x,u)\partial_1 + C(x,u)\partial_u \tag{38}$$

with some smooth functions A, B, C.

Theorem 6 [19]. *The equation (36) is Q-conditionally-invariant under the operator (38) if functions A, B, C satisfy differential equations:*

Case 1: $A = 1$

$$\begin{aligned} &B_{uu} = 0, \quad C_{uu} = 2(B_{1u} + BB_u), \\ &3B_uF = 2(C_{1u} + CB_u) - (B_0 + B_{11} + 2BB_1), \\ &CF_u - (C_u - 2B_1)F = C_0 + C_{11} + 2CB_1. \end{aligned} \tag{39}$$

Hereafter subscript, mean differentiation with respect to the corresponding variables (x_0, x_1, u).

Case 2. $A = 0$, $B = 0$

$$CF_u - C_uF = C_0 + C_{11} + 2CC_{1u} + C^2C_{uu}. \tag{40}$$

Having constructed the general solutions of nonlinear systems (39), (40), we shall obtain the general operator of conditional symmetry of equation.

Theorem 7 [19]. *The equation (36) is Q-conditionally-invariant under the operator (38) ($A = 1$, $B_u \neq 0$) iff it, is locally equivalent to the equation*

$$u_0 + u_{11} = b_3u^3 + b_1u + b_0, \quad b_0, b_1, b_3 = \text{const}, \tag{41}$$

the operator (38) having the form

$$Q = \partial_0 + \frac{3}{2}\sqrt{2b_3}u\partial_1 + \frac{3}{2}\left(b_3u^3 + b_1u - b_0\right)\partial_u. \tag{42}$$

The equation (41) is reduced to one of the following canonical equations:

$$u_0 + u_{11} = \lambda u(u^2 - 1), \tag{43}$$

$$u_0 + u_{11} = \lambda(u^3 - 3u + 2), \tag{44}$$

$$u_0 + u_{11} = \lambda u^3, \tag{45}$$

$$u_0 + u_{11} = \lambda u(u^2 + 1). \tag{46}$$

Ansätzes constructed by means of the operator (42) have the from

$$\varphi(\omega) = 2\tan^{-1} u + \sqrt{2\lambda}x_1, \quad \omega = -\ln(1 - u^{-2}) + 2\lambda x_0, \tag{47}$$

$$\begin{gathered}\varphi(\omega) = -\frac{4}{9}\ln\frac{u+2}{u-1} - \frac{2}{3}(u-1)^{-1} - \sqrt{2\lambda}x_1, \\ \omega = \frac{2}{9}\ln\frac{u+2}{u-1} - \frac{2}{3}(u-1)^{-1} - 3\lambda x_0,\end{gathered} \tag{48}$$

$$\varphi(\omega) = -2u^{-1} + \sqrt{2\lambda}x_1, \quad \omega = -u^{-2} - 3\lambda x_0, \tag{49}$$

$$\varphi(\omega) = 2\tan^{-1} u - \sqrt{2\lambda}x_1, \quad \omega = -\ln(1 + u^{-2}) - 3\lambda x_0. \tag{50}$$

The anzätzes (47)–(50) reduce the equations (43)–(46) to ODE:

$$2\ddot{\varphi} = (\dot{\varphi}^2 - 1)\dot{\varphi}, \quad 2\ddot{\varphi} = \dot{\varphi}^3 - 3\dot{\varphi} + 2, \tag{51}$$

$$2\ddot{\varphi} = \dot{\varphi}^3, \quad 2\ddot{\varphi} = \dot{\varphi}(\dot{\varphi}^2 + 1). \tag{52}$$

It is evident from the above equations that ansätzes generated by the operator of conditional invariance (42) change essentially their nonlinearities in second parts. This fact allows to integrate the ODE (51), (52) in elementary functions

$$\varphi(\omega) = -2\tan^{-1}\left(\sqrt{C_1\exp\omega + 1}\right) + C_2, \tag{53}$$

$$\ln\left[C_1 - \frac{3}{2}(\varphi + 2\omega)\right] = \ln C_2 - \frac{3}{2}(\varphi - \omega), \tag{54}$$

$$\varphi(\omega) = 2\sqrt{C_1 - \omega} + C_2, \tag{55}$$

$$\varphi(\omega) = 2\tan^{-1}\left(\sqrt{C_1\exp\omega - 1}\right) + C_2, \tag{56}$$

where C_1, C_2 are constants.

Thus, substitute (53)–(56) into (47)–(50), we get families of exact solutions of the equations (43)–(46). These solutions cannot be obtained within the framework of the Lie method.

Theorem 8 [20]. *The equation (37) is Q-conditionally-invariant under the operator (38) with $A = 1$ if functions B, C satisfy the following system of equations:*

$$uC_{uu} = 2(BB_u + uB_{u1}), \quad B_{uu} = 0, \tag{57}$$

$$B_0 + uB_{11} - CBu^{-1} - 2uC_{u1} + 2BB_1 - 2CB_u = 0, \tag{58}$$

$$C_0 + uC_{11} - C^2u^{-1} + 2CB_1 = 0. \tag{59}$$

Solving equations (57)–(59), we get an explicit form of the operator (38)

$$Q = b_1Q_1 + b_2Q_2 + b_3D_1 + b_4D_2 + b_5\partial_0 + b_6\partial_1, \tag{60}$$

$$\begin{gathered} Q_1 = x_1\partial_0 + u\partial_1, \quad Q_2 = x_0^2\partial_0 + 2x_1u\partial_1 + 2u^2\partial_u, \\ D_1 = 2x_0\partial_0 + x_1\partial_1, \quad D_2 = x_1\partial_1 + 2u\partial_u, \quad b_i = \mathrm{const}, \quad i = \overline{1,6}. \end{gathered} \tag{61}$$

Theorem 9 [20]. *The equation (37) is Q-conditionally-invariant under the operator*

$$Q = \partial_1 + C(x,u)\partial_u, \tag{62}$$

if $C(x,u)$ satisfies the condition

$$C_0 + u(C_{11} + 2CC_{1u} + C^2C_{uu}) + CC_1 + C^2C_u = 0. \tag{63}$$

Partial solutions or the equation (63) give rise to explicit form of operators of conditional symmetry. Below we adduce some of them

$$Q_3 = \sqrt{x_0}\partial_1 + \sqrt{2u}\partial_u, \tag{64}$$

$$Q_4 = \sqrt{2x_0}\partial_1 + R(u)\partial_u, \tag{65}$$

$$Q_5 = \partial_1 + \ln u\partial_u, \tag{66}$$

$$Q_6 = x_0\partial_1 + x_1\partial_u, \tag{67}$$

where $R(u)$ a solution of ODE $u\ddot{R} + \dot{R} = R^{-1}$.

Let us adduce some ansätzes generated by operators Q_1, Q_2, Q_3

$$x_0u - \frac{1}{2}x_1^2 = \varphi(u), \tag{68}$$

$$\frac{2ux_0}{x_1} - x_1 = \varphi\left(\frac{u}{x_1}\right), \tag{69}$$

$$u = \frac{1}{2}\left(\frac{x_1}{\sqrt{x_0}} + \varphi(x_0)\right)^2. \tag{70}$$

Reduced equations have the form

$$\begin{aligned} &\ddot{\varphi}(u) = 0 && \text{for the ansatz (68)},\\ &\ddot{\varphi}\left(\frac{u}{x_1}\right) = 0 && \text{for the ansatz (69)},\\ &2x_0\dot{\varphi}(x_0) + \varphi(x_0) = 0 && \text{for the ansatz (70)}. \end{aligned}$$

Thus the ansätzes (68)–(70) reduce nonlinear heat equations to linear ODE.

6. An equation of the Korteweg-de Vries type. Let us consider a non-linear equation [23]

$$u_0 + F(u)u_1^k + u_{111} = 0, \tag{71}$$

$u_{111} = \frac{\partial^3 u}{\partial x^3}$, k is an arbitrary real parameter. With $F(u) = u$, $k = 1$ (71) coincides with the classical KdV equation.

Theorem 10 [23]. *The equation (71) is Q-conditionally invariant under the following Galilei-type operator:*

$$Q = x_0^r\partial_1 + H(x,u)\partial_u, \tag{72}$$

r is an arbitrary real parameter, if

$$\begin{aligned} 1)\quad & F(u) = \lambda_1 u^{\frac{2-k}{u}} + \lambda_2 u^{\frac{1-k}{2}}, \\ & H(x,u) = \left(\frac{k\lambda_1}{2}\right)^{-1/k} u^{1/2}, \end{aligned} \tag{73}$$

$$\begin{aligned} 2)\quad & F(u) = (\lambda_1 \ln u)^{1-k}, \\ & H(x,u) = (k\lambda_1)^{-1/k}u, \end{aligned} \tag{74}$$

$$\begin{aligned} 3)\quad & F(u) = (\lambda_1 \arcsin u + \lambda_2)(1-u^2)^{\frac{1-k}{2}}, \\ & H(x,u) = (k\lambda_1)^{-1/k}(1-u^2)^{1/2}, \end{aligned} \tag{75}$$

$$\begin{aligned} 4)\quad & F(u) = (\lambda_1 \operatorname{Arsh} u + \lambda_2)(1-u^2)^{\frac{1-k}{2}}, \\ & H(x,u) = (k\lambda_1)^{-1/k}(1+u^2)^{1/2}, \end{aligned} \tag{76}$$

$$\begin{aligned} 5)\quad & F(u) = \lambda_1 u, \\ & H(x,u) = (k\lambda_1)^{-1/k}, \end{aligned} \tag{77}$$

where $r \neq k^{-1}$, $k \neq 0$, λ_1, λ_2 are arbitrary constants.

By means of operators of conditional invariance (72) we reduce the equation (71) to ODE and construct the following exact solutions:

$$u = \left\{\frac{x_1}{2}\left(\frac{k\lambda_1 x_0}{2}\right)^{-1/k} + \lambda x_0^{-1/k} - \frac{\lambda_2}{\lambda}\right\}^2,$$

when $F(u)$ is of the form (73);

$$u = \exp\left\{-\frac{k(k\lambda_1)^{-3/k}}{k-2}x_0^{-\frac{3}{k}+1} + \lambda x_0^{-1/k} + (k\lambda_1 x_0)^{-1/k}x_1 - \frac{\lambda_2}{\lambda}\right\},$$

when $k \neq 2$, $F(u)$ is of the form (74); when $k = 2$

$$u = \exp\left\{-(2\lambda_1)^{-3/2}x_0^{-1/2}\ln x_0 + \lambda x_0^{-1/2} + (2\lambda_1 x_0)^{-1/2}x_1 - \frac{\lambda_2}{\lambda}\right\};$$

$$u = \sin\left\{\frac{k(k\lambda_1)^{-3/k}}{k-2}x_0^{-\frac{3}{k}+1} + \lambda x_0^{-1/k} + (k\lambda_1 x_0)^{-1/k}x_1 - \frac{\lambda_2}{\lambda}\right\}, \quad k \neq 2,$$

$$u = \sin\left\{(2\lambda_1)^{-3/2}\frac{\ln x_0}{\sqrt{x_0}} + \lambda x_0^{-1/2} + (2\lambda_1 x_0)^{-1/2}x_1 - \frac{\lambda_2}{\lambda}\right\}, \quad k = 2,$$

when $F(u)$ is of the form (75);

$$u = \mathrm{sh}\left\{-\frac{k(k\lambda_1)^{-3/k}}{k-2}x_0^{-\frac{3}{k}+1} + \lambda x_0^{-1/k} + (k\lambda_1 x_0)^{-1/2}x_1\right\}, \quad k \neq 2,$$

$$u = \mathrm{sh}\left\{-(2\lambda_1)^{-3/2}x_0^{-1/2}\ln x_0 + \lambda x_0^{-1/2} + (2\lambda_1 x_0)^{-1/2}x_1\right\}, \quad k = 2,$$

when $F(u)$ is of the form (76). In all formulae λ is an arbitrary parameter.

Thus, having investigated the conditional symmetry of the equation (71), we construct nontrivial classes of exact solutions.

7. Nonlinear wave equation. An equation of the form

$$u_{00} - (F(u)u_1)_1 = 0 \tag{78}$$

is widely used for description of nonlinear wave processes. The group properties of the equation (78) were investigated in detail by means of Lie method in [24]. Depending on explicit form of the function $F(u)$ the equation (78) has wide conditional symmetry.

Theorem 11 [25]. *The equation (78) is Q-conditionally invariant under the operator*

$$Q = A(x,u)\partial_0 + B(x,u)\partial_1 + H(x,u)\partial_u,$$

if functions $A(x,u)$, $B(x,u)$, $H(x,u)$, $F(u)$ satisfy the following systems of equations:

Case 1: $A = 1$, $D = F - B^2$

$$(B_u D^{-1})_u = 0, \quad F(H_1 D^{-1})_1 - (H_0 D^{-1})_0 - H^2 = 0,$$

$$(H_u D^{-1})_u - H(H_0 D^{-1})_u - H(H_u D^{-1})_0 + \\ + D^2\{2F(B_0 D_1 - B_1 H_0 + H[B_u H_1 - B_1 H_u]) - BHH_1F\} = 0,$$

$$D^2 H_{uu} + D\{(H\dot{F})_u + 2B(B_u H_u - B_{uu}H) - 2FB_{1u} - 2B_{0u}\} - \\ - HD^2 + 2BB_0 D_u + 2\dot{B}B_1(BF - 2B_u F) = 0,$$

$$D\{B_{00} + 2(B_0 H)_u - 2(BH_{0u} - B_u H_0) + 2(H_1 F)_u - B_{11}F + B_{uu}H^2 + \\ + 2BHH_{uu}\} - D_u\{B_0 H + B_u H^2 + 2BHH_u\} + \\ + B\{B_1 H\dot{F} + 2B_0^2 + 2B_0 B_u H + 4BB_0 H_u + 4B_1 H_u F - 2B_1^2 F\} = 0.$$

Case 2: $A = 1$, $B = F^{1/2}$

1) $\dot{B}H + 2BH_u = 0, \quad H_0 + HH_u - BH_1 = 0;$

2) $\dot{B}H + 2BH_u \neq 0, \quad H_0 + HH_u - BH_1 = 0;$

$$[\ddot{B}H^2 + 2\dot{B}(BH_1 + HH_u) + 2B(H_{0u} + HH_{uu} + BH_{1u})] = (H_0 + HH_u - \\ - HH_1) - [H_{00} + H^2 H_{uu} - B^2 H_{11} + 2HH_{0u} - 2\dot{B}HH_1](\dot{B}H + 2BH_u) = 0.$$

Case 3: $A = 0$, $B = 1$

$$H_{00} - H^3\dot{F} - (3HH_1 + 2H^2 H_u)\dot{F} - (H_{11} + 2HH_{1u})F = 0.$$

Having solved these systems we constructed explicit forms of operators Q for special forms of the function $F(u)$. Let us adduce some of obtained operators and ansätzes:

$$F(u) = \exp u,$$

$$Q_1 = x_1\partial_1 + \partial_u, \quad u = \ln x_1 + \varphi(x_0),$$

$$Q_2 = \partial_0 + 2\,\mathrm{tg}\,x_0\partial_u, \quad \exp u = \frac{\varphi(x_0)}{\cos^2 x_0};$$

$$F(u) = u^k,$$
$$Q_1 = \partial_0 + \exp\left(\frac{u}{2}\right)\partial_1 - 4x_0^{-1}\partial_u, \quad x_0\exp\left(\frac{u}{2}\right) + x_1 + \varphi\left(x_0^2\exp\frac{u}{2}\right) = 0,$$
$$Q_2 = (k+1)x_1\partial_1 + u\partial_u, \quad u^{k+1} = x_1\varphi^{k+1}(x_0);$$
$$F(u) = u^{-1/2},$$
$$Q_1 = \partial_0 + x_1u^{1/2}\partial_u, \quad 2u^{1/2} = x_0x_1 + \varphi(x_1),$$
$$Q_2 = x_1^2\partial_0 + (4x_0 + a_1x_1^5)u^{1/2}\partial_u, \quad u^{1/2} = x_0^2x_1^{-2} + \frac{a_1^2}{2}x_0x_1^3 + \varphi(x_1),$$

where a_1, a_2, a_3 are constants.

The most simple solutions of the equation (78), constructed by means of the above ansätzes are or the form

$$\exp u = (x_1^2 + a_1)\cos^{-2}x_0, \quad \exp u = x_1\exp x_0,$$

if $F(u) = \exp u$;

$$u^{k+1} = x_0^{k+1}x_1,$$

if $F(u) = u^k$;

$$u = x_0x_1 + \frac{x_0^4}{12} + a_1, \quad u = W(x_0)x_1^2,$$

if $F(u) = u$;

$$u^{1/2} = W(x_1)x_0^2, \quad 2u^{1/2} = x_0x_1 + \frac{x_1^4}{24} + a_1,$$
$$u^{1/2} = x_0^2x_1^{-2} + 3a_1x_0x_1^3 + \frac{a_1}{6}x_1^3 + a_2x_1^{-1} + a_3x_1^2,$$

if $F(u) = u^{1/2}$.

So we had classified and reduced the nonlinear wave equations (78) by means of conditional symmetry.

8. Three-dimensional acoustics equation. Bounded sound beams are described by a nonlinear equation of the form [26]

$$u_{00} - (F(u)u_1)_1 - u_{22} - u_{33} = 0. \tag{79}$$

In the case when $F(u) = u$ it coincides with the Khokhlov–Zabolotskaya equation

$$u_{00} - (uu_1)_1 - u_{22} - u_{33} = 0. \tag{80}$$

Let us add to (79) an additional condition in the form of a first-order nonlinear equation

$$u_0u_1 - F(u)u_1^2 - u_2^2 - u_3^2 - 0. \tag{81}$$

Theorem 12 [26]. *The equation (80) with the condition (81) is invariant under the infinite-dimensional algebra with the operator*

$$X = a_i(u)R_i, \quad i = \overline{1,12}, \tag{82}$$

where $a_i(u)$ are arbitrary smooth functions of the dependent variable u,

$$R_{\mu+1} = \partial_\mu, \quad \mu = \overline{0,3}, \quad R_5 = x_3\partial_2 - x_2\partial_3,$$
$$R_6 = x_2\partial_1 + 2x_0\partial_2, \quad R_7 = x_3\partial_1 + 2x_0\partial_3, \quad R_8 = x^\mu\partial_\mu,$$
$$R_9 = 4x_0\partial_0 + 2x_1\partial_1 + 3x_2\partial_2 + 3x_3\partial_3 - 2\frac{F(u)}{F'(u)}\partial_u, \quad R_{10} = F'(u)x_0\partial_1 - \partial_u,$$
$$R_{11} = x_2\partial_0 + 2(x_1 + F(u)x_0)\partial_2, \quad R_{12} = x_3\partial_0 + 2(x_1 + 2F(u)x_0)\partial_3.$$

Operators $R_1, \ldots, R_8$ are Lie symmetry operators for the equation (80), $R_9, \ldots$, R_{12} are operators of the conditional symmetry for the equation (79). Using conditional symmetry operators of the equation (79) $R_9, \ldots, R_{12}$ it is possible to construct wide classes of exact solutions. For example, the operator $X = \partial_0 + a(u)\partial_1$ generates the following ansätzes:

$$u = \varphi(\omega_1, \omega_2, \omega_3), \quad \omega_1 = a(u)x_0 + x_3, \quad \omega_2 = x_2, \quad \omega_3 = x_3. \tag{83}$$

The ansatz (83) reduces the four-dimensional equation (79), (81) to three-dimansional ones

$$\begin{aligned} &(a(\varphi) - \varphi)\varphi_{11} - \varphi_{22} - \varphi_{33} + \left(\frac{da(\varphi)}{d\varphi} - 1\right)\varphi_1^2 = 0, \\ &(a(\varphi) - \varphi)\varphi_1^2 - \varphi_2^2 - \varphi_3^2 = 0, \quad \varphi_i = \frac{\partial\varphi}{\partial\omega_i}, \quad i = \overline{1,3}. \end{aligned} \tag{84}$$

Taking $a(u)$ in some concrete form it is possible in some cases to construct the general solution of (84). Let $a(u) = u + 1$, then we get a system

$$\varphi_{11} - \varphi_{22} - \varphi_{33} = 0, \tag{85}$$

$$\varphi_1^2 - \varphi_2^2 - \varphi_3^2 = 0. \tag{86}$$

The system (85), (86) can be naturally called the Bateman (1914) — Sobolev–Smirnov (1932–1933) equations, because Bateman, Sobolev and Smirnov investigated this system in detail. The equations (85), (86) has the general solution which is given by Sobolev–Smirnov formula

$$\varphi = c_1(\varphi)\omega_1 + c_2(\varphi)\omega_2 + c_3(\varphi)\omega_3, \tag{87}$$

where c_1, c_2, c_3 are arbitrary functions satisfying the following conditions:

$$c_1^2 - c_2^2 - c_3^2 = 0, \quad c_2^2 + c_3^2 \neq 0.$$

Thus the formula (87) gives the class of exact solutions for the three-dimensional nonlinear equations (85), (86).

9. Conditional symmetry of the Dirac equation. Let us consider the nonlinear Dirac equation

$$\{\gamma_\mu p^\mu - \lambda(\bar\varphi\varphi)\}\varphi(x) = 0 \tag{88}$$

and put on its solutions a condition $\bar\varphi\varphi = 1$. Then (88) becomes a linear equation with a nonlinear additional condition:

$$(\gamma_\mu p^\mu - \lambda)\psi = 0, \quad \bar\psi\psi = 1. \tag{89}$$

The system (89) is conditionally invariant under the operators [9]

$$Q_1 = P_0 - \lambda\gamma_0, \quad Q_2 = P_3 - \lambda\gamma_3. \tag{90}$$

In the case under consideration the equation of the type (6) has the form

$$Q_1\psi = 0 \quad \text{and} \quad Q_2\psi = 0. \tag{91}$$

The operator Q_1 generates the ansatz

$$\psi(x) = \exp(-i\lambda\gamma_0 x_0)\varphi(x_1, x_2, x_3), \tag{92}$$

where $\varphi(x_1, x_2, x_3)$ is a four-component vector-function depending on three variables only.

10. Conditional symmetry of Maxwell's equation. Let us consider a linear system [5]

$$\frac{\partial\vec{E}}{\partial t} = \operatorname{rot}\vec{H}, \quad \frac{\partial\vec{H}}{\partial t} = -\operatorname{rot}\vec{E}. \tag{93}$$

It can be verified directly that the system (93) is not invariant under the Lorentz transformations. However if we add to the system (93) the well-known additional conditions

$$\operatorname{div}\vec{E} = 0, \quad \operatorname{div}\vec{H} = 0, \tag{94}$$

the system (93), (94) becomes a Lorentz-invariant one. The point of view on Maxwell's equations which was set forth [1–9, 21] stressesthe naturality of the notion of conditional invariance and its importance for a wide class of equations of mathematical physics [22].

Conclusion. Investigation of conditional symmetry of partial differential equation has been started recently. The adduced results show that we can anticipate on this way the qualitatively new understanding of symmetry of an equation, of symmetry classification or partial differential equations, of reduction of multi-dimensional nonlinear equations to equations with less number of independent variables, of process of linearization of nonlinear equations.

The principle of relativity, or equivalence of all inertial reference frames, is one of the most fundamental laws of physics, mechanics, hydromechanics, biophysics. Saying in the language of mathematics this principle represents the invariance of an equation of motion whether under the Galilei transformations or under the Lorentz ones. Partial differential equations which do not, satisfy this principle usually are not considered in physical theories. Such equations cannot be used for mathematical description of motion of real physical systems.

The concept of conditional invariance enables to get essentially wider classes of equations satisfying relativity principle. Equations which are non-compatible in usual sense with the relativity principle can satisfy it conditionally, that is, non-trivial conditions on solutions of these equations exist, which pick out subsets of solutions of the initial equation, invariant or under Galilei transformations, or tinder Lorentz ones. Description and detailed investigation of classes of equations conditionally invariant, under Galilei and Poincaré groups and their subgroup seem to the author a rather significant problem of mathematical physics.

Conditional symmetry, for example, of a scalar equation, enables to construct ansätzes which increase the number of dependent variables (antireduction). It allows not only to carry out reduction by number of independent variables but to increase the number of dependent variables. We should like to stress that such ansätzes change essentially the structure of nonlinearity of the initial equation. And, certainly, they cannot be constructed by means of the classical Lie method. The process of linearization, for example, of the nonlinear Navier–Stokes system in our approach is considered as change of a nonlinear equation for a linear system

$$\frac{\partial \vec{u}}{\partial t} + \Delta \vec{u} + \vec{\nabla} p = 0, \quad \operatorname{div} \vec{u} = 0, \tag{95}$$

with a nonlinear additional condition

$$(\vec{u}\,\vec{\nabla})\vec{u} = 0 \quad \text{or} \quad \{(\vec{u}\,\vec{\nabla})\vec{u}\}^2 = 0. \tag{96}$$

The linear Navier–Stokes equation with the nonlinear additional conditions has a nontrivial conditional symmetry. Evidently it is also possible to choose as an additional condition for the Stokes–Stokes equation the following equations:

$$(\vec{u}\,\vec{\nabla})\vec{u} + \vec{\nabla} p = 0.$$

We are going to devote further papers to detailed investigation of conditional linerisation of nonlinear partial differential equations.

In conclusion I adduce the list (which is far from being complete) of nonlinear equations having nontrivial conditional symmetry

$$u_0 + u_{11} = F(u), \quad u_0 + uu_{11} = 0, \qquad (1988,\ 1990)$$

$$Su + F(|u|)u = 0, \quad S = i\frac{\partial}{\partial x_0} - \Delta, \qquad (1990)$$

$$u_{00} = u\Delta u, \qquad (1988)$$

$$\Box u = F(u), \qquad (1989)$$

$$u_{01} - (F(u)u_1)_1 - u_{22} - u_{33} = 0, \qquad (1990)$$

$$u_{00} - (F(u)u_1)_1 = 0, \qquad (1991)$$

$$u_{00} = C(x, u, \underset{1}{u})\Delta u, \qquad (1987)$$

$$u_0 - \vec{\nabla}[F(u)\vec{\nabla} u] = 0, \qquad (1988)$$

$$u_0 + F(u)u_1^k + u_{111} = 0, \qquad (1991)$$

$$u_0 + \frac{\partial^2 \varphi(u)}{\partial x_1^2} + \frac{N}{x_1}\frac{\partial \varphi(u)}{\partial x_1} = F(u), \qquad (1992)$$

$$u_0 + u_{11} + \frac{3}{2x_1}u_1 = \lambda u^3, \qquad (1992)$$

$$u_0 + uu_{11} + \frac{N}{x_1} uu_1 = \lambda_1 u + \lambda_2, \qquad (1992)$$

$$\begin{aligned} &\vec{u}_0 + (\vec{u}\,\vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla}p, \\ &\rho_0 + \mathrm{div}\,(\rho\vec{u}) = 0, \qquad (1992) \\ &p = f(\rho), \quad p = \frac{1}{2}\lambda\rho^2, \end{aligned}$$

$$(1 - u_\alpha u^\alpha)\Box u + u_\mu u_\nu u^{\mu\nu} = 0. \qquad (1989)$$

In the brackets we indicated the years when the conditional symmetry of the corresponding equation had been investigated.

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