Conditional symmetry and reduction of partial differential equations

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Sufficient reduction conditions for partial differential equations possessing nontrivial conditional symmetry are established. The results obtained generalize the classical reduction conditions of differential equations by means of group-invariant solutions. A number of examples illustrating the reduction in the number of independent and dependent variables of systems of partial differential equations are considered.

An analysis of well-known methods for the construction of exact solutions of nonlinear partial differential equations (PDE) (e.g., method of group-theoretic reduction [1, 2], method of differential constraints [3], method of ansatz [4–6]) led us to conclude that most of these methods involve narrowing the set of solutions, i.e., out of the whole set of solutions of the particular equations specific subsets are selected that admit analytic description. In order to implement this approach, certain additional constraints (expressed in the form of equations) that enable us to distinguish these subsets must be imposed on the solution set. For obvious reasons, these additional equations are assumed to be simpler than the initial equations. By complementing the initial equation with additional constraints, we are usually led to an over-determined system of PDE. Consequently, there arises the problem of investigating the consistency of a system of PDE. A second restriction on the choice of these additional constraints is that the resulting system of PDE possesses broader symmetry than the initial system of PDE (or simply a different type of symmetry).

In the present paper we establish sufficient conditions for the reduction of differential equations that generalize the classical reduction conditions of PDE possessing a nontrivial Lie transformation group. Our concern will be with the following:

$$U_A(x,u,\underset{1}{u},\dots,\underset{r}{u})=0,\quad A=\overline{1,M},\tag{1}$$

$$\xi_{a\mu}(x,u)u^\alpha_{x_\mu}-\eta^\alpha_a(x,u)=0,\quad a=\overline{1,N},\tag{2}$$

where $x=(x_0,x_1,\dots,x_{n-1})$, $u(x)=(u^0(x),\dots,u^{m-1}(x))$, $\underset{s}{u}=\{\partial^s u^\alpha/\partial x_{\mu_1}\dots\partial x_{\mu_s}$, $0\le\mu_i\le n-1\}$, $s=\overline{1,r}$, U_A, $\xi_{a\mu}$, η^α_a are sufficiently smooth functions, $N\le n-1$.

Below summation over repeated indices is understood. Let us introduce the notation

$$R_1=\operatorname{rank}\|\xi_{a\mu}(x,u)\|^{N\ n-1}_{a=1\ \mu=0},$$

$$R_2=\operatorname{rank}\|\xi_{a\mu}(x,u),\ \eta^\alpha_a(x,u)\|^{N\ n-1\ m-1}_{a=1\ \mu=0\ \alpha=0}.$$

It is self-evident that $R_1\le R_2$. We shall prove that the case $R_1=R_2$ leads to a reduction in the number of independent variables of the PDE (1), while the case

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$R_1 < R_2$ leads to a reduction in the number of independent and the number of dependent variables of the PDE (1).

1. Reduction of number of independent variables of PDE. In this section we assume that $R_1 = R_2$.

Definition 1. *The set of first-order differential equations*

$$Q_a = \xi_{a\mu}(x,u)\partial_{x_\mu} + \eta_a^\alpha(x,u)\partial_{u^\alpha}, \tag{3}$$

where $\partial_{x_\mu} = \partial/\partial_{x_\mu}$, $\partial_{u^\alpha} = \partial/\partial u^\alpha$; $\xi_{a\mu}$, η_a^α *are smooth functions, is said to be involutive if there exist function* $f_{ab}^c(x,u)$ *such that:*

$$[Q_a, Q_b] = f_{ab}^c Q_c, \quad a,b = \overline{1,N}. \tag{4}$$

Here $[Q_1,Q_2] = Q_1Q_2 - Q_2Q_1$.

The simplest example of an involutive set of operators is a Lie algebra.

It is well-known that conditions (4) ensure that the over-determined system of PDE (2) is consistent (Frobenius theorem [7]). The general solution of the system (2) is given by the formulas

$$F^\alpha(\omega_1,\omega_2,\ldots,\omega_{n+m-R_1}) = 0, \quad \alpha = \overline{0,m-1}, \tag{5}$$

where $\omega_j = \omega_j(x,u)$ are functionally independent first integrals of the system of PDE (2) and F_α are arbitrary smooth functions.

By virtue of the condition $R_1 = R_2$, first integrals (say, $\omega_1,\ldots,\omega_m$) may be chosen that satisfy the condition

$$\det \|\partial\omega_j/\partial u^\alpha\|_{j=1\ \alpha=0}^{m\ m-1} \neq 0. \tag{6}$$

By solving (5) with respect to ω_j, $j = 1,\ldots,m$, we have

$$\omega_j = \varphi_j(\omega_{m+1},\omega_{m+2},\ldots,\omega_{m+n-R_1}), \quad j = \overline{1,m}, \tag{7}$$

where φ_j are arbitrary smooth functions

Definition 2. *Formula (7) is called the ansatz of the field* $u^\alpha = u^\alpha(x)$ *invariant with respect to the involutive set of operators (3) provided (6) is satisfied.*

Formula (7) become especially simple and self-evident if

$$\begin{gathered}\partial\xi_{a\mu}/\partial u^\alpha = 0, \quad \eta_a^\alpha = f_a^{\alpha\beta}(x)u^\beta, \\ a = \overline{1,N}, \quad \mu = \overline{0,n-1}, \quad \alpha,\beta,\gamma = \overline{0,m-1}.\end{gathered} \tag{8}$$

Under conditions (8) the operators in (3) may be rewritten in the following non-Lie form [8]:

$$Q_a = \xi_{a\mu}(x)\partial x_\mu + \eta_a(x), \quad a = \overline{1,N}, \tag{9}$$

where $\eta_a = \| -\partial\eta_a^\alpha/\partial u^\beta\|_{\alpha,\beta=0}^{m-1}$ are $(m\times m)$ matrices and the system (2) takes the form

$$\xi_{a\mu}(x)u_{x_\mu} + \eta_a(x)u = 0, \quad a = \overline{1,N}. \tag{10}$$

Here $u = (u^0,u^1,\ldots,u^{m-1})^T$ is a column function.

In this case, the set of functionally independent first integrals of the system (2) with $R_1 = R_2$ may be chosen as follows [7]:

$$\begin{aligned} &\omega_j = b_{j\alpha}(x)u^\alpha, \quad j = \overline{1, m}, \\ &\omega_i = \omega_i(x), \quad i = \overline{m+1, m+n-R_1} \end{aligned} \tag{11}$$

and, moreover, $\det \|b_{j\alpha}(x)\|_{i=1\ \alpha=0}^{m\ m-1} \neq 0$.

Substituting (11) in (7) and solving for the variables u^α, $\alpha = 0, \dots, m-1$, we have

$$u^\alpha = A^{\alpha\beta}(x)\varphi^\beta(\omega_{m+1}, \omega_{m+2}, \dots, \omega_{m+n-R_1})$$

or (in matrix notation)

$$u = A(x)\varphi(\omega_{m+1}, \omega_{m+2}, \dots, \omega_{m+n-R_1}). \tag{12}$$

It is easily verified that the matrix

$$(x) = \left(\|b_{j\alpha}(x)\|_{j=1\ \alpha=0}^{m\ m-1}\right)^{-1}$$

satisfies the following system of PDE:

$$Q_a A \equiv \xi_{a\mu}(x)A_{x_\mu} + \eta_a(x)A = 0, \quad a = \overline{1, N}, \tag{13}$$

and that the functions $\omega_{m+1}(x), \omega_{m+2}(x), \dots, \omega_{m+n-R_1}(x)$ form a complete set of functionally independent first integrals of the system of PDE

$$\xi_{a\mu}(x)\omega_{x_\mu} = 0, \quad a = \overline{1, N}. \tag{14}$$

The ansatz (7) is said to *reduce* the system of PDE (1) if substitution of (7) in (1) yields a system of PDE for the functions $\varphi^0, \varphi^1, \dots, \varphi^{m-1}$ that contains only the new independent variables $\omega_{m+1}, \omega_{m+2}, \dots, \omega_{m+1-R_1}$.

Definition 3. *The system of PDE (1) is conditionally invariant with respect to the involutive set of differential operators (3) if the over-determined system of PDE (1), (2) is Lie invariant with respect to a one-parameter transformation group with generators Q_a, $a = 1, \dots, N$.*

Before stating the reduction theorem, we prove several auxiliary assertions.

Lemma 1. *Suppose that the operators (3) form an involutive set. Then the set of differential operators*

$$Q'_a = \lambda_{ab}(x)Q_b, \quad a = \overline{1, N} \tag{15}$$

with $\det \|\lambda_{ab}(x, u)\|_{a,b=1}^N \neq 0$ *is also involutive.*

We prove the assertion by direct computation. In fact,

$$\begin{aligned} [Q'_a, Q'_b] = [\lambda_{ac}Q_c, \lambda_{bd}Q_d] = \lambda_{ac}(Q_c\lambda_{bd})Q_d - \lambda_{bd}(Q_d\lambda_{ac})Q_c + \lambda_{ac}\lambda_{bd}f_{cd}^{d_1}Q_{d_1} = \\ = \tilde{f}_{ab}^c Q_c = \tilde{f}_{ab}^c \lambda_{cd}^{-1} Q'_d. \end{aligned}$$

Here λ_{cd}^{-1} are the elements of the inverse of the matrix $\|\lambda_{ab}(x, u)\|_{a,b=1}^N$.

Lemma 2. *Suppose that the differential operators (3) satisfy the condition $R_1 = R_2$ and that the conditions*

$$[Q_a, Q_b] = 0, \quad a, b = \overline{1, N} \tag{16}$$

are satisfied. Then there exists a change of variables

$$x'_\mu = f_\mu(x,u), \quad \mu=\overline{0,n-1}, \quad u'^\alpha = g^\alpha(x,u), \quad \alpha=\overline{0,m-1} \tag{17}$$

that reduces the operators Q_a to the form $Q'_a = \partial_{x'_{a-1}}$.

Proof. It is known that for any first-order differential operator

$$Q = \xi_\mu(x,u)\partial_{x_\mu} + \eta^\alpha(x,u)\partial_{u^\alpha},$$

where ξ_μ and η^α are sufficiently smooth functions, there exists a change of variables (17) that reduces the operator Q to the form $Q' = \partial_{x'_0}$ (cf. [1]). Consequently, the operator Q_1 from the set (3) is reduced to the form $Q'_1 = \partial_{x'_0}$ by means of the change of variables (17). From the condition $[Q_1, Q_a] = 0$, $a = 2,\dots,N$, it follows that the coefficients of the operators $Q'_2, Q'_3, \dots, Q'_N$ do not depend on the variable x'_0, whence the operator Q'_2 reduces to the operator $Q''_2 = \partial_{x''_1}$ under the change of variables

$$x''_0 = x'_0, \quad x''_\mu = f'_\mu(x'_1,\dots,x'_{n-1},u'), \quad \mu=\overline{1,n-1},$$
$$u''_\alpha = g'^\alpha(x'_1,\dots,x'_{n-1},u'), \quad \alpha=\overline{0,m-1},$$

without the form of the operator Q'_1 changing.

Repeating the above procedure $N-2$ times completes the proof.

Lemma 3. *A system of PDE of the form (1) that is conditionally invariant with respect to a set of differential operators $\partial_{x'_\mu}$, $\mu = 0, N-1$, possesses the structure*

$$\begin{gathered} U_A = F_{AB}W_B(x_N, x_{N+1},\dots,x_{n-1},u,\underset{1}{u},\dots,\underset{r}{u}) + F^\alpha_{A\mu}u^\alpha_{x_\mu}, \\ A=\overline{1,M}, \quad \alpha=\overline{0,m-1}, \quad \mu=\overline{0,N-1}, \end{gathered} \tag{18}$$

where F_{AB} and $F^\alpha_{A\mu}$ are arbitrary smooth functions of x and $u,\underset{1}{u},\dots,\underset{r}{u}$, W_B are arbitrary smooth functions, and, moreover, $\|F_{AB}\|^M_{A,B=1} \neq 0$.

We shall prove the lemma with $N = 1$. By Definition 3, the system (1) is conditionally invariant under the operator $Q = \partial_{x_0}$ if the system

$$\begin{gathered} U_A(x,u,\underset{1}{u},\dots,\underset{r}{u}) = 0, \quad A=\overline{1,M}, \\ u^\alpha_{x_0} = 0, \quad \alpha=\overline{1,m-1} \end{gathered} \tag{19}$$

is Lie invariant with respect to a one-parameter translation group with respect to the variable x_0. Denoting by $\tilde{Q}$ the r-th extension of Q, the Lie invariant criteria for the system of PDE (19) under this group assume the form (cf. [1, 2])

$$\tilde{Q}U_A\Big|_{\substack{U_B=0\\ u^\alpha_{x_0}=0}} = 0, \quad A,B=\overline{1,N}, \quad \alpha=\overline{0,m-1}, \tag{19a}$$

$$\tilde{Q}u^\alpha_{x_0}\Big|_{\substack{U_B=0\\ u^\beta_{x_0}=0}} = 0, \quad B=\overline{1,N}, \quad \alpha,\beta=\overline{0,m-1}. \tag{19b}$$

Direct computation shows that the relations

$$\tilde{Q} \equiv \partial_{x_0}, \quad \tilde{Q}u^\alpha_{x_0} \equiv \partial_{x_0}(u^\alpha_{x_0}) = 0$$

hold (recall that in the extended space of the variables $x, u, \underset{1}{u}, \ldots, \underset{r}{u}$ variables x_0 and $u^\alpha_{x_0}$ are independent), whence, using the method of undetermined coefficients, we may rewrite (19a) and (19b) in the form

$$\partial U_A/\partial x_0 = R_{AB}U_B + P^\alpha_A u^\alpha_{x_0}, \quad A = \overline{1, M}, \tag{19c}$$

where R_{AB} and P^α_A are arbitrary smooth functions of $x, u, \underset{1}{u}, \ldots, \underset{r}{u}$.

The system (19c) may be considered a system of inhomogeneous ordinary differential equations for the functions U_A, $A = 1, \ldots, M$. Integrating (19c) with respect to $P^\alpha_A = 0$, we have

$$U^{(0)}_A = F_{AB}W_B, \quad A = \overline{1, M},$$

where W_B, $B = 1, \ldots, M$, are arbitrary smooth functions of the variables $x_1, x_2, \ldots$, $x_{n-1}, u, \underset{1}{u}, \ldots, \underset{r}{u}$; $F = \|F_{AB}\|^M_{A,B=1}$ is the fundamental matrix of the system (19c) (which is known to satisfy the condition $\det F \neq 0$).

Further, by applying the method of variation of an arbitrary parameter, we deduce (18) with $N = 1$, where

$$F^\alpha_{A0} = F_{AB}\int (F)^{-1}_{BC}P^\alpha_c dx_0, \quad A = \overline{1, M}, \quad \alpha = \overline{0, m-1}.$$

The lemma is proved.

Theorem 1. *Suppose that the system of PDE (1) is conditionally invariant with respect to the involutive set of operators (3). Then the ansatz invariant with respect to the set of operators (3) reduces this system.*

Proof. By the definition of the quantity $R_1, R_1 \leq N$. We denote by δ the difference $N - R_1$. Then R_1 equations of the system (2) are linearly independent (without loss of generality, we may assume that it is the first R_1 equations which are linearly independent), and the other δ equations are linear combinations of these first R_1 equations.

By the condition that $R_1 = R_2$, there exists a nonsingular $(R_1 \times R_1)$ matrix $||\lambda_{ab}(x, u)||^{R_1}_{a,b=1}$ such that

$$\lambda_{ab}(\xi_{b\mu}u^\alpha_{x_\mu} - \eta^\alpha_b) = u^\alpha_{x_{a-1}} + \sum_{\mu=R_1}^{n-1}\tilde{\xi}_{a\mu}u^\alpha_{x_\mu} - \tilde{\eta}^\alpha_a, \quad a = \overline{1, R_1} \quad \alpha = \overline{0, m-1}.$$

By the definition of conditional invariance, the system of PDE (1), (2) is invariant with respect to one-parameter transformation groups with generators (3), whence the equivalent system of PDE

$$\begin{aligned} &U_A(x, u, \underset{1}{u}, \ldots, \underset{r}{u}) = 0, \quad A = \overline{1, M}, \\ &u^\alpha_{x_{a-1}} + \sum_{\mu=R_1}^{n-1}\tilde{\xi}_{a\mu}u^\alpha_{x_\mu} - \tilde{\eta}^\alpha_a = 0, \quad a = \overline{1, R_1}, \quad \alpha = \overline{0, m-1} \end{aligned} \tag{20}$$

is invariant with respect to a one-parameter group with generators

$$Q'_a = \lambda_{ab}Q_b = \partial_{x_{a-1}} + \sum_{\mu=R_1}^{n-1}\tilde{\xi}_{a\mu}\partial_{x_\mu} + \tilde{\eta}^\alpha_a\partial_{u^\alpha}. \tag{21}$$

In fact, the action of a one-parameter transformation group with infinitesimal operator Q_a on the solution manifold of the system (20) is equivalent to an identity transformation.

Since the set of operators (21) is involutive (Lemma 1), there exist functions $f^c_{ab}(x,u)$ such that

$$[Q'_a, Q'_b] = f^c_{ab} Q'_c, \quad a,b,c = \overline{1,R_1}. \tag{22}$$

Computing the commutators on the left side of (22) and equating the coefficients of the linearly independent operators ∂_{x_0}, ∂_{x_1}, $\partial_{x_{R_1-1}}$ gives us $f^c_{ab} = 0$, with $a,b,c = 1,\dots,R_1$. Consequently, the operators Q'_a commute. Hence, by Lemma 2, there exists a change of variables (17) that reduces these operators to the form $Q''_a = \partial/\partial x'_{a-1}$.

Expressed in terms of the new variables x' and $u'(x')$, the system (20) takes the form

$$\begin{aligned} &U'_A(x', u', \underset{1}{u'}, \dots, \underset{r}{u'}) = 0, \quad A = \overline{1,M}, \\ &u'^{\alpha}_{x_{a-1}} = 0, \quad \alpha = \overline{0,m-1}, \quad a = \overline{1,R_1}. \end{aligned} \tag{23}$$

Moreover, the system of PDE (23) is conditionally invariant with respect to the set of operators $Q''_a = \partial'_{x_{a-1}}$, $a = 1,\dots,R_1$, whence, by Lemma 3, the system (23) may be rewritten in the form

$$\begin{aligned} &U'_A = F_{AB} W_B(x'_{R_1}, \dots, x'_{n-1}, u', \underset{1}{u'}, \dots, \underset{r}{u'}) + F^{\alpha}_{A\mu} u'^{\alpha}_{x'_\mu}, \\ &A = \overline{1,M}, \quad \alpha = \overline{0,m-1}, \quad \mu = \overline{0,R_1-1}, \\ &u'^{\alpha}_{x'_{a-1}} = 0, \quad \alpha = \overline{0,m-1}, \quad a = \overline{1,R_1}, \end{aligned}$$

where $\det \|F_{AB}\|^{R_1}_{A,B=1} \neq 0$, whence

$$\begin{aligned} &W_A(x'_{R_1}, \dots, x'_{n-1}, u', \underset{1}{u'}, \dots, \underset{r}{u'}) = 0, \\ &u'^{\alpha}_{x'_{a-1}} = 0, \quad A = \overline{1,R_1}, \quad \alpha = \overline{0,m-1}, \quad a = \overline{1,R_1}. \end{aligned} \tag{24}$$

The ansatz of the field $u'^{\alpha} = u'^{\alpha}(x')$ invariant under the involutive set of operators $Q''_c = \partial_{x'_{a-1}}$, $a = 1,\dots,R_1$, is given by the formulas

$$u'^{\alpha} = \varphi^{\alpha}(x'_{R_1}, x'_{R_1+1}, \dots, x'_{n-1}), \quad \alpha = \overline{0,m-1}. \tag{25}$$

Here φ^{α} are arbitrary sufficiently smooth functions.

Substituting (25) in (24), we obtain

$$W_A(x'_{R_1}, \dots, x'_{n-1}, u', \underset{1}{u'}, \dots, \underset{r}{u'}) \equiv W'_A(x'_{R_1}, \dots, x'_{n-1}, \varphi, \underset{1}{\varphi}, \dots, \underset{r}{\varphi}) = 0, \tag{26}$$

where $\underset{s}{\varphi}$ is the set of partial derivatives of the functions $\varphi^{\alpha} = \varphi^{\alpha}(x'_{R_1}, \dots, x'_{n-1})$ of order s.

Rewriting ansatz (25) in terms of the initial variables x and $u(x)$

$$g^{\alpha}(x,u) = \varphi^{\alpha}(f_{R_i}(x,u), \dots, f_{n-1}(x,u)), \quad \alpha = \overline{0,m-1}, \tag{27}$$

yields the ansatz for the field $u^{\alpha} = u^{\alpha}(x), \alpha = 0,\dots,m-1$, invariant with respect to the involutive set of operators (3) that reduces the system (1) to a system of PDE with $n - R_1$ independent variables. The theorem is proved.

Corollary. *Suppose that the operators*

$$Q_a = \xi_{a\mu}(x,u)\partial_{x_\mu} + \eta_a^\alpha(x,u)\partial_{u^\alpha}, \quad a = \overline{1,N}, \quad N \le n-1$$

are the basis elements of a subalgebra of the invariance algebra of the system of equations (1) and, moreover, that $R_1 = R_2$. Then the ansatz invariant in the Lie algebra $\langle Q_1, Q_2, \ldots, Q_N\rangle$ reduces the system (1) to a system of PDE having $n-N$ independent variables.

Proof. From the definition of a Lie algebra it follows that the operators Q_a satisfy (4) with $f_{ab}^c = \mathrm{const}$. Consequently, they form an involutive set of first-order differential operators, which renders the above assertion a direct consequence of Theorem 1.

By the above assertion, the classical reduction theorem for differential equations by means of group-invariant solutions [1, 2, 9] is a special case of Theorem 1. If any one of the operators Q_a does not belong to the invariance algebra of the given equation and if the conditions of Theorem 1 hold, a reduction via Q_a-conditionally invariant ansätzes is obtained (numerous examples of conditionally invariant solutions are constructed in [4–6, 10–14]).

We shall now consider several examples.

Example 1. The Lie-maximal invariance algebra of the Schrodinger equation

$$\Delta_3 u + U(\vec{x}^{\,2})u = 0 \tag{28}$$

with arbitrary function U is the Lie algebra of the rotation group having basis elements

$$J_{ab} = x_a\partial_{x_b} - x_b\partial x_a, \quad a,b = \overline{1,3}. \tag{29}$$

To obtain the ansatz invariant relative to the set of operators (29), the complete set of first integrals of the following system of PDE must be constructed:

$$x_a u_{x_b} - x_b u_{x_a} = 0, \quad a,b = \overline{1,3}. \tag{30}$$

This set contains $3 - R_1$ functionally invariant first integrals, where

$$R_1 = \operatorname{rank}\|\xi_{ab}(x)\|_{a,b=1}^3 = \operatorname{rank}\left\|\begin{matrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{matrix}\right\| = 2.$$

Consequently, the ansatz for the field $u = u(\vec{x})$ invariant with respect to a Lie algebra having basis elements (29) has the form

$$u(\vec{x}) = \varphi(\omega), \tag{31}$$

where $\varphi \in C^2(\mathbb{R}^1, \mathbb{C}^1)$ is an arbitrary smooth function and $\omega = \omega(\vec{x})$ is the first integral of the system of PDE (30). It is not hard to see that $\omega = \vec{x}^{\,2}$ satisfies (30) and, consequently, is the first integral. Substitution of (31) in (28) yields an ordinary differential equation for the function $\varphi(\omega)$:

$$4\omega\ddot{\varphi} + 6\dot{\varphi} + U(\omega)\varphi = 0.$$

Thus, the ansatz for the field $u = u(\vec{x})$ invariant with respect to a three-dimensional Lie algebra with basis elements (29) reduces (28) to a $(3-R_1)$-dimensional PDE (in this case, to an ordinary differential equation).

Example 2. Consider the nonlinear eikonal equation

$$u_{x_0}^2 - u_{x_1}^2 - u_{x_2}^2 - u_{x_3}^2 + 1 = 0. \tag{32}$$

As shown in [15], the maximal invariance algebra of (32) is the 21-parameter conformal algebra $AC(2,3)$. This algebra contains, in particular, a one-dimensional subalgebra generated by the operator $Q = x_0\partial_u - u\partial_{x_0}$.

To obtain the ansatz invariant under the operator Q, the complete set of first integrals of the following PDE must be constructed:

$$uu_{x_0} + x_0 = 0. \tag{33}$$

The solution of (33) is sought for in the implicit form $f(x,u) = 0$, whence $uf_{x_0} - x_0 f_u = 0$.

The complete set of first integrals of the latter PDE is $\omega_0 = u^2 + x_0^2$, $\omega_1 = x_1$, $\omega_2 = x_2$, $\omega_3 = x_3$. Solving $f(\omega_0, \omega_1, \omega_2, \omega_3) = 0$ with respect to ω_0, we have

$$u^2 + x_0^2 = \varphi(\omega_1, \omega_2, \omega_3) \tag{34}$$

Consequently, (34) gives the ansatz of the field $u^\alpha = u^\alpha(x)$ invariant under the operator Q. Solving (34) for u yields

$$u = \{-x_0^2 + \varphi(\omega_1, \omega_2, \omega_3)\}^{1/2}. \tag{35}$$

Let us emphasize that ansatz (34) cannot be represented in the form (12), since the coefficients of Q do not satisfy condition (8).

Substituting (35) in (32) gives us a three-dimensional PDE for the function $\varphi = \varphi(\vec{\omega})$:

$$\varphi_{\omega_1}^2 + \varphi_{\omega_2}^2 + \varphi_{\omega_3}^2 - \varphi^2 = 0.$$

Example 3. A detailed group-theoretic analysis of the nonlinear wave equation

$$u_{tt} = (a^2(u)u_x)_x, \tag{36}$$

where $a(u)$ is some smooth function, was performed in [16]. It was established that the maximal invariance algebra of (36) has the basis operators

$$Q_1 = \partial_t, \quad Q_2 = \partial_x, \quad Q_3 = t\partial_t + x\partial_x, \tag{37}$$

whence the most general group-invariant ansatz for the PDE (36) is given by the formula $u = \varphi(\omega)$, where $\omega = \omega(t,x)$ is the first integral of the PDE

$$\{\alpha\partial_t + \beta\partial_x + \delta(t\partial_t + x\partial_x)\}\omega(t,x) = 0. \tag{38}$$

Here α, β, and δ are arbitrary real constants. Using transformations from the group G with generators of the form (37), Eq. (38) may be reduced to either one of the following equations:

1) $\alpha\omega_t + \beta\omega_x = 0 \quad$ (under $\delta = 0$);
2) $t\omega_t + x\omega_x = 0 \quad$ (under $\delta \neq 0$),

The first integrals of these equations are given by the formulas $\omega = \alpha x - \beta t$ and $\omega = xt^{-1}$, respectively.

Thus, there are two distinct group-invariant ansätzes of the PDE (36) with arbitrary function $a(u)$:

$$\begin{aligned} &1) \quad u(t,x) = \varphi(\alpha x - \beta t), \\ &2) \quad u(t,x) = \varphi(xt^{-1}). \end{aligned} \tag{39}$$

Substitution of the above ansätzes in (36) yields the ordinary differential equations

$$\begin{aligned} &1) \quad (\beta^2 - \alpha^2 a^2(\varphi))\ddot{\varphi} - 2\alpha^2 a(\varphi)\dot{a}(\varphi)\dot{\varphi}^2 = 0, \\ &2) \quad (\omega^2 - a^2(\varphi))\ddot{\varphi} - 2\omega\dot{\varphi} - 2a(\varphi)\dot{a}(\varphi)\dot{\varphi}^2 = 0. \end{aligned}$$

It was established recently [17] that ansätzes (39) do not exhaust the complete set of ansätzes reducing the PDE (36) to ordinary differential equations. This result is a consequence of conditional symmetry, a property that is not found within the framework of the infinitesimal Lie method.

Let us show, following [17], that (36) is conditionally invariant under the operator

$$Q = \partial_t - \varepsilon a(u)\partial_x, \tag{40}$$

where $\varepsilon = \pm 1$.

Proceeding on the basis of the second extension of Q in (36), we have

$$\tilde{Q}\{u_{tt} - (a^2(u)u_x)_x\} = \varepsilon\dot{a}u_x\{u_{tt} - (a^2u_x)_x\} + \varepsilon(\dot{a}\dot{u}_x + \dot{a}\partial_x)(u_t^2 - a^2u_x^2), \tag{41}$$

whence it follows that the PDE (36) is Lie-noninvariant with respect to a group with infinitesimal operator (40). But if the additional constraint

$$Q_u \equiv u_t - \varepsilon a(u)u_x = 0 \tag{42}$$

is imposed on $u(t,x)$, the right side of (41) vanishes. Consequently, the system (36), (42) is Lie-invariant with respect to a group with generator (40), whence we conclude that the initial PDE (36) is conditionally invariant under the operator Q.

The complete set of functionally independent first integrals of (42) may be chosen in the form $\omega_1 = u$, $\omega_2 = x + \varepsilon a(u)t$.

Consequently, the ansatz invariant under the operator Q is given by the formula $\omega_2 = \varphi(\omega^1)$, or

$$x + \varepsilon a(u)t = \varphi(u), \tag{43}$$

where $\varphi(u)$ is an arbitrary sufficiently smooth function.

Substituting (43) in (36) leads us to conclude that the PDE (36) is satisfied identically. Put differently, (43) gives a solution of the nonlinear equation (36) for an arbitrary function $\varphi(u)$. Recall that solutions that are obtained by means of the group-invariant ansätzes (39) contain two arbitrary constants of integration, and cannot, in theory, contain arbitrary functions.

Thus, the conditional symmetry of PDE enlarges the range of possibilities for reduction of PDE in an essential way.

Example 4. Consider the system of nonlinear Dirac equations

$$\{i\gamma_\mu\partial_\mu - \lambda(\bar{\psi}\psi)^{1/2k}\}\psi = 0, \tag{44}$$

where γ_μ, $\mu = 0, \ldots, 3$, are (4×4) Dirac matrices, $\psi = \psi(x_0, x_1, x_2, x_3)$ a four-dimensional complex column function, $\bar{\psi} = (\psi^*)^T \gamma_0$, λ, k real constants, and $\partial_\mu = \partial/\partial x_\mu$, $\mu = 0, \ldots, 3$.

It is well known (cf. [5]) that the Lie-maximal invariance group of the system of PDE (44) is the 11-parameter extended Poincaré group complemented with the 3-parameter group of linear transformations in the space ψ^α, $\psi^{*\alpha}$. In [5, 10] it is established that the conditional symmetry of the nonlinear Dirac equation is essentially broader. From [10], it follows that the system: (44) is conditionally invariant with respect to the involutive set of operators

$$
\begin{aligned}
&Q_1 = \frac{1}{2}(\partial_0 - \partial_3), \quad Q_2 = \omega_1 \partial_2 - \{B_1 \psi\}^\alpha \partial_{\psi^\alpha}, \\
&Q_3 = \frac{1}{2}(\partial_0 + \partial_3) - \dot{\omega}_1 (x_1 \partial_1 + x_2 \partial_2) - \dot{\omega}_2 \partial_1 - \{B_2 \psi\}^\alpha \partial_{\psi^\alpha},
\end{aligned}
\tag{45}
$$

where B_1 and B_2 are (4×4) matrices of the form

$$
\begin{aligned}
B_1 &= \frac{1}{2}(1 - 2k)\dot{\omega}_1 \gamma_2 (\gamma_0 + \gamma_3), \\
B_2 &= -k\dot{\omega}_1 + (2\omega_1)(2\dot{\omega}_1^2 - \omega_1 \ddot{\omega}_1)(\gamma_1 x_1 + 2(k-1)\gamma_2 x_2)(\gamma_0 + \gamma_3) + (2\omega_1)^{-1} \times \\
&\quad \times ((2\dot{\omega}_1 \dot{\omega}_2 - \omega_1 \ddot{\omega}_2)\gamma_1 + 2(\omega_3 \dot{\omega}_1 - \omega_1 \dot{\omega}_3)\gamma_2)(\gamma_0 + \gamma_3),
\end{aligned}
$$

ω_1, ω_2, and ω_3 are arbitrary smooth functions of $x_0 + x_3$, and $\{\psi\}^\alpha$ denotes the α-th component of the function ψ. Since the coefficients of the operators (45) satisfy conditions (8), they may be rewritten in non-Lie form:

$$
\begin{aligned}
&Q_1 = \frac{1}{2}(\partial_0 - \partial_3), \quad Q_2 = \omega_1 \partial_2 + B_1, \\
&Q_3 = \tfrac{1}{2}(\partial_0 + \partial_3) - \dot{\omega}_1 (x_1 \partial_1 + x_2 \partial_2) - \dot{\omega}_2 \partial_1 + B_2.
\end{aligned}
$$

Consequently, the ansatz of the field $\psi(x)$ invariant with respect to the set of operators Q_1, Q_2, Q_3 must be found in the form (12), where $A(x)$ is a (4×4) matrix and $\omega = \omega(x)$ a real function satisfying the following system of PDE

$$
\begin{aligned}
&\frac{1}{2}(A_{x_0} - A_{x_2}) = 0, \quad \omega_1 A_{x_2} + B_1 A = 0, \\
&\frac{1}{2}(A_{x_0} + A_{x_3}) - (\dot{\omega}_1 x_1 + \dot{\omega}_2) A_{x_1} - \dot{\omega}_1 x_2 A_{x_2} - B_2 A = 0, \\
&\omega_{x_0} - \omega_{x_3} = 0, \quad \omega_{x_2} = 0, \\
&\omega_{x_0} + \omega_{x_3} - 2(\dot{\omega}_1 x_1 + \dot{\omega}_2)\omega_{x_1} - 2\dot{\omega}_1 x_2 \omega_{x_2} = 0.
\end{aligned}
$$

Omitting the steps in integration of the above system, let us write down the final result, the ansatz for the field $\psi = \psi(x)$ invariant with respect to the involutive set of operators (45):

$$
\begin{aligned}
\psi(x) = \omega_1^k \exp\{&(2\omega_1)^{-1}(\dot{\omega}_1 x_1 + \dot{\omega}_2)\gamma_1(\gamma_0 + \gamma_3) + \\
&+ (2\omega_1)^{-1}((2k-1)\dot{\omega}_1 x_2 + \omega_3)\gamma_2(\gamma_0 + \gamma_3)\}\varphi(\omega_1 x_1 + \omega_2).
\end{aligned}
\tag{46}
$$

This ansatz reduces me system of PDE (44) to a system of ordinary differential equations for the 4-component function $\varphi = \varphi(\omega)$,

$$
i\gamma_1 \dot{\varphi} - \lambda(\bar{\varphi}\varphi)^{1/2k}\varphi = 0. \tag{47}
$$

The general solution of the system (47) has the form [5]

$$\varphi = \exp\{i\lambda\gamma_1(\bar{\chi}\chi)^{1/2k}\omega\}\chi,$$

where χ is an arbitrary constant 4-component column. Substituting the resulting expression for $\varphi = \varphi(\omega)$ in (46) gives us the class of exact solutions of the nonlinear Dirac equation containing three arbitrary functions.

Nonlinear equations of mathematical and theoretical physics that admit nontrivial conditional symmetry have been analyzed in [14].

3. Reduction of number of independent and number of dependent variables of PDE. Suppose (3) is an involutive set of operators that satisfy the condition $R_2 - R_1 = \delta > 0$. In this case we have to modify somewhat the above technique of reducing PDE by means of ansätzes invariant with respect to the involutive set (3). Note that the case in which (3) are basis operators of a subalgebra of the Lie invariance algebra of a given equation satisfying the condition $R_1 < R_2$ leads to "partially invariant" solutions [18].

We wish to solve the initial system of PDE in implicit form:

$$\omega^\alpha(x, u) = 0, \quad \alpha = \overline{0, m-1}, \tag{48}$$

where ω^α are smooth functions satisfying the condition

$$\det \|\partial\omega^\alpha/\partial u^\beta\|_{\alpha,\beta=0}^{m-1} \neq 0. \tag{49}$$

As a result, (1) and (2) assume the form

$$H_A(x, u, \omega, \underset{1}{\omega}, \ldots, \underset{r}{\omega}) = 0, \quad A = \overline{1, M}, \tag{50}$$

$$\xi_{a\mu}(x, u)\omega^\alpha_{x_\mu} + \eta^\beta_a(x, u)\omega^\alpha_{u^\beta} = 0, \quad a = \overline{1, N}, \tag{51}$$

where $\underset{s}{\omega} = \{\partial^s\omega/\partial x_{\mu_1}\cdots\partial_{x_{\mu_p}}\partial u^{\alpha_1}\cdots\partial u^{\alpha_q},\ p + q = s\}$.

It is clear that, as they are defined in the space of the variables x, u, $\omega(x, u)$, the operators (3) satisfy the condition $R'_1 = R'_2$ (since the coefficients of ∂_{ω^α} are all zero). By means of the same reasoning as in the proof of Theorem 1, we may establish the following result. There exists a change of variables (17) that reduces the system (51) to the form

$$\omega^\alpha_{x'_\mu} = 0, \quad \mu = \overline{0, R_1 - 1}, \quad \omega^\alpha_{u'^\beta} = 0, \quad \beta = \overline{0, \delta - 1}. \tag{52}$$

If the system (48), (50) is conditionally invariant with respect to the set of operators (3) and if condition (52) holds, it may be rewritten as follows:

$$\begin{gathered} \omega^a(x', u') = 0, \quad \alpha = \overline{0, m-1}, \\ H'_A(x'_{R_1}, \ldots, x'_{n-1}, u'^\delta, \ldots, u'^{m-1}, \omega, \underset{1}{\omega}, \ldots, \underset{r}{\omega}) = 0, \end{gathered} \tag{53}$$

where the symbol $\underset{s}{\omega}$ denotes the collection of partial derivatives of the function ω of order s with respect to the variables $x'_{R_1}, \ldots, x'_{n-1}, u'^\delta, \ldots, u'^{m-1}$.

Integrating (52) yields the ansatz of the field w^α:

$$\omega^\alpha = F^\alpha(x'_{R_1}, \ldots, x'_{n-1}, u'^\delta, \ldots, u'^{m-1}), \quad \alpha = \overline{0, m-1}, \tag{54}$$

where F^α are arbitrary smooth functions. But the ansatz of the field $u'^\alpha(x')$ cannot be obtained by substituting (54) in the relations $\omega^\alpha(x', u'(x')) = 0$, $\alpha = 0, \ldots, m-1$, since the inequality $R_2 - R_1 = \delta > 0$ violates the condition (49) (if $\delta > 0$, the matrix $\|\partial\omega^\alpha/\partial u^{i\beta}\|_{\alpha,\beta=0}^{m-1}$ has null columns).

To overcome this problem, we shall, by definition, let the expressions

$$F^\alpha(x'_{R_1}, \ldots, x'_{n-1}, u'^\delta, \ldots, u'^{m-1}) = 0, \quad \alpha = \overline{\delta, m-1},$$
$$u'^j = C_j, \quad j = \overline{0, \delta-1}$$

be the ansatz of the field $u'^\alpha = u'^\alpha(x')$ invariant with respect to the set of operators

$$Q_j = \partial_{x'_{j-1}}, \quad j = \overline{1, R_1}, \quad X_i = \partial_{u'^{i-1}}, \quad i = \overline{1, \delta}. \tag{55}$$

The latter ansatz may be rewritten in the form

$$\begin{aligned} &u'^\alpha = C_\alpha, \quad \alpha = \overline{0, \delta-1}, \\ &u'^{\alpha+\beta} = \varphi^\beta(x'_{R_1}, \ldots, x'_{n-1}), \quad \beta = \overline{0, m-\delta-1}, \end{aligned} \tag{56}$$

where φ^β are arbitrary smooth functions and C_α are arbitrary constants.

Rewriting (56) in terms of the initial variables gives us

$$\begin{aligned} &g^\alpha(x, u) = C_\alpha, \quad \alpha = \overline{0, \delta-1}, \\ &g^{\beta+\delta}(x, u) = \varphi^\beta(f_{R_1}(x, u), \ldots, f_{n-1}(x, u)), \quad \beta = \overline{0, m-\delta-1}. \end{aligned} \tag{57}$$

Moreover, substituting (57) in the initial system of PDE (1) or, equivalently, substituting the expressions $\omega^\alpha = g^\alpha - C_\alpha$, $\alpha = 0, \ldots, \delta-1$, $\omega^\beta = g^{\beta+\delta} - \varphi^\beta$, $0 \le \beta \le m-\delta-1$ in the PDE (50) yields a system of M differential equations for $m-\delta$ functions. Consequently, the dimension of the system (1) decreases by R_1 independent and δ dependent variables.

Let us rewrite (57) in a form more convenient in applications. For this purpose, note that, without loss of generality, we may renumber the operators (3) satisfying the condition $R_2 - R_1 = \delta > 0$ in such a way that the first R_1 operators satisfy the condition

$$\operatorname{rank} \|\xi_{a\mu}\|_{a=1}^{R_1}{}_{\mu=0}^{n-1} = \operatorname{rank} \|\xi_{a\mu}, \eta_a^\alpha\|_{a=1}^{R_1}{}_{\alpha=0}^{m-1}{}_{\mu=0}^{n-1}$$

and the last $N - R_2$ operators are linear combinations of the previous R_2 operators.

Let $\omega_j(x, u)$, $j = 1, \ldots, m+n-R_2$, be the complete set of functionally independent first integrals of the system (51) and, moreover,

$$\operatorname{rank} \|\partial\omega_j/\partial u^\alpha\|_{j=1}^{m-\delta}{}_{\alpha=0}^{m-1} = m-\delta$$

and let $\rho_j(x, u)$ be the solutions of the equations $Q_{1+R_1}\rho(x, u) = 1$ with $i = 1, 2, \ldots, \delta$. Then (57) may be expressed in the following equivalent form:

$$\begin{aligned} &\rho_i(x, u) = C_i, \quad i = \overline{1, \delta}, \\ &\omega_j(x, u) = \varphi^j(\omega_{R_1}(x, u), \ldots, \omega_{n-1}(x, u)), \quad j = \overline{1, m-\delta}. \end{aligned} \tag{58}$$

Definition 4. *Expressions (58) are called the ansatz of the field $u^\alpha = u^\alpha(x)$ invariant with respect to the involutive set of operators (3) provided $R_2 - R_1 \equiv \delta > 0$.*

The above reasoning may be summarized in the form of a theorem.

Theorem 2. *Suppose that the system of PDE (1) is conditionally invariant with respect to the involutive system of operators (3) and, moreover, that $R_1 < R_2$. Then the system (1) is reduced by the ansatz invariant with respect to the set of operators (3).*

Example 1. The system of two wave equations

$$\Box u = 0, \quad \Box v = 0 \tag{59}$$

is invariant with respect to a one-parameter group with infinitesimal operator $Q = \partial_v$. Since $R_1 = 0$ and $R_2 = 1$, the parameter δ is equal to 1. The complete set of first integrals of the equation $\partial\omega(x,u,v)/\partial v = 0$ is given by the functions

$$\omega_\mu = x_\mu, \quad \mu = \overline{0,3}, \quad \omega_4 = u,$$

whence the ansatz for the field $u(x)$, $v(x)$ invariant under the operator Q has the form (58),

$$u = \varphi(\omega_0, \omega_1, \omega_2, \omega_3), \quad v = C, \quad C = \text{const}.$$

Substituting the above expressions in (59) yields

$$\varphi_{\omega_0\omega_0} - \varphi_{\omega_1\omega_1} - \varphi_{\omega_2\omega_2} - \varphi_{\omega_3\omega_3} = 0$$

i.e., the number of dependent variables of the initial system (59) is reduced.

Example 2. Consider the system of nonlinear Thirring equations

$$iv_x = mu + \lambda_1|u|^2v, \quad iu_y = mv + \lambda_2|v|^2u, \tag{60}$$

where u, v are complex functions of x, y and λ_1, λ_2 are real constants.

The above system admits a one-parameter transformation group with generator

$$Q = iu\partial_u + iv\partial_v - iu^*\partial_{u^*} - iv^*\partial_{v^*}.$$

Following the change of variables

$$u(x,y) = H_1(x,y)\exp\{iZ_1(x,y) + iZ_2(x,y)\},$$
$$v(x,y) = H_2(x,y)\exp\{iZ_1(x,y) - iZ_2(x,y)\},$$

where H_j and Z_j are the new dependent variables, Q assumes the form $Q' = \partial_{Z_1}$. Consequently, the ansatz invariant under Q has the form

$$\begin{aligned} u(x,y) &= H_1(x,y)\exp\{iC + iZ_2(x,y)\},\\ v(x,y) &= H_2(x,y)\exp\{iC - iZ_2(x,y)\}. \end{aligned} \tag{61}$$

Substitution of (61) in (60) yields a system of four PDE for the three functions H_1, H_2, and Z_2,

$$H_{2x} = mH_{1x}\sin 2Z_2, \quad H_{1y} = -mH_2\sin 2Z_2,$$
$$H_2Z_{2x} = mH_1\cos 2Z_2 + \lambda_1H_1H_2^2,$$
$$-H_1Z_{2y} = mH_2\cos 2Z_2 + \lambda_2H_2H_1^2.$$

Example 3. A group analysis of the one-dimensional gas dynamics equations

$$u_t + uu_x + \rho^{-1}p_x = 0, \quad \rho_t + (u\rho)_x = 0, \quad p_t + (up)_x + (\gamma-1)pu_x = 0 \tag{62}$$

has been carried out by Ovsyannikov [1], who established, in particular, that the invariance algebra of the system of PDE (62) contains the basis element

$$Q = p\partial_p + \rho\partial_\rho. \tag{63}$$

The complete set of functionally independent first integrals of the equation $Qw(t, x, u, p, \rho) = 0$ is: $\omega_1 = u$, $\omega_2 = p\rho^{-1}$, $\omega_3 = t$, and $\omega_4 = x$. Consequently, the ansatz invariant under Q (63) may be chosen in the form

$$u = \varphi^1(t,x), \quad p\rho^{-1} = \varphi^2(t,x), \quad \ln\rho + F(p\rho^{-1}) = C, \tag{64}$$

where $C = \mathrm{const}$ and F is some smooth function.

Substituting the ansatz (64) in the system of PDE (62) yields a system of three differential equations for the two unknown functions $\varphi^1(t,x)$ and $\varphi^2(t,x)$:

$$\begin{aligned} &\varphi^1_t + \varphi^1\varphi^1_x - \varphi^2\dot{F}(\varphi^2)\varphi^2_x = 0, \\ &\varphi^2_t + \varphi^1\varphi^2_x + (\gamma-1)\varphi^2\varphi^1_x = 0, \\ &\varphi^1_x((1-\gamma)\varphi^2\dot{F}(\varphi^2) - 1) = 0, \end{aligned} \tag{65}$$

Thus we have achieved a reduction of the number of dependent variables of the gas dynamics equations.

It is of interest that if $\varphi^1_x \neq 0$, it follows from the third equation of the system (65) that $F = \lambda + (1-\gamma)^{-1}\ln(\rho^{-1}p)$. Substituting this expression in (62) yields $p = k\rho^\gamma$, $k \in \mathbb{R}^1$, which is the relation that characterizes a polytropic gas.

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