On the new exact solutions of the nonlinear Maxwell–Born–Infeld equations

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Предложены две нелокальные подстановки, сводящие уравнения Максвелла с материальными уравнениями Борца–Инфельда к скалярным уравнениям. С использованием этих подстановок построены многопараметрические семейства точных решений нелинейных уравнений Максвелла–Борна–Инфельда.

In the present work we obtain broad classes of exact solutions of the Maxwell equations

$$\partial_t\vec{D} = \operatorname{rot}\vec{H}, \quad \partial_t\vec{B} = -\operatorname{rot}\vec{E}, \tag{1a}$$

$$\operatorname{div}\vec{D} = 0, \quad \operatorname{div}\vec{B} = 0, \tag{1a}$$

with constitutive equations suggested by Born and Infeld [1] in 1934

$$\begin{gathered}\vec{D} = \tau\vec{E} + \tau_1\vec{B}, \quad \vec{H} = \tau\vec{B} - \tau_1\vec{E}, \\ \tau = \left\{1 + \vec{B}^2 - \vec{E}^2 - (\vec{B}\vec{E})^2\right\}^{-1/2}, \quad \tau_1 = (\vec{B}\vec{E})\tau.\end{gathered} \tag{1c}$$

Here E_a, H_a, B_a, D_a are smooth functions on $t \equiv x_0 \in \mathbb{R}^1$, $\vec{x} \in \mathbb{R}^3$, $a = \overline{1,3}$.

Symmetry properties of equations (1a–c) are investigated in [2] but we do not apply symmetry reduction procedure to construct their exact solutions. Our approach generalizes that of papers [1, 3] and is based on the fact that the general solution of equations (1a,b) can be represented in the form

$$\vec{B} = \operatorname{rot}\vec{U}, \quad \vec{D} = \operatorname{rot}\vec{W}, \quad \vec{H} = \partial_t\vec{W}, \quad \vec{E} = -\partial_t\vec{U}, \tag{2}$$

where $U = (U_1, U_2, U_3)$, $W = (W_1, W_2, W_3)$ are arbitrary smooth vector functions.

Substitution of formulas (2) into (1c) gives rise to the first-order system of partial differential equations (PDE)

$$\begin{gathered}\operatorname{rot}\vec{W} = -\tau\left[\partial_t\vec{U} + ((\partial_t\vec{U})(\operatorname{rot}\vec{U}))\operatorname{rot}\vec{U}\right], \\ \partial_t\vec{W} = \tau\left[\operatorname{rot}\vec{U} + ((\partial_t\vec{U})(\operatorname{rot}\vec{U}))\partial_t\vec{U}\right],\end{gathered} \tag{3}$$

where $\tau = \left\{1 + (\operatorname{rot}\vec{U})^2 - (\partial_t\vec{U})^2 - ((\partial_t\vec{U})(\operatorname{rot}\vec{U}))^2\right\}^{-1/2}$.

To obtain exact solutions of system (3) we use the ansatz

$$\vec{U} = \vec{a}\varphi(\omega_0, \omega_1, \omega_2), \tag{4}$$

where $\omega_0 = t$, $\omega_1 = \vec{b}\vec{x}$, $\omega_2 = \vec{c}\vec{x}$; $\varphi \in C^2(\mathbb{R}^3, \mathbb{R}^1)$; $\vec{a}$, $\vec{b}$, $\vec{c}$ are arbitrary constant vectors in the space $\mathbb{R}^3$ satisfying conditions

$$\vec{a}^{\,2} = \vec{b}^{\,2} = \vec{c}^{\,2} = 1, \quad \vec{a}\vec{b} = \vec{b}\vec{c} = \vec{c}\vec{a} = 0.$$

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Since $\operatorname{rot}\vec{U} = -\vec{c}\frac{\partial\varphi}{\partial\omega_1} + \vec{b}\frac{\partial\varphi}{\partial\omega_2}$, the relation $(\partial_t\vec{U})\operatorname{rot}\vec{U} = 0$ holds. Consequently, system (3) is rewritten in the form

$$\operatorname{rot}\vec{W} = -\tau\vec{a}\frac{\partial\varphi}{\partial\omega_0}, \quad \partial_t\vec{W} = \tau\left(-\vec{c}\frac{\partial\varphi}{\partial\omega_1} + \vec{b}\frac{\partial\varphi}{\partial\omega_2}\right), \tag{5}$$

where $\tau = \left(1 - \varphi_0^2 + \varphi_1^2 + \varphi_2^2\right)^{-1/2}$, $\varphi_\mu \equiv \partial\varphi/\partial\omega_\mu$.

Since $\partial_t\operatorname{rot}\vec{W} = \operatorname{rot}\partial_t\vec{W}$, the equality

$$\partial_t(-\tau\vec{a}\varphi_0) = \operatorname{rot}\left[\tau\left(-\vec{c}\frac{\partial\varphi}{\partial\omega_1} + \vec{b}\frac{\partial\varphi}{\partial\omega_2}\right)\right]$$

holds. The above equation after making some manipulations takes the form

$$\vec{a}(1 - \varphi_\mu\varphi^\mu)^{-3/2}\left[(1 - \varphi_\mu\varphi^\mu)\Box\varphi + \varphi_{\mu\nu}\varphi^\mu\varphi^\nu\right] = \vec{0}.$$

Here $\varphi_{\mu\nu} \equiv \frac{\partial^2\varphi}{\partial\omega_\mu\partial\omega_\nu}$; $\mu,\nu = \overline{0,2}$, the summation over the repeated indices in the Minkowski space $\mathbb{R}(1,2)$ with the metric tensor $g_{\mu\nu} = \operatorname{diag}(1,-1,-1)$ is implied.

Thus provided the function $\varphi(\omega)$ is a solution of the nonlinear PDE

$$(1 - \varphi_\mu\varphi^\mu)\Box\varphi + \varphi_{\mu\nu}\varphi^\mu\varphi^\nu = 0, \tag{6}$$

and what is more $1 - \varphi_\mu\varphi^\mu \neq 0$, formulas (2), (4), (5) give particular solutions of system of nonlinear PDE (1).

We look for a solution of equation (6) in the form

$$\varphi = \Phi(y_1, y_2), \tag{7}$$

where $y_1 = \omega_0 + \omega_1$, $y_2 = \omega_2$.

Substitution of (7) into (6) gives rise to the following equation for Φ:

$$\frac{\partial^2\Phi}{\partial y_2^2} = 0, \tag{8}$$

whence $\Phi = h_1(y_1)y_2 + h_2(y_1)$, $h_i \subset C^2(\mathbb{R}^1,\mathbb{R}^1)$ being arbitrary functions. Substituting the obtained expressions into (7) we get the class of exact solutions of nonlinear PDE (6) containing two arbitrary functions on $\omega_0 + \omega_1 \equiv t + \vec{b}\vec{x}$

$$\varphi = \omega_2 h_1(\omega_0 + \omega_1) + h_2(\omega_0 + \omega_1).$$

Hence by using formulas (2), (4), (5) we obtain a family of exact solutions of the nonlinear Maxwell–Born–Infeld equations

$$\begin{aligned}
&\vec{E} = -\vec{a}(\dot{h}_1\vec{c}\vec{x} + \dot{h}_2), \quad \vec{H} = (1 + h_1^2)^{-1/2}[\vec{b}h_1 - \vec{c}(\dot{h}_1\vec{c}\vec{x} + \dot{h}_2)],\\
&\vec{D} = -\vec{a}(1 + h_1^2)^{-1/2}[\dot{h}_1\vec{c}\vec{x} + \dot{h}_2], \quad \vec{B} = \vec{b}h_1 - \vec{c}(\dot{h}_1\vec{c}\vec{x} + \dot{h}_2).
\end{aligned} \tag{9}$$

Similarly, using exact solutions of equation (6) constructed in [4, 5], that satisfy the condition $1 - \varphi_\mu\varphi^\mu \neq 0$ we shall write down corresponding particular solutions of system (1) (as earlier, we shall use the designations $\omega_0 = t$, $\omega_1 = \vec{b}\vec{x}$, $\omega_2 = \vec{c}\vec{x}$, $\omega_2 = \omega_0^2 - \omega_1^2 - \omega_2^2$).

$$\text{1.}\quad \varphi(\omega_0,\omega_1,\omega_2) = c_1\int_0^{\sqrt{\omega^2}} \frac{dy}{\sqrt{1+c_2y^4}}, \quad \tau = \left(1 + \frac{c_1^2}{1+c_2\omega^4 - c_1^2}\right)^{1/2},$$

$$\vec{E} = -\frac{c_1 t\vec{a}}{\sqrt{\omega^2(1+c_2\omega^4)}}, \quad \vec{B} = \frac{c_1[-\vec{b}(\vec{c}\,\vec{x}) + \vec{c}(\vec{b}\,\vec{x})]}{\sqrt{\omega^2(1+c_2\omega^4)}}, \tag{10}$$

$$\vec{D} = -\frac{c_1 t\vec{a}}{\sqrt{\omega^2(1+c_2\omega^4 - c_1^2)}}, \quad \vec{H} = \frac{c_1[-\vec{b}(\vec{c}\,\vec{x}) + \vec{c}(\vec{b}\,\vec{x})]}{\sqrt{\omega^2(1+c_2\omega^4 - c_1^2)}}.$$

$$\text{2.}\quad \varphi(\omega_0,\omega_1,\omega_2) = \pm\left[\frac{\omega_0 - \omega_1}{c_1}\,\mathrm{th}(c_1(\omega_0+\omega_1)+c_2)\right]^{1/2} + c_3,$$

$$\tau = \mathrm{ch}(c_1(\omega_0+\omega_1)+c_2),$$

$$\vec{E} = \mp\frac{\vec{a}}{4}\left[\frac{\mathrm{cth}(c_1(t+\vec{b}\,\vec{x})+c_2)}{c_1(t-\vec{b}\,\vec{x})}\right]^{1/2} \times$$
$$\times\left[\frac{2c_1(t-\vec{b}\,\vec{x}) + \mathrm{sh}\,2(c_1(t+\vec{b}\,\vec{x})+c_2)}{\mathrm{ch}^2(c_1(t+\vec{c}\vec{x})+c_2)}\right],$$

$$\vec{B} = \mp\frac{\vec{c}}{4}\left[\frac{\mathrm{cth}(c_1(t+\vec{b}\,\vec{x})+c_2)}{c_1(t-\vec{b}\,\vec{x})}\right]^{1/2} \times \tag{11}$$
$$\times\left[\frac{2c_1(t-\vec{b}\,\vec{x}) - \mathrm{sh}\,2(c_1(t+\vec{b}\,\vec{x})+c_2)}{\mathrm{ch}^2(c_1(t+\vec{b}\vec{x})+c_2)}\right],$$

$$\vec{D} = \mp\frac{\vec{a}}{4}\left[\frac{2\{2c_1(t-\vec{b}\,\vec{x}) + \mathrm{sh}\,2(c_1(t+\vec{b}\,\vec{x})+c_2)\}^2}{c_1(t-\vec{b}\,\vec{x})\,\mathrm{sh}\,2(c_1(t+\vec{b}\,\vec{x})+c_2)}\right]^{1/2},$$

$$\vec{H} = \mp\frac{\vec{c}}{4}\left[\frac{2\{2c_1(t-\vec{b}\,\vec{x}) - \mathrm{sh}\,2(c_1(t+\vec{b}\,\vec{x})+c_2)\}^2}{c_1(t-\vec{b}\,\vec{x})\,\mathrm{sh}\,2(c_1(t+\vec{b}\,\vec{x})+c_2)}\right]^{1/2},$$

$$\text{3.}\quad \varphi(\omega_0,\omega_1,\omega_2) = \pm\left[c_2e^{c_3(\omega_0-\omega_1)} + \frac{2}{c_3}(\omega_0+\omega_1)\right]^{1/2} + c_1,$$

$$\tau = \left[\frac{2(\omega_0+\omega_1) + c_2c_1e^{c_3(\omega_0-\omega_1)}}{2(\omega_0+\omega_1) - c_2c_1e^{c_3(\omega_0-\omega_1)}}\right]^{1/2},$$

$$\vec{E} = \mp\frac{\vec{a}}{2}\frac{\left(\frac{2}{c_3} + c_2c_3e^{c_3(t-\vec{b}\,\vec{x})}\right)}{\left[c_2e^{c_3(t-\vec{c}\,\vec{x})} + \frac{2}{c_3}(t+\vec{b}\,\vec{x})\right]^{1/2}}, \tag{12}$$

$$\vec{B} = \mp\frac{\vec{c}}{2}\frac{\left(\frac{2}{c_3} - c_2c_3e^{c_3(t-\vec{b}\,\vec{x})}\right)}{\left[c_2e^{c_3(t-\vec{b}\,\vec{x})} + \frac{2}{c_3}(t+\vec{b}\,\vec{x})\right]^{1/2}},$$

$$\vec{H} = \mp\frac{\vec{a}}{2}\frac{\left(\frac{2}{c_3} - c_2c_3e^{c_3(t-\vec{b}\,\vec{x})}\right)}{\left[-c_2e^{c_3(t-\vec{b}\,\vec{x})} + \frac{2}{c_3}(t+\vec{b}\,\vec{x})\right]^{1/2}},$$

$$\vec{D} = \mp\frac{\vec{a}}{2}\frac{\left(\frac{2}{c_3} + c_2c_3e^{c_3(t-\vec{b}\,\vec{x})}\right)}{\left[-c_2e^{c_2(t-\vec{b}\,\vec{x})} + \frac{2}{c_3}(t+\vec{b}\,\vec{x})\right]^{1/2}}.$$

By direct check one can become convinced of the fact that solutions (9)–(12) are such that the vectors $\vec{E}$ and $\vec{D}$ as well as $\vec{B}$ and $\vec{H}$ are parallel. Besides, the conditions

$$\vec{E}\vec{B} = \vec{D}\vec{B} = \vec{E}\vec{H} = \vec{D}\vec{H} = 0$$

hold.

Some other classes of exact solutions of system (3) are obtained by putting in (3)

$$\operatorname{rot}\vec{U} = \vec{0}, \quad \vec{U}_{tt} = \vec{0}. \tag{13}$$

By force of (13) equation (3) reads

$$\operatorname{rot}\vec{W} = -\frac{\vec{\nabla}\varphi}{\sqrt{1-(\vec{\nabla}\varphi)^2}}, \quad \vec{U} = \vec{\nabla}(t\varphi(\vec{x}) + V(\vec{x})), \tag{14}$$

where $\varphi(\vec{x}), V(\vec{x}) \in C^2(\mathbb{R}^3, \mathbb{R}^1)$ are arbitrary functions.

From the integrability condition of system (14): $\vec{\nabla}\,[\vec{\nabla}\times\vec{W}] = 0$ it follows that

$$-\operatorname{div}\left(\frac{\vec{\nabla}\varphi}{\sqrt{1-(\vec{\nabla}\varphi)^2}}\right) = 0,$$

whence

$$\frac{(1-\vec{\nabla}\varphi)^2)\Delta\varphi + \varphi_{x_a}\varphi_{x_b}\varphi_{x_ax_b}}{[1-\vec{\nabla}\varphi)^2]^{3/2}} = 0.$$

The above equation, provided $(\vec{\nabla}\varphi)^2 \neq 1$, takes the form

$$[1-(\vec{\nabla}\varphi)^2]\Delta\varphi + \varphi_{x_a}\varphi_{x_b}\varphi_{x_ax_b} = 0. \tag{15}$$

In [4, 5] the following classes of exact solutions of equation (15)

$$\varphi(\vec{x}) = c_1\ln\left[\sqrt{(\vec{a}\,\vec{x}+c_2)^2 + (\vec{b}\,\vec{x}+c_3)^2} + \right.$$
$$\left. + \sqrt{(\vec{a}\,\vec{x}+c_2)^2 + (\vec{b}\,\vec{x}+c_3)^2 + c_1^2}\right] + c, \tag{16}$$

$$\varphi(\vec{x}) = \int_{\omega_0}^{\sqrt{\vec{x}^2+c_2\vec{a}\,\vec{x}+c_2^2}}(1+c_1\tau^4)^{-1/2}d\tau$$

were constructed. Omitting intermidiate computations we write down the exact solutions of the Maxwell–Born–Infeld equations (1) obtained by substituting (16) into (2), (14)

$$\vec{B} = \vec{0}, \quad \vec{H} = \vec{0}, \quad \vec{D} = -\frac{c_1[\vec{a}(\vec{a}\,\vec{x} + c_2) + \vec{b}(\vec{b}\,\vec{x} + c_3)]}{(\vec{a}\,\vec{x} + c_2)^2 + (\vec{b}\,\vec{x} + c_3)^2},$$

$$\vec{E} = -\frac{c_1[\vec{a}(\vec{a}\,\vec{x} + c_2) + \vec{b}(\vec{b}\,\vec{x} + c_3)]}{\sqrt{(\vec{a}\,\vec{x} + c_2)^2 + (\vec{b}\,\vec{x} + c_3)^2}}[(\vec{a}\,\vec{x} + c_2)^2 + (\vec{b}\,\vec{x} + c_3)^2 + c_1^2]^{-1/2},$$

$$\vec{B} = \vec{0}, \quad \vec{H} = \vec{0}, \quad \vec{D} = -\frac{\vec{a}(\vec{a}\,\vec{x} + c_2/2) + \vec{b}(\vec{b}\,\vec{x}) + \vec{c}(\vec{c}\,\vec{x})}{\sqrt{c_1(\vec{x}^{\,2} + c_2\vec{a}\,\vec{x} + c_2^2)^2(\vec{x}^{\,2} + c_2\vec{a}\,\vec{x} + c_2^2) + \frac{3}{4}c_2^2}},$$

$$\vec{E} = -\frac{\vec{a}(\vec{a}\,\vec{x} + c_2/2) + \vec{b}(\vec{b}\,\vec{x}) + \vec{c}(\vec{c}\,\vec{x})}{\sqrt{[1 + c_1(\vec{x}^{\,2} + c_2\vec{a}\,\vec{x} + c_2^2)^2](\vec{x}^{\,2} + c_2\vec{a}\,\vec{x} + c_2^2)}}.$$

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