On the general solution of the d'Alembert equation with nonlinear eikonal constraint

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We construct general solutions of the system of nonlinear differential equations $\Box u = 0$, $u_\mu u^\mu = 0$ in the four- and five-dimensional complex pseudo-Euclidian spaces. The obtained results are used to reduce multi-dimensional nonlinear wave equation to ordinary differential equations.

1. Introduction. In the present paper we construct general solution of the multi-dimensional system of partial differential equations

$$\begin{gathered}\Box_n u \equiv 0,\\ u_\mu u^\mu \equiv u_{x_1}^2 - u_{x_2}^2 - \cdots - u_{x_{n-1}}^2 = 0\end{gathered} \tag{1}$$

in the four- and five-dimensional pseudo-Euclidian space. In (1) $u = u(x_0, x_1, \ldots, x_{n-1}) \in C^2(\mathbb{C}^n, \mathbb{C}^1)$. Hereafter the summation over the repeated indices in the pseudo-Euclidian space $M(1,n)$ with the metric tensor $g_{\mu\nu} = \operatorname{diag}(1, -1, \ldots, -1)$ is understood.

We suggest a new algorithm of construction of exact solutions of the nonlinear d'Alembert equation

$$\Box_4 u = \lambda u^k, \quad \lambda, k \in \mathbb{R}^1 \tag{2}$$

via solutions of the system of PDE (1).

2. Integration of the system (1): The list of principal results. Below we adduce assertions giving general solutions of the system of PDE (1) with arbitrary $n \in \mathbb{N}$ provided $u(x) \in C^2(\mathbb{R}^n, \mathbb{R}^1)$, and with $n = 4, 5$, provided $u(x) \in C^2(\mathbb{C}^n, \mathbb{C}^1)$.

Theorem 1. *Let $u(x)$ be sufficiently smooth real function on n real variables $x_0, \ldots, x_{n-1}$. Then the general solution of the system of nonlinear PDE (1) is given by the following formula:*

$$C_\mu(u) x_\mu + C_n(u) = 0, \tag{3}$$

where $C_\mu(u)$, $C_n(u)$ are arbitrary real functions that satisfy the condition

$$C_\mu(u) C_\mu(u) = 0 \tag{4}$$

(the condition (4) means that n-vector $(C_0, C_1, \ldots, C_{n-1})$ is an isotropic one).

Note 1. As far as we know Jacobi, Smirnov and Sobolev were the first to obtain the formulae (3), (4) when $n = 3$ [1, 2]. That is why it is natural to call (3), (4) the Jacoby–Smirnov–Sobolev formulae (JSSF). Later on, in 1944 Yerugin generalized JSSF up to the case $n = 4$ [3]. Recently, Collins [4] proved that JSSF give the general solution of system (1) under arbitrary $n \in \mathbb{N}$. He applied rather complicated

in Symmetry Analysis of Equations of Mathematical Physics, Kyiv, Institute Mathematics, 1992, P. 68–90.

differential geometry technique. Below we show that to integrate Eqs. (1) it is quite enough to apply only classical methods of mathematical physics.

Theorem 2. *The general solution of the system of nonlinear PDE (1) in the class of functions $u = u(x_0, x_1, x_2, x_3) \in C^2(\mathbb{C}^4, \mathbb{C}^1)$ is given by the following formula:*

$$F(A_\mu(u)x_\mu, B_\nu(u)x_\nu, u) = 0, \tag{5}$$

where $F \in C^2(\mathbb{C}^3, \mathbb{C}^1)$ is an arbitrary function, $A_\mu, B_\mu \in C^2(\mathbb{C}^1, \mathbb{C}^1)$ are arbitrary smooth functions satisfying the conditions

$$A_\mu A_\mu = A_\mu B_\mu = B_\mu B_\mu = 0. \tag{6}$$

Theorem 3. *The general solution of the system of nonlinear PDE (1) in the class of functions $u = u(x_0, x_1, x_2, x_3, x_4) \in C^2(\mathbb{C}^5, \mathbb{C}^1)$ is given by one of the following formulae:*

$$1)\ A_\mu(\tau, u)x_\mu + C_1(\tau, u) = 0, \tag{7}$$

where $\tau = \tau(u, x)$ is a complex function determined by the equation

$$B_\mu(\tau, u)x_\mu + C_2(\tau, u) = 0, \tag{8}$$

and $A_\mu, B_\mu, C_1, C_2 \in C^2(\mathbb{C}^2, \mathbb{C}^1)$ are arbitrary functions satisfying the conditions

$$A_\mu A_\mu = A_\mu B_\mu = B_\mu B_\mu = 0, \quad B_\mu \frac{\partial A_\mu}{\partial \tau} = A_\mu \frac{\partial B_\mu}{\partial \tau} = 0 \tag{9}$$

and what is more

$$\Delta = \det \begin{Vmatrix} x_\mu \dfrac{\partial A_\mu}{\partial \tau} + \dfrac{\partial C_1}{\partial \tau} & x_\mu \dfrac{\partial A_\mu}{\partial \tau} + \dfrac{\partial C_1}{\partial \tau} \\ x_\mu \dfrac{\partial B_\mu}{\partial \tau} + \dfrac{\partial C_2}{\partial \tau} & x_\mu \dfrac{\partial B_\mu}{\partial \tau} + \dfrac{\partial C_2}{\partial \tau} \end{Vmatrix} \neq 0. \tag{10}$$

$$2)\ A_\mu(x)x_\mu + C_1(u) = 0, \tag{11}$$

where $A_\mu(u)$, $C_1(u)$ are arbitrary smooth functions satisfying relations

$$A_\mu A_\mu = 0 \tag{12}$$

(in the formulae (7)–(12) the index μ takes the values $0, 1, 2, 3, 4$).

Note 2. In 1915 Bateman [5] investigating particular solutions of the Maxwell equations came to the problem of integrating the d'Alembert equation $\Box_4 u = 0$ with additional nonlinear condition (the eikonal equation) $u_{x_\mu} u_{x_\mu} = 0$. He obtained the following class of exact solutions of the above system:

$$u(x) = C_\mu(\tau)x_\mu + C_4(\tau), \tag{13}$$

where $\tau = \tau(x)$ is a smooth function determined from the equation

$$\dot{C}_\mu(\tau)x_\mu + \dot{C}_4(\tau) = 0, \tag{14}$$

$c_\mu(\tau)$, $C_4(\tau)$ are arbitrary smooth functions satisfying the conditions

$$C_\mu C_\mu = \dot{C}_\mu \dot{C}_\mu = 0. \tag{15}$$

It is not difficult to show that the solutions (13)–(15) are complex (see Lemma 1 below). Another class of complex solutions of the system (1) with $n=4$ was constructed by Yerugin [3]. But neither Bateman formulae (13)–(15) not Yerugin's results give the general solution of the system (1) with $n=4$.

3. Proof of Theorems 1–3. It is well-known that the system of PDE (1) admits an infinite-dimensional Lie algebra [6]. It is this very fact that enables us to construct its general solution.

Proof of the Theorem 1. Let us make in (1) the hodograph transformation

$$z_0 = u(x), \quad z_a = x_a, \quad a=\overline{1,n-1}, \quad w(z) = x_0. \tag{16}$$

Evidently, the transformation (16) is defined for all functions $u(x)$, such that $u_{x_0} \not\equiv 0$.
But the system (1) with $u_{x_0} =$ takes the form

$$\sum_{a=1}^{n-1} u_{x_a x_a} = 0, \quad \sum_{a=1}^{n-1} u_{x_a}^2 = 0,$$

whence $u_{x_a} \equiv 0$, $a=\overline{1,n-1}$ or $u(x) = \text{const}$.

Consequently, the change of variables (16) is defined on the whole set of solutions of the system with the only exception $u(x) = \text{const}$.

Being rewritten in the new variables z, $w(z)$ the system (1) takes the form

$$\sum_{a=1}^{n-1} w_{z_a z_a} = 0, \quad \sum_{a=1}^{n-1} w_{z_a}^2 = 1. \tag{17}$$

Differentiating the second equation with respect to z_b, z_c we get

$$\sum_{a=1}^{n-1} (w_{z_a z_b z_c} w_{z_a} + w_{z_a z_b} w_{z_a z_c}) = 0.$$

Choosing in the above equality $c=b$ and summing we have

$$\sum_{a,b=1}^{n-1} (w_{z_a z_b z_b} w_{z_a} + w_{z_a z_b} w_{z_a z_b}) = 0,$$

whence, by force of (17),

$$\sum_{a,b=1}^{n-1} w_{z_a z_b}^2 = 0. \tag{18}$$

Since $w(z)$ is a real valued function from (18) it follows that $w_{z_a z_b} = 0$, $a,b = \overline{1,n-1}$, whence

$$w(z) = \sum_{a=1}^{n-1} \alpha_a(z_0) z_a + \alpha(z_0). \tag{19}$$

In (19) $\alpha_a, \alpha \in C^2(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions.

Substituting (19) into the second equation of system (17), we have

$$\sum_{a=1}^{n-1} \alpha_a^2(z_0) = 1. \tag{20}$$

Thus, the formulae (19), (20) give the general solution of the system of nonlinear PDE (17). Rewriting (19), (20) in the initial variables, we get

$$x_0 = \sum_{a=1}^{n-1} \alpha_a(u)x_a + \alpha(u), \quad \sum_{a=1}^{n-1} \alpha_a^2(u) = 1. \tag{21}$$

To represent the formula (21) in the manifestly covariant form (3) we redefine the functions $\alpha_a(u)$ in the following way:

$$\alpha_a(u) = \frac{A_a(u)}{A_0(u)}, \quad \alpha(u) = -\frac{B(u)}{A_0(u)}, \quad a = \overline{1, n-1}.$$

Substituting the above expressions into (21) we come to the formulae (7).

Further, since $u = \mathrm{const}$ is contained in the class of functions $u(x)$ determined by the formulae (7) under $A_\mu \equiv 0$, $\mu = \overline{0, n-1}$, $B(u) = u + \mathrm{const}$, JSSF (7) give the general solution of the system of the PDE (1) with an arbitrary $n \in \mathbb{N}$. The theorem is proved.

Let us emphasize that the above used arguments can be applied only to the case of real-valued function $u(x)$. It a solution of the system (1) is looked for in the class of complex-valued functions $u(x)$. JSSF (7) do not give its general solution with $n > 3$. Each case $n = 4, 5, \ldots$ requires a special consideration.

Further we shall adduce the proof of Theorem 3 (Theorem 2 is proved in the same way).

Case 1. $u_{x_0} \neq 0$. In this case the hodograph transformation (16) reducing the system (1) with $n = 5$ to the form

$$\sum_{a=1}^{4} w_{z_a z_a} = 0, \quad \sum_{a=1}^{4} w_{z_a}^2 = 1, \quad w_{z_0} \neq 0 \tag{22}$$

is defined.

The general solution of nonlinear complex Eqs. (22) was constructed by the authors in [7]. It is given by the following formulae:

$$1)\ w(z) = \sum_{a=1}^{4} \alpha_a(\tau, z_0) z_a + \gamma_1(\tau, z_0), \tag{23}$$

where $\tau = \tau(z_0, \ldots, z_4)$ is the function determined from the equation

$$\sum_{a=1}^{4} \beta_a(\tau, z_0) z_a + \gamma_2(\tau, z_0) = 0 \tag{24}$$

and $\alpha_a, \beta_a, \gamma_1, \gamma_2 \in C^2(\mathbb{C}^2, \mathbb{C}^1)$ are arbitrary functions satisfying the relations

$$\sum_{a=1}^{4} \alpha_a^2 = 1, \quad \sum_{a=1}^{4} \alpha_a \beta_a = \sum_{a=1}^{4} \beta_a^2 = 0, \quad \sum_{a=1}^{4} \alpha_a \frac{\partial \beta_a}{\partial \tau} = 0. \tag{25}$$

$$2)\ w(z) = \sum_{a=1}^{4} \alpha_a(z_0) z_a + \gamma_1(z_0), \tag{26}$$

where $\alpha_a, \gamma_1 \in C^2(\mathbb{C}^2, \mathbb{C}^1)$ are arbitrary functions satisfying the relation

$$\sum_{a=1}^{4} \alpha_a^2 = 1. \tag{27}$$

Rewriting the formulae (24), (25) in the initial variables x, $u(x)$, we have

$$x_0 = \sum_{a=1}^{4} \alpha_a(\tau, u) x_a + \gamma_1(\tau, u), \tag{28}$$

where $\tau = \tau(u, x)$ is a function determined from the equation

$$\sum_{a=1}^{4} \beta_a(\tau, u) x_a + \gamma_2(\tau, u) = 0 \tag{29}$$

and the relations (25) hold.

Evidently, the formulae (7) under

$$\begin{aligned} &A_0 = 1, \quad A_a = \alpha_a, \quad C_1 = -\gamma_1, \\ &B_0 = 0, \quad B_a = \beta_a, \quad C_2 = -\gamma_1, \quad a = \overline{1,4}. \end{aligned} \tag{30}$$

Further, by force of inequality $w_{z_a} \not\equiv 0$ we get from (23)

$$\sum_{a=1}^{4} (\alpha_{az_0} + \alpha_{a\tau} \tau_{z_0}) x_a + \gamma_{1z_0} + \gamma_{1\tau} \tau_{z_0} \neq 0. \tag{31}$$

Differentiation of (24) with respect to z_0 yields the following expression for τ_{z_0}:

$$\tau_{z_0} = -\left(\sum_{a=1}^{4} \beta_{az_0} x_a + \gamma_{2z_0} \right) \left(\sum_{a=1}^{4} \beta_{a\tau} x_a + \gamma_{2\tau} \right)^{-1}.$$

Substitution of the above result, into (31) yields relation of the form

$$\left(\sum_{a=1}^{4} \beta_{a\tau} x_a + \gamma_{2\tau} \right)^{-1} \begin{vmatrix} \displaystyle\sum_{a=1}^{4} \alpha_{az_0} x_a + \gamma_{1z_0} & \displaystyle\sum_{a=1}^{4} \alpha_{a\tau} x_a + \gamma_{1\tau} \\ \displaystyle\sum_{a=1}^{4} \beta_{az_0} x_a + \gamma_{2z_0} & \displaystyle\sum_{a=1}^{4} \beta_{a\tau} x_a + \gamma_{2\tau} \end{vmatrix} \neq 0.$$

As the direct, check shows the above inequality follows from (10) with the conditions (30).

Now we turn to solutions of the system (22) of the form (26). Rewriting the formulae (26), (27) in the initial variables x, $u(x)$ we get

$$x_0 = \sum_{a=1}^{4} \alpha_a(u) x_a + \gamma_1(u), \quad \sum_{a=1}^{4} \alpha_a^2(u) = 1.$$

After making in the obtained equalities the chance $\alpha_a = A_a A_0^{-1}$, $a = \overline{1,4}$, $\gamma_1 = -C_1 A_0^{-1}$, we arrive at the formulae (11), (12).

Thus, under $u_{x_0} \not\equiv 0$ the general solution of the system (1) is contained in the class of functions $u(x)$ given by the formulae (7)–(10) or (11), (12).

Case 2. $u_{x_0} \equiv 0$, $u \not\equiv \text{const}$. It is well-known that the system of PDE (1) is invariant under the generalized Poincaré group $P(1, n-1)$ (see, e.g. [8])

$$x'_\mu = \Lambda_{\mu\nu} x_\nu + \Lambda_\mu, \quad u'(x') = u(x),$$

where $\Lambda_{\mu\nu}$, Λ_μ are arbitrary complex parameters satisfying the relations $\Lambda_{\mu\alpha}\Lambda_{\alpha\nu} = g_{\mu\nu}$, $\mu, \nu = \overline{0, n-1}$. Hence, it follows that, the transformation

$$u(x) + u(x') = u(\Lambda_{\mu\nu} x_\nu) \tag{32}$$

leaves the set of solutions of the system (1) invariant. So when $u(x) \not\equiv \text{const}$ we can obtain $u_{x_0} \not\equiv 0$ by using the transformation (32). Consequently, in the case 2 the general solution is also given by the formulae (7)–(12) up to the transformation (32).

Case 3. $u = \text{const}$. Choosing in (11), (12) $A_\mu = 0$, $\mu = \overline{0,4}$, $C_1 = u + \text{const}$ we come to the condition that this solution is described by the formulae (7)–(12).

Thus, we have proved that, up to transformations from the group $P(1,4)$ (32), the general solution of the system of PDE (1) with $n = 5$ is given by the formulae (7)–(12). But these formulae are not changed with the transformation (32). So to complete the proof of the theorem it is enough to demonstrate that each function $u = u(x)$, determined by the equalities (7)–(12), is a solution of the system of equations (1).

Differentiating the relations (7), (8) with respect to x_μ, we have

$$\begin{aligned} &A^\mu + \tau_{x_\mu}(A_{\nu\tau} x_\nu + C_{1\tau}) + u_{x_\mu}(A_{\nu u} x_\nu + C_{1u}) = 0, \\ &B^\mu + \tau_{x_\mu}(B_{\nu\tau} x_\nu + C_{2\tau}) + u_{x_\mu}(B_{\nu u} x_\nu + C_{2u}) = 0. \end{aligned}$$

Resolving the above system of linear algebraic equations with respect to u_{x_μ}, τ_{x_μ}, we get

$$\begin{aligned} u_{x_\mu} &= \frac{1}{\Delta}(B_\mu(A_{\nu\tau} x_\nu + C_{1\tau}) - A_\mu(B_{\nu\tau} x_\nu + C_{2\tau})), \\ \tau_{x_\mu} &= \frac{1}{\Delta}(A_\mu(B_{\nu u} x_\nu + C_{1u}) - B_\mu(A_{\nu u} x_\nu + C_{2u})), \end{aligned} \tag{33}$$

where $\Delta \neq 0$ by force of (10). Consequently,

$$\begin{aligned} u_{x_\mu} u_{x_\mu} = \Delta^{-2}\Big[&B_\mu B_\mu (A_{\nu\tau} x_\nu + C_{1\tau})^2 - \\ &- 2A_\mu B_\mu (A_{\nu\tau} x_\nu + C_{1\tau})(B_{\nu\tau} x_\nu + C_{2\tau}) + A_\mu A_\mu (B_{\nu\tau} x_\nu + C_{2\tau})^2\Big] = 0. \end{aligned}$$

Analogously, differentiating (33) with respect to x_ν and convoluting the obtained expression with the metric tensor $g_{\mu\nu}$, we get

$$g^{\mu\nu} u_{x_\mu x_\nu} = \Box_5 u = 0.$$

Further, differentiating (11) with respect to x_μ, we have

$$u_{x_\mu} = -A_\mu (\dot{A}_\nu x_\nu + \dot{C}_1)^{-1}, \quad \mu = \overline{0,4},$$

whence

$$u_{x_\mu x_\nu} = -(\dot{A}_\mu A_\nu + \dot{A}_\nu A_\mu)(\dot{A}_\nu x_\nu + \dot{C}_1)^{-2} + A_\mu A_\nu(\ddot{A}_\nu x_\nu + \ddot{C}_1)(\dot{A}_\nu x_\nu + \dot{C}_1)^{-2}.$$

Consequently,

$$\begin{aligned} &u_{x_\mu} u_{x_\mu} = A_\mu A_\mu(\dot{A}_\nu x_\nu + \dot{C}_1)^{-2} = 0, \\ &\Box_5 u \equiv u_{x_\mu x_\mu} = -(\dot{A}_\mu \dot{A}_\mu)(\dot{A}_\nu x_\nu + \dot{C}_1)^{-2} + \\ &\qquad + A_\mu A_\mu(\ddot{A}_\nu x_\nu + \ddot{C}_1)(\dot{A}_\nu x_\nu + \dot{C}_1)^{-2} = 0. \end{aligned}$$

Theorem 3 is proved.

4. Applications: reduction of the nonlinear wave equation (2). Following [7, 8] we look for a solution of the nonlinear wave equation

$$\Box_4 w = F(w), \quad F \in C^1(\mathbb{R}^1, \mathbb{R}^1) \tag{34}$$

in the form

$$w = \varphi(w_1, w_2), \tag{35}$$

where $w_i = w_i(x) \in C^2(\mathbb{R}^4, \mathbb{R}^1)$ are functionally-independent. The functions $w_1(x)$, $w_2(x)$ are determined by the demand that the substitution of (35) into (34) yields two-dimensional PDE for a function $\varphi(w_1, w_2)$. As a result we obtain an over-determined system of PDE [8]

$$\begin{aligned} &\Box_4 w_1 = f_1(w_1, w_2), \quad \Box_4 w_2 = f_2(w_1, w_2), \\ &w_{1x_\mu} w_{1x_\mu} = g_1(w_1, w_2), \quad w_{2x_\mu} w_{2x_\mu} = g_2(w_1, w_2), \\ &w_{1x_\mu} w_{2x_\mu} = g_3(w_1, w_2), \quad \operatorname{rank} \|\partial w_i/\partial x_\mu\|_{i=1}^{2}{}_{\mu=0}^{3} = 2 \end{aligned} \tag{36}$$

and besides the function $\varphi(w_1, w_2)$ satisfies the two-dimensional PDE

$$g_1 \varphi_{w_1 w_1} + g_2 \varphi_{w_2 w_2} + 2 g_3 \varphi_{w_1 w_2} + f_1 \varphi_{w_1} + f_2 \varphi_{w_2} = F(\varphi). \tag{37}$$

Let us consider the following problem: to describe all smooth real functions $w_1(x)$, $w_2(x)$ such that the ansatz (35) reduces Eq. (34) to ordinary differential equation (ODE) with respect to the variable w_1. It means that one has to put coefficients g_2, g_3, f_2 in (37) equal to zero. In other words, it is necessary to construct the general solution of the system of nonlinear PDE

$$\begin{aligned} &\Box_4 w_1 = f_1(w_1, w_2), \quad w_{1x_\mu} w_{1x_\mu} = g_1(w_1, w_2), \\ &w_{1x_\mu} w_{2x_\mu} = 0, \quad w_{2x_\mu} w_{2x_\mu} = 0, \quad \Box_4 w_2 = 0. \end{aligned} \tag{38}$$

The above system contains Eqs. (1) as a subsystem. So, the d'Alembert–eikonal system (1) arises in a natural way when solving the problem of reduction of Eq. (34) be PDE having the smaller dimension (see, also [7, 9]).

Under the appropriate choice of the function $G(w_1, w_2)$ the change of variables

$$v = G(w_1, w_2), \quad u = w_2$$

reduces the system (38) to the form

$$\Box_4 v = f(v, u), \quad v_{x_\mu} v_{x_\mu} = \lambda, \tag{39a}$$

$$v_{x_\mu} u_{x_\mu} = 0, \quad u_{x_\mu} u_{x_\mu} = 0, \quad \Box_4 u = 0, \tag{39b}$$

$$\operatorname{rank} \left\| \begin{matrix} v_{x_0} & v_{x_1} & v_{x_2} & v_{x_3} \\ u_{x_0} & u_{x_1} & u_{x_2} & u_{x_3} \end{matrix} \right\| = 2, \tag{39c}$$

where λ is a real parameter taking the values -1, 0, 1.

Before formulating the principal assertion, we shall prove an auxiliary lemma.

Lemma. *Let $a = (a_0, a_1, a_2, a_3)$, $b = (b_0, b_1, b_2, b_3)$ be four-vectors defined in the real Minkowski space $M(1,3)$. Suppose they satisfy the relations*

$$a_\mu b_\mu = b_\mu b_\mu = 0, \quad \sum_{\mu=0}^{3} b_\mu^2 \neq 0. \tag{40}$$

Then the inequality $a_\mu a_\mu \leq 0$ holds.

Proof. It is known that any isotropic vector b in the space $M(1,3)$ can be reduced to the form $b = (\alpha, \alpha, 0, 0)$, $\alpha \neq 0$ by means of transformations from the group $P(1,3)$. Substituting $b = (\alpha, \alpha, 0, 0)$ into the first equality from (40), we get

$$\alpha(a_0 - a_3) = 0 \quad \longleftrightarrow \quad a_0 = a_3.$$

Consequently, the vector a has the following component: a_0, a_1, a_2, a_0. That is why $a_\mu a_\mu = a_0^2 - a_1^2 - a_2^2 - a - 0^2 = -(a_1^2 + a - 2^2) \leq 0$.

Let us note that $a_\mu a_\mu = 0$ iff $a_2 = a_3$, i.e. $a_\mu a_\mu = 0$ iff the vectors a and b are parallel.

Theorem 4. *Eqs. (39a-c) are compatible iff*

$$\lambda = -1, \quad f = -N(v + h(u))^{-1}, \tag{41}$$

where $h \in C^1(\mathbb{R}^1, \mathbb{R}^1)$ is an arbitrary function, $N = 0, 1, 2, 3$.

Theorem 5. *The general solution of the system of Eqs. (39a-c) being determined up to the transformation from the group $P(1,3)$ is given by the following formulae:*

a) under $f = -3(v + h(u))^{-1}$, $\lambda = -1$

$$\begin{aligned} &(v + h(u))^2 = -(\dot{A}_\nu \dot{A}_\nu)^{-1} (\dot{A}_\mu x_\mu + \dot{B})^2 + \\ &\qquad\qquad + (\dot{A}_\nu \dot{A}_\nu)^{-3} (E_{\mu\nu\alpha\beta} A_\mu \dot{A}_\nu \ddot{A}_\alpha x_\beta + C)^2, \\ &A_\mu(u) x_\mu + B(u) = 0; \end{aligned} \tag{42}$$

b) under $f = -2(v + h(u))^{-1}$, $\lambda = -1$

$$(v + h(u))^2 = -(\dot{A}_\nu \dot{A}_\nu)^{-1} (\dot{A}_\mu x_\mu + \dot{B}), \quad A_\mu x_\mu + B = 0, \tag{43a}$$

where $A_\mu(u)$, $B(u)$, $C(u)$ are arbitrary smooth functions satisfying the relations

$$A_\mu A_\mu = 0, \quad \dot{A}_\mu \dot{A}_\mu \neq 0; \tag{43b}$$

c) under under $f = -(v + h(u))^{-1}$, $\lambda = -1$

$$\begin{aligned} &u = C_0(x_0 - x_3), \\ &(v + h(x_0 - x_3))^2 = (x_1 + C_1(x_0 - x_3))^2 + (x_2 + C_2(x_0 - x_3))^2, \end{aligned} \tag{44}$$

where C_0, C_1, C_2 are arbitrary smooth functions;

d) under $f = 0$, $\lambda = -1$

$$1) \quad v = (-\dot{A}_\nu \dot{A}_\nu)^{-3/2} E_{\mu\nu\alpha\beta} A_\mu \dot{A}_\nu \ddot{A}_\alpha x_\beta + C, \quad A_\mu x_\mu + B = 0, \tag{45}$$

where $A_\mu(u)$, $B(u)$, $C(u)$ are arbitrary smooth functions satisfying the relations (43b);

$$2) \quad u = C_0(x_0 - x_3), \tag{46}$$

$$v = x_1 \cos C_1(x_0 - x_3) + x_2 \sin C_1(x_0 - x_3) + C_2(x_0 - x_3), \tag{47}$$

where C_0, C_1, C_2 are arbitrary smooth functions.

In the above formulae (42), (43a), (45) we denote by the symbol $E_{\mu\nu\alpha\beta}$ the components of antisymmetrical fourth-order tensor, i.e.

$$E_{\mu\nu\alpha\beta} = \begin{cases} 1, & (\mu, \nu, \alpha, \beta) = \text{cycle}\,(0, 1, 2, 3), \\ -1, & (\mu, \nu, \alpha, \beta) = \text{cycle}\,(1, 0, 2, 3), \\ 0, & \textit{in the remaining cases}. \end{cases} \tag{48}$$

Proof of Theorems 4, 5. By force of (39c) $u \not\equiv \text{const}$. Consequently, up to transformations from the group $P(1,3)$ $u_{x_0} \not\equiv 0$. That is why one can apply to Eqs. (39) the hodograph transformation

$$\begin{aligned} &z_0 = u(x), \quad z_a = x_a, \quad a = \overline{1,3}, \\ &w(z) = x_0, \quad v = v(z_0, z_a). \end{aligned} \tag{49}$$

As a result the system (39a,b) reads

$$\sum_{a=1}^{3} w_{z_a}^2 = 1, \quad \sum_{a=1}^{3} w_{z_a z_a} = 0, \tag{50a}$$

$$\sum_{a=1}^{3} v_{z_a} w_{z_a} = 0, \tag{50b}$$

$$\sum_{a=1}^{3} v_{z_a}^2 = -\lambda, \quad \sum_{a=1}^{3} (v_{z_a z_a} + 2 w_{z_0}^{-1} v_{z_a} w_{z_a z_0}) = -f(v, z_0). \tag{50c}$$

Since $v(z)$ is a real-valued function, $\lambda = -1$ or $\lambda = 0$.

Case 1. $\lambda = -1$. As it is shown in the Section 1, the general solution of the system (50a) in the class of real-valued functions $w(z)$ is given by the formulae (19), (20) with $n = 4$. On substituting (19) into (50b), we obtain the linear first-order PDE

$$\sum_{a=1}^{3} \alpha_a(z_0) v_{z_a} = 0, \tag{51}$$

the general solution of which is represented in the form

$$v = v(z_0, \rho_1, \rho_2). \tag{52}$$

In (52)

$$z_0, \quad \rho_1 = \left(\sum_{a=1}^{3}\alpha_a^2\right)^{-1/2}\left(\sum_{a=1}^{3}\dot{\alpha}_a z_a + \alpha\right),$$

$$\rho_2 = \left(\sum_{a=1}^{3}\dot{\alpha}_a^2\right)^{-1/2}\sum_{a,b,c=1}^{3} E_{abc} z_a \alpha_b \dot{\alpha}_c$$

are first integrals of Eq. (51) and what is more $\sum\limits_{a=1}^{3}\alpha_a^2 \neq 0$ (the case $\alpha_a = \mathrm{const}$, $a = \overline{1,3}$ will be considered separately).

Substitution of the expression (52) into (50c) yields the system of two PDE for a function $v = v(z_0, \rho_1, \rho_2)$

$$v_{\rho_1\rho_2} + v_{\rho_2\rho_2} + 2\rho_1^{-1} v_{\rho_1} = -f(v, z_0), \tag{53a}$$

$$v_{\rho_1}^2 + v_{\rho_2}^2 = 1. \tag{53b}$$

Let us exclude function $f(v, z_0)$ from (53) by considering of the third-order differential consequence of (53)

$$v_{\rho_2}(v_{\rho_1\rho_1} + v_{\rho_2\rho_2} + 2\rho_1^{-1}v_{\rho_1})_{\rho_1} - v_{\rho_1}(v_{\rho_1\rho_1} + v_{\rho_2\rho_2} + 2\rho_1^{-1}v_{\rho_1})_{\rho_2} = 0, \tag{54a}$$

$$v_{\rho_1}^2 + v_{\rho_2}^2 = 1. \tag{54b}$$

Further we shall consider the cases $v_{\rho_2\rho_2} = 0$ and $v_{\rho_2\rho_2} \neq 0$ separately.

A. $v_{\rho_2\rho_2} = 0$. Then

$$v = g_1(z_0, \rho_1)\rho_2 + g_2(z_0, \rho_1), \tag{55}$$

where $g_1, g_2 \in C^2(\mathbb{R}^2, \mathbb{R}^1)$ are arbitrary functions.

Substituting (55) into (54b) and splitting the obtained quality by the powers of ρ_2, we have

$$g_{1\rho_1} = 0, \quad g_1^2 + (g_{2\rho_2})^2 = 1,$$

whence

$$v = \alpha\rho_1 \pm \sqrt{1-\alpha^2}\rho_2 - h(z_0). \tag{56}$$

Here $\alpha \in \mathbb{R}^1$ is an arbitrary smooth function.

Substituting (56) into (53a), we get an algebraic equation $\alpha\sqrt{1-\alpha^2} = 0$, whence $\alpha = 0, \pm 1$.

Finally, substitution of (56) into (53a) yields an equation for $f(v, z_0)$

$$2\alpha\rho_1^{-1} = -f\left(\alpha\rho_1 \pm \sqrt{1-\alpha^2}\rho_2 - h(z_0), z_0\right). \tag{57}$$

From Eq. (57) it follows that under $\alpha = 0$

$$f = 0, \quad v = \pm\rho_2 - h(z_0) \tag{58}$$

and under $\alpha = \pm 1$

$$f = -2(v + h(z_0))^{-1}, \quad v = \pm\rho_1 - h(z_0). \tag{59}$$

B. $v_{\rho_2\rho_2} \neq 0$. In such a case one can apply the Euler transformation to Eqs. (54)

$$\begin{aligned}
&z_0 = y_0, \quad \rho_1 = y_1, \quad \rho_2 = G_{y_2}, \quad v + G = \rho_2 y_2, \quad v_{\rho_1} = -G_{y_1}, \quad v_{\rho_2} = y_2,\\
&v_{\rho_2\rho_2} = (G_{y_2y_2})^{-1}, \quad v_{\rho_1\rho_2} = -G_{y_1y_2}(G_{y_2y_2})^{-1},\\
&v_{\rho_1\rho_1} = (G^2_{y_1y_2} - G_{y_1y_1}G_{y_2y_2})(G_{y_2y_2})^{-1}.
\end{aligned} \tag{60}$$

Here y_0, y_1, y_2 are new independent variables, $G = G(y_0, y_1, y_2)$ is a new function.

In the new variables y, $G(y)$ the equation (54b) is linearized

$$G_{y_1} = \pm\sqrt{1 - y_2^2},$$

whence

$$G = \pm y_1\sqrt{1 - y_2^2} + H(y_0, y_2), \quad H \in C^2(\mathbb{R}^2, \mathbb{R}^1). \tag{61}$$

The equation (54a) after the change of variables (60) and substitution of the formula (61) takes the form

$$\left[y_1 - (1 - y_2^2)^{3/2}H_{y_2y_2}\right]^{-2}\left[3y_2H_{y_2y_2} + (y_2^2 - 1)H_{y_2y_2y_2}\right] + 2y_1^2H_{y_2y_2} = 0. \tag{62}$$

Splitting (62) by the powers of y_1 and integrating the obtained equations, we get

$$H = h_1(y_0)y_2 + h_2(y_0).$$

Substituting the above result into (61) and returning to the initial variables z_0, ρ_1, ρ_2, $v(z_0, \rho_1, \rho_2)$, we have the general solution of the system of PDE (54)

$$v + h_2(z_0) = \pm\left[(\rho_2 - h_1(z_0))^2 + \rho_1^2\right]^{1/2}. \tag{63}$$

At last, substituting (63) into the equation (53a), we come to conclusion that the function f is determined by the formula

$$f(v, z_0) = -3(v + h_2(z_0))^{-1}.$$

Let, us consider now the case $\alpha_a = \text{const}$, $a = \overline{1,3}$. Then the equality $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ holds. That is why, using transformations from the group $P(1,3)$, one can obtain $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$, i.e. $u = C_0(x_0 - x_3)$, $C_0 \in C^2(\mathbb{R}^1, \mathbb{R}^1)$. Then, from Eqs. (39b) it follows that $v = v(\xi, x_1, x_2)$, $\xi = x_0 - x_3$ and what is more Eqs. (39a) take the form

$$v_{x_1}^2 + v_{x_2}^2 = 1, \quad v_{x_1x_1} + v_{x_2x_2} = -f(v, C_0(\xi)). \tag{64}$$

It, is known [7, 10] that Eqs. (64) are compatible iff $f = 0$ or $f = -(v + h(\xi))^{-1}$, $h \in C^1(\mathbb{R}^1, \mathbb{R}^1)$. And besides the general solution of (64) is given by the formulae (46) and (44) respectively.

Thus we have completely investigated the case $\lambda = -1$.

Case 2. $\lambda = 0$. By force of the fact that the function v is a real one, from (50b) it follows that $v = v(z_0)$. Consequently, the equality $v = v(u)$ holds that breaks the condition (39c). So under $\lambda = 0$ the system (39a-c) is incompatible.

So, we have proved that the system of nonlinear PDE (39a-c) is compatible iff the relations (41) hold and its general solution is given by one of the formulae (44), (46), (58), (59), (63). To complete the proof, one has to rewrite the expressions (58), (59), (63) in the manifestly covariant form (42), (43a), (45).

Let us consider as an example the formula (59)

$$v = \pm\rho_1 - h(z_0) = \pm\left(\sum_{a=1}^{3}\dot{\alpha}_a^2\right)^{-1/2}\left(\sum_{a=1}^{3}x_a\dot{\alpha}_a(u) + \dot{\alpha}(u)\right) - h(u), \tag{65}$$

the function $u(x)$ being determined by the formula (12)

$$\sum_{a=1}^{3}\alpha_a(u)x_a + \alpha(u) = 0, \quad \sum_{a=1}^{3}\alpha_a^2(u) = 1. \tag{66}$$

Let us make in (65), (66) the change $\alpha_a = A_aA_0^{-1}$, $\alpha = -BA_0^{-1}$, whence

$$\begin{aligned}
&A_\mu(u)x_\mu + B(u) = 0, \quad A_\mu A_\mu = 0,\\
&h(u) + v = \pm\left[\sum_{a=1}^{3}\dot{A}_aA_0^{-1} - A_a\dot{A}_0A_0^{-2}\right]^{-1/2}\times\\
&\quad\times\left[\sum_{a=1}^{3}x_a(\dot{A}_aA_0^{-1} - A_a\dot{A}_0A_0^{-2}) + B\dot{A}_0A_0^{-2} - \dot{B}A_0^{-1}\right] =\\
&\quad= \pm\left[\sum_{a=1}^{3}(\dot{A}_a^2A_0^{-2} + A_a^2\dot{A}_0^2A_0^{-1} - 2\dot{A}_aA_a\dot{A}_0A_0^{-3})^{-1/2}\right]\times\\
&\quad\times\left[\sum_{a=1}^{3}x_a(\dot{A}_aA_0^{-1} - A_a\dot{A}_0A_0^{-2}) + B\dot{A}_0A_0^{-2} - \dot{B}A_0\right] =\\
&\quad= \pm\left[-\dot{A}_\mu\dot{A}_\mu A_0^{-2} - A_\mu A_\mu\dot{A}_0^2A_0^{-4} + 2\dot{A}_\mu A_\mu\dot{A}_0A_0^{-3}\right]^{-1/2}\times\\
&\quad\times\left[-A_0^{-1}(x_\mu\dot{A}_\mu + \dot{B}) + A_0^{-2}\dot{A}_0(x_\mu A_\mu + B)\right] =\\
&\quad= \mp(-\dot{A}_\mu\dot{A}_\mu)^{-1/2}(x_\mu\dot{A}_\mu + \dot{B}).
\end{aligned}$$

The only thing left is to prove that $\dot{A}_\mu\dot{A}_\mu < 0$. Since $A_\mu A_\mu = 0$, the equality $\dot{A}_\mu A_\mu = 0$ holds. Consequently, by force of the lemma $-A_\mu\dot{A}_\mu \geq 0$ and what is more the equality $\dot{A}_\mu\dot{A}_\mu = 0$ holds iff $\dot{A}_\mu = k(u)A_\mu$. The general solution of the above system of ordinary differential equations reads $A_\mu = \tilde{k}(u)\theta_\mu$, where $\tilde{k}(u)$ is an arbitrary function, $\theta_\mu \in \mathbb{R}^1$, $\theta_\mu\theta_\mu = 0$. Whence it follows that $\alpha_a = A_aA_0^{-1} = \theta_a\theta_0^{-1} =$ const and the condition $\sum\limits_{a=1}^{3}\dot{\alpha}_a^2 \neq 0$ does not hold. We come to the contradiction whence it follows that $\dot{A}_\mu\dot{A}_\mu < 0$.

Thus we have obtained the formula (43a). Derivation of the remaining formulae from (42), (45) is carried out in the same way. The theorems are proved.

Substitution of the above obtained results into the formula $w=\varphi(v,u)$ yields the following collection of ansätzes for the nonlinear wave equation (34)

$$\begin{aligned}
&1. \quad w=\varphi\Big(-h(u)\pm\Big[(-\dot{A}_\nu\dot{A}_\nu)^{-1}(\dot{A}_\mu x_\mu+\dot{B})^2-\\
&\qquad -(\dot{A}_\nu\dot{A}_\nu)^{-3}(E_{\mu\nu\alpha\beta}A_\mu\dot{A}_\nu\ddot{A}_\alpha x_\beta+C(u))^2\Big]^{1/2},u\Big);\\
&2. \quad w=\varphi\Big(-h(u)\pm(-\dot{A}_\nu\dot{A}_\nu)^{1/2}(\dot{A}_\mu x_\mu+\dot{B}),u\Big);\\
&3. \quad w=\varphi\Big(h(x_0-x_3)\pm([x_1+C_1(x_0-x_3)]^2+\\
&\qquad +[x_2+C_2(x_0-x_3)]^2)^{1/2},x_0-x_3\Big);\\
&4. \quad w=\varphi\Big((-\dot{A}_\nu\dot{A}_\nu)^{-3/2}(E_{\mu\nu\alpha\beta}A_\mu\dot{A}_\nu\ddot{A}_\alpha x_\beta+C(u)),u\Big);\\
&5. \quad w=\varphi\Big(x_1\cos C_1(x_0-x_3)+x_2\sin C_1(x_0-x_3)+C_2(x_0-x_3),x_0-x_3\Big).
\end{aligned}\tag{67}$$

Here $u=u(x)$ is determined by JSSF (8) with $n=4$.

Substitution of the expressions (67) into (34) gives the following equations for $\varphi=\varphi(u,v)$:

$$1. \quad \varphi_{vv}+3(v+h(u))^{-1}\varphi_v=-F(\varphi), \tag{68}$$

$$2. \quad \varphi_{vv}+2(v+h(u))^{-1}\varphi_v=-F(\varphi), \tag{69}$$

$$\begin{aligned}
&3. \quad \varphi_{vv}+(v+h(u))^{-1}\varphi_v=-F(\varphi),\\
&4. \quad \varphi_{vv}=-F(\varphi),\\
&5. \quad \varphi_{vv}=F(\varphi).
\end{aligned}\tag{70}$$

Eqs. 4, 5 from (68)–(70) are known to be integrable in quadratures. Therefore, any solution of the d'Alembert–eikonal system (1) corresponds to some class of exact solutions of the nonlinear wave equation (34) that contains arbitrary functions. Saying it in another way, the formulae (67) make it possible to construct wide families of exact solutions of the nonlinear PDE (34) using exact solutions of the linear d'Alembert equation $\Box_4 u=0$ satisfying the additional constraint $u_{x_\mu}u_{x_\mu}=0$.

It is interesting to compare our approach to the problem of reduction of Eq. (34) with classical Lie approach. In the framework of the Lie approach the functions $w_1(x)$, $w_2(x)$ from (35) are looked for as invariants of the symmetry group of the equation under study (in the case involved it is the Poincaré group $P(1,3)$). Since the group $P(1,3)$ is a finite-parameter group, its invariants cannot contain an arbitrary function (complete description of invariants of the group $P(1,3)$ had been carried out in [11]). So the ansätzes (67) cannot be obtained by means of Lie symmetry of the PDE (34).

The ansätzes (67) correspond to conditional invariance of the nonlinear wave equation (34). It means that there exist two differential operators $Q_a=\xi_{a\mu}(x)\partial_{x_\mu}$, $a=\overline{1,2}$ such that

$$Q_a w\equiv Q_a\varphi(w_1,w_2)=0, \quad a=\overline{1,2}$$

and besides the system of PDE

$$Q_a w=0, \quad a=\overline{1,2}, \quad \Box_4 w-F(w)=0$$

is invariant in Lie's sense under the one-parameter groups having generators Q_1, Q_2 (on the conditional invariance of mathematical and theoretical physics equations see [8, 12, 13]).

It is worth noting that the ansätzes 2, 5 from (67) were obtained in [14] without using the concept of conditional invariance.

5. On the new exact, solutions of the nonlinear wave equation. The general solution of Eqs. (70) is given by the following quadrature [15]:

$$v + D(u) = \int_0^{\varphi(u,v)} \left[-\int_0^{\tau} F(z)dz + C(u) \right]^{-1/2} d\tau, \tag{71}$$

where $D(u), C(u) \in C^2(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions.

Substituting into (71) expressions for $u(x)$, $v(x)$ given by the formulae 4, 5 from (67), we get two classes of exact solutions of the nonlinear wave equation (34) that contain several arbitrary functions of one variable.

Eqs. (68), (69) are Emden–Fauler type equations. They were investigated by many authors see, e.g. [15]). In particular, it is known that the equations

$$\varphi_{vv} + 2v^{-1}\varphi_v = -\lambda\varphi^5, \tag{72}$$

$$\varphi_{vv} + 3v^{-1}\varphi_v = -\lambda\varphi^3 \tag{73}$$

are integrated in quadratures. In the paper [11] it had been established that Eqs. (72), (73) possess the Painleve property. This fact made it possible to integrate them by applying rather complicated technique. We shall demonstrate how to integrate Eqs. (72), (73) by using their symmetry properties.

It occurs that Eq. (72) admits the symmetry operator $Q = 2v\partial_v - \varphi\partial_\varphi$. Following [15] we find the change of the variables

$$\varphi = z(\tau)v^{-1/2}, \quad \tau = \ln v$$

that reduce the operator Q to the form $Q' = \partial_\tau$. Eq. (72) in the new variables reads

$$z_{\tau\tau} = \frac{1}{4}z - \lambda z^5,$$

whence

$$z_\tau^2 = \frac{1}{4}z^2 - \frac{\lambda}{3}z^6 + \frac{1}{4}D(u), \tag{74}$$

where $D(u) \in C^1(\mathbb{R}^1, \mathbb{R}^1)$ is an arbitrary function. Further we consider in detail the case $D(u) = \delta \equiv \text{const}$.

On putting $z^2 = R(\tau)$ we get the following equation:

$$R_\tau^2 = -\frac{4\lambda}{3}R^4 + R^2 + \delta R \equiv S(R). \tag{75}$$

Integration of (75) yields

$$\int_0^{z^2} \frac{dR}{\sqrt{S(R)}} = \pm(\ln v + \ln C(u)). \tag{76}$$

Here $C(u)$ is an arbitrary smooth function.

Let us represent the polynomial $S(R)$ in the form

$$S(R) = -\frac{4}{3}\lambda R(R-\theta_1)(R-\theta_2)(R-\theta_3),$$

where θ_i are the roots of the polynomial $S(R)$ that satisfy equations (the Vieta's theorem)

$$\theta_1+\theta_2+\theta_3 = 0, \quad \theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1 = -\frac{3}{4\lambda}, \quad \theta_1\theta_2\theta_3 = \frac{3\delta}{4}.$$

The explicit form of the integral in the left side of Eq. (76) depends on relations connecting the roots θ_i.

Case 1. $\theta_1 = 0$, $\theta_2 \neq \theta_3$, $\theta_2 \neq 0$, $\theta_3 \neq 0$. Such a case taken place under $\delta = 0$, solution of Eq. (72) being given by the formulae

$$\varphi = \left\{\frac{\sqrt{3}C(u)}{a(1+C^2(u)v)^2}\right\}^{1/2} \quad \text{under } \lambda = a^2 > 0, \tag{77}$$

$$\varphi = \left\{\frac{\sqrt{3}C(u)}{a(1-C^2(u)v)^2}\right\}^{1/2} \quad \text{under } \lambda = -a^2 < 0, \tag{78}$$

Case 2. $\theta_1 = \theta_2$, $\theta_2 \neq 0$, $\theta_3 \neq 0$, $\theta_3 \neq \theta_2$. Such relations are satisfied provided $\lambda = a^2 > 0$, $\delta = \pm(3a)^{-1}$, solution of Eq. (72) taking the form

$$\varphi = \left\{\frac{\sin(\ln(vC(u)))+1}{av(2\sin(\ln(vC(u)))-4)}\right\}^{1/2}. \tag{79}$$

Case 3. $\theta_1 \neq \theta_2$, $\theta_2 \neq \theta_3$, $\theta_3 \neq \theta_1$. $\lambda = -a^2 < 0$. In such a case the polynomial $S(R)$ has two real and two complex roots. Therefore it is represented in the form

$$S(R) = \frac{4a^2}{3}R(R+\theta_1)((R+\theta_2)^2+\theta_3^2),$$

solution of Eq. (72) taking the form

$$\varphi = \left\{\frac{p\theta_1\left(1-\mathrm{cn}\left[\frac{2a}{\sqrt{3}}\sqrt{pq}\ln(vC(u))\right]\right)}{v\left[(p+q)\mathrm{cn}\left[\frac{2a}{\sqrt{3}}\sqrt{pq}\ln(vC(u))\right]+q-p\right]}\right\}^{1/2}. \tag{80}$$

Here

$$p = \sqrt{\theta_2^2+\theta_3^2}, \quad q = \sqrt{(\theta_1+\theta_2)^2+\theta_3^2}, \quad h = \frac{1}{2}\frac{\sqrt{(p+q)^2+\theta_1^2}}{pq}.$$

Case 4. $\theta_1 \neq \theta_2$, $\theta_2 \neq \theta_3$, $\theta_3 \neq \theta_1$, $0 < \frac{1}{\lambda} < (3\delta)^2$, $\lambda = a^2$. The polynomial $S(R)$ has two real and two complex roots and is given by the formula

$$S(R) = \frac{4a^2}{3}R(\theta_1-R)((R+\theta_2)^2+\theta_3^2).$$

The solution of Eq. (72) has the form

$$\varphi = \left\{ \frac{q\theta_1 \left(1 + \mathrm{cn}\left[\frac{2a}{\sqrt{3}}\sqrt{pq}\,\ln(vC(u))\right]\right)}{v\left[(p+q) + (q-p)\mathrm{cn}\left[\frac{2a}{\sqrt{3}}\sqrt{pq}\,\ln(vC(u))\right]\right]} \right\}^{1/2}, \tag{81}$$

where

$$p = \sqrt{(\theta_2-\theta_1)^2+\theta_3^2}, \quad q = \sqrt{\theta_2^2+\theta_3^2}, \quad h = \frac{1}{2}\frac{\sqrt{\theta_1^2-(p+q)^2}}{pq}.$$

Case 5. $\theta_1 \neq \theta_2$, $\theta_2 \neq \theta_3$, $\theta_3 \neq \theta_1$, $\lambda = a^2 > 0$, $\lambda(3\delta)^2 < 1$. Is this case the polynomial $S(R)$ has four real roots $\theta_0 < \theta_1 < \theta_2 < \theta_3$ (one of them is equal to zero) and is represented in the form

$$S(R) = \frac{4a^2}{3}(\theta_0 - R)(R-\theta_1)(R-\theta_2)(R-\theta_3).$$

Solution of Eq. (72) reads

$$\varphi = \left\{ \frac{\theta_0(\theta_1-\theta_3) - \theta_3(\theta_1-\theta_0)\,\mathrm{sn}^2\left[\frac{a}{\sqrt{3}}\sqrt{(\theta_0-\theta_2)(\theta_1-\theta_3)}\,\ln(vC(u))\right]}{v\left(\theta_1-\theta_3-(\theta_1-\theta_0)\,\mathrm{sn}^2\left[\frac{a}{\sqrt{3}}\sqrt{(\theta_0-\theta_2)(\theta_1-\theta_3)}\,\ln(vC(u))\right]\right)} \right\}^{1/2}. \tag{82}$$

In the above formulae (80)–(82) cn, sn are elliptic functions of the order k.

Substituting the formulae (77)–(82) into the ansatz 2 from (67) with $h \equiv 0$, where $u = u(x)$ is determined by JSSF (43a) we obtain wide families of new exact solutions of the nonlinear PDE (34) under $F(w) = \lambda w^5$.

Eq. (73) is integrated in analogous way. As a result we have

1. $\lambda = -a^2 < 0$,
$$\varphi = \frac{1}{av}\,\mathrm{tg}\left(\pm\frac{\sqrt{2}}{a^2}\ln(vC(u))\right); \tag{83}$$

2. $\lambda = a^2 > 0$,
$$\varphi = \frac{2\sqrt{2}C(u)}{a(1+v^2C^2(u))}; \tag{84}$$

3. $\lambda = -a^2 < 0$,
$$\varphi = \frac{2\sqrt{2}C(u)}{a(1-v^2C^2(u))}; \tag{85}$$

4. $\lambda = 2a^{-2} > 0$, $a > 0$,
$$\varphi = \frac{b}{v}\,\mathrm{cn}\left[\frac{\sqrt{b^2+d^2}}{a}\ln(vC(u))\right], \tag{86}$$

where

$$b = \left(a^2 + a\sqrt{a^2+4\delta}\right)^{1/2}, \quad d = \left(-a^2 + a\sqrt{a^2+4\delta}\right)^{1/2},$$
$$\delta \in \mathbb{R}^1, \quad k = b^{-1}\sqrt{b^2-d^2};$$

5. $\lambda = 2a^{-2} > 0, \quad a > 0,$

$$\varphi = \frac{b}{v}\,\mathrm{dn}\left(\frac{b}{a}\ln(vC(u))\right), \tag{87}$$

where

$$b = \left(a^2 + a\sqrt{a^2+4\delta}\right)^{1/2}, \quad d = \left(a^2 - a\sqrt{a^2+4\delta}\right)^{1/2}, \quad k = b^{-1}\sqrt{b^2-d^2};$$

6. $\lambda = -2a^{-2} < 0, \quad a > 0,$

$$\varphi = \frac{b}{v}\left[\mathrm{cn}\left(\frac{\sqrt{b^2-d^2}}{a}\ln(vC(u))\right)\right], \tag{88}$$

where

$$b = \left(a^2 + a\sqrt{a^2+4\delta}\right)^{1/2}, \quad d = \left(a\sqrt{a^2+4\delta} - a^2\right)^{1/2},$$
$$\delta > 0, \quad k = d(b^2+d^2)^{-1/2};$$

7. $\lambda = -2a^{-2} < 0, \quad -\frac{a^2}{4} < \delta < 0, \quad a > 0,$

$$\varphi = \frac{b}{v}\,\mathrm{tn}\left(\frac{b}{a}\ln(vC(u))\right), \tag{89}$$

where

$$b = \left(a^2 + a\sqrt{a^2+4\delta}\right)^{1/2}, \quad d = \left(a^2 - a\sqrt{a^2+4\delta}\right)^{1/2}, \quad k = b^{-1}\sqrt{b^2-d^2};$$

8. $\lambda = -2a^{-2} < 0,$

$$\varphi = \frac{b}{v}\left(\frac{1+\mathrm{cn}\left(\frac{2b}{a}\ln(vC(u))\right)}{1-\mathrm{cn}\left(\frac{2b}{a}\ln(vC(u))\right)}\right)^{1/2}, \tag{90}$$

where

$$b = \sqrt[4]{-4\delta a^2}, \quad \delta < -\frac{a^2}{4}, \quad k = \frac{\sqrt{b^2-d^2}}{\sqrt{2}b}.$$

In the above formulae (83)–(90) cn, dn, tn are elliptic functions of the order k.

Substituting the formulae (83)–(90) into the ansatz 1 from (67) with $h = 0$, where $u = u(x)$ is determined by JSSP (43a) we get ad families of exact solutions of the nonlinear Eq. (34) under $F(w) = \lambda w^3$.

Let us emphasize once more that solutions of nonlinear PDE (34) obtained in the above described manner contain several arbitrary functions and cannot in principle be constructed by means of symmetry reduction procedure.

In conclusion, we adduce two examples of exact solutions of Eq. (34) with $F(w) = \lambda w^3$ that can be written down in the explicit form

$$u(x) = \left(x_1^2+x_2^2+x_3^2-x_0^2\right)^{1/2}\mathrm{tg}\left\{\sqrt{2}\left[\ln\left(x_1^2+x_2^2+x_3^2-x_0^2\right)^{1/2} + \right.\right.$$
$$\left.\left. + \ln\left[C\left(\frac{x_0x_1 \pm x_2\sqrt{x_1^2+x_2^2-x_0^2}}{x_1^2+x_2^2}\right)\right]\right]\right\},$$

$$u(x) = \left(x_1^2 + x_2^2 + x_3^2 - x_0^2\right)^{1/2} \operatorname{tg} \left\{ \sqrt{2} \left[\ln \left(x_1^2 + x_2^2 + x_3^2 - x_0^2\right)^{1/2} + \right.\right.$$
$$\left.\left. + \ln \left[C \left(\frac{x_1 x_2 \pm x_0 \sqrt{x_1^2 + x_2^2 - x_0^2}}{x_1^2 + x_2^2} \right) \right] \right] \right\}. \tag{91}$$

Here C is an arbitrary smooth function.

It is important to note that the formulae (91) under $C \equiv \mathrm{const}$ give the already known solutions (see, e.g. [8, 11]).

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