# Symmetry analysis. Preface

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Till now there are no general methods of investigation of arbitrary nonlinear partial differential equations (PDE). However if a nonlinear equation is beautiful, that is to say it possesses a non-trivial symmetry, than it is possible to obtain rather wide and rich information about its solutions; to carry out reduction of multidimensional equations to ordinary differential equations [1], to construct classes of exact and approximate solutions, investigate asymptotic of special classes of solutions, etc. [1–6].

It is important to point out that beautiful equations are these ones which are widely used in mathematical and theoretical physics, and in applied mathematics. It is connected with the fact that mathematical models of real processes must be of such a form that e.g. conservation laws of energy, momentum, angular momentum of motion or relativity principles [1, 3] and other important principles of physics are satisfied in these models. The beauty (or symmetry, or approximate symmetry) of an equation has a lot of forms: local (or Lie, as we call it in [2, 6]), nonlocal (non-Lie [2]), discrete, non-group, non-algebraic. Therefore it is not simple to give a mathematically correct, effective and general enough definition of beauty of an equation.

The principal ideas and methods of investigation of group (local) properties of partial differential equations PDE are developed by Sophus Lie. These methods enable to study group properties of an arbitrary partial differential equation. The above methods form a part of modern theory of differential equations called on L.V. Ovsyannikov and N.Kh. Ibragimov suggestion “Group analysis of PDE” [4, 5].

Since PDE prove rather often to possess symmetry that cannot be presented in terms of Lie groups or Lie algebras, we use a more general term “Symmetry analysis” suggested in [2, 6]. Symmetry analysis is the aggregate of mathematical methods for investigating local, geometry, non-geometry, discrete, inner and dynamic symmetries of PDE.

The present collection contains papers by participants of the Seminar “Symmetry analysis of mathematical physics equations” (Institute of Mathematics of the Academy of Sciences of Ukraine) in which two scientific directions are considered:

1. Conditional symmetry of equations of nonlinear mathematical physics.
2. Local, non-local symmetry and construction of classes of exact solutions of nonlinear PDE.

In conclusion, I adduce several beautiful second-order PDE, that have not been investigated yet

$$a(u,\omega,\Box u) + b(u,\omega,u_\alpha u_\beta u^{\alpha\beta}) + c(u,\omega,\det u_{\mu\nu}) = F(u,\omega),$$

$$\omega \equiv u_\alpha u^\alpha = \left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \cdots - \left(\frac{\partial u}{\partial x_n}\right)^2,$$

in Symmetry Analysis of Equations of Mathematical Physics, Kyiv, Institute Mathematics, 1992, P. 5–6.

$$u = u(x_0, x_1, \dots, x_n), \quad u_\alpha u_\beta u^{\alpha\beta} = \frac{\partial u}{\partial x_\alpha}\frac{\partial u}{\partial x_\beta}\frac{\partial^2 u}{\partial x^\alpha \partial x^\beta},$$

$$\det u_{\mu\nu} \equiv \det\left(\frac{\partial^2 u}{\partial x_\mu \partial x_\nu}\right);$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} - v^2\Delta\vec{E} = \vec{0}, \qquad \frac{\partial^2 \vec{H}}{\partial t^2} - v^2\Delta\vec{H} = 0, \quad v = v\left(\vec{E}, \vec{H}, \frac{\partial \vec{E}}{\partial x}, \frac{\partial \vec{H}}{\partial x}\right);$$

$$v_\alpha \frac{\partial v_\mu}{\partial x_\alpha} = 0, \quad v^2 \equiv v_\alpha v^\alpha = v_0^2 - v_1^2 - v_2^2 - v_3^3;$$

$$\left(\frac{\partial}{\partial t} + v_k\frac{\partial}{\partial x_k}\right) v_l = F_l(v_1, v_2, v_3), \quad l = 1, 2, 3;$$

$$\frac{\partial^2 u}{\partial x_\mu \partial x_\nu}\frac{\partial^2 u}{\partial x^\mu \partial x^\nu} = F(u, \omega, \Box u);$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial u}{\partial x_b}\frac{\partial}{\partial x_b}\right)\left(\frac{\partial}{\partial t} + \frac{\partial u}{\partial x_a}\frac{\partial}{\partial x_a}\right) u = F(u).$$

In the above formulae $a$, $b$, $c$, $F$, $F_1$ arbitrary smooth


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