Nonlocal symmetry and generating solutions for Harry–Dym type equations

W.I. FUSHCHYCH, V.A. TYCHININ

Изучена нелиевская симметрия уравнений $u_0 = f(u)u_{111}$, $w_0 = g(w_1)w_{111}$, выделены уравнения, допускающие нелокальную линеаризацию; установлены формулы размножения решений. Для редукции нелинейных уравнений применяется нелиевский анзац $u = h(x)\dot\varphi(\omega) + f(x)\varphi(\omega) + g(x)$.

1. Let us consider two classes of one-dimensional third order nonlinear equations

$$u_0 - f(u)u_{111} = 0, \tag{1}$$

$$w_0 - g(w_{11})w_{111} = 0, \tag{2}$$

$$u_\mu = \partial_\mu u = \frac{\partial u}{\partial x_\mu}, \quad u_{\underbrace{1\ldots1}_{n}} = \partial_1^n u = \frac{\partial^n u}{\partial x_1^n}, \quad w_\mu = \partial_\mu w = \frac{\partial w}{\partial x_\mu},$$

$$w_{\underbrace{1\ldots1}_{n}} = \partial_1^n w = \frac{\partial^n w}{\partial x_1^n} \quad (\mu = 0, 1,\ n \in \mathbb{N}),$$

where $f(u)$, $g(w_{11})$ are arbitrary smooth functions.

In the present paper linearizable equations are picked out from the sets of equations (1) and (2) by means of nonlocal transformations. Non-Lie symmetry of equations (1), (2) is investigated. The formulas of generating solutions for nonlinear equations belonging to classes (1), (2) are obtained. Non-Lie ansatz

$$u = h(x)\dot\varphi(\omega) + f(x)\varphi(\omega) + g(x), \quad x = (x_0, x_1), \quad \dot\varphi(\omega) = \frac{d\varphi}{d\omega}, \tag{3}$$

which should be consider as the generalization of ansatz [1]

$$u = f(x)\varphi(\omega) + g(x)$$

is used for reducing equations (1), (2) to ODE. Some sets of exact partial solutions for nonlinear equations are constructed.

Note 1. The Eq. (1) is equivalent to equation

$$z_0 - \partial_1^3 c(z) = 0 \tag{4}$$

The connection between these equations is given by transformation

$$c(z) = u. \tag{4a}$$

Thereby, the equality

$$f(u) = \dot c(c^{-1}[u])$$

Доклады АН Украины, 1992, № 9, С. 24–30.

holds. Here $c^{-1}[u]$ is the function inverse to $c(u)$. In that case, when $f(u) = u^3$, $c(z) = z^{-\frac{1}{2}}$, the Eq. (4) coincides with the known Harry–Dym equation [2].

2. Nonlocal symmetry. Consider the Eq. (4)

$$z_0 = \partial_1^3 c(z) = \partial_1^2(\dot{c}(z)z_1).$$

The substitution

$$z = w_{11} \tag{5a}$$

reduces (4) to equation

$$w_0 = \dot{c}(w_{11})w_{111}. \tag{5}$$

Making use the Euler–Ampere transformation

$$w = y_1 v_1 - v, \quad x_1 = v_1, \quad x_0 = y_0, \quad v = v(y_0, y_1), \quad v_{11} \neq 0, \tag{6}$$

under Eq. (5), we get

$$v_0 = \dot{c}(v_{11}^{-1})v_{11}^{-3}v_{111}. \tag{7}$$

Using the substitution

$$v_{11} = z(y_0, y_1) \tag{7a}$$

in equation (7), twice differentiated on y_1, we get

$$z_0 = \partial_1^2\left(\dot{c}(z^{-1})z^{-3}z_1\right). \tag{8}$$

It follows from (8), that transformations (5a), (6), (7a) do not take out any Eq. (4) beyond the this class of equations, none the less the set of Eq. (4) is not invariant under these transformations. If function $\dot{c}(z^{-1})z^{-3}$ in (8) satisfies the condition

$$\dot{c}(z^{-1})z^{-3} = \lambda, \quad \lambda = \text{const.} \tag{9a}$$

then Eq. (4) is linearisable. When the condition

$$\dot{c}(z) = \dot{c}(z^{-1})z^{-3} \tag{9b}$$

holds, then the Eq. (8) coincides with initial equation (4), i.e. these equations are invariant with respect to nonlocal transformations (5a), (6), (7a).

The condition (9b) allows to describe all the equations of the class (4) which are invariant with respect to transformations (5a), (6), (7a).

Theorem 1. *The Eq. (4) is invariant with respect to transformations. (5a), (6), (7a), if it has the form*

$$z_0 = \partial_1^2\left(z^{-\frac{1}{2}}\varphi(\ln z)z_1\right). \tag{10}$$

Here $\varphi(\alpha)$ is an arbitrary smooth even function.

Corollary 1. *The Eq. (1) is invariant with respect to transformations (4a), (5a), (6), (7a), (4a) if it has the form*

$$u_0 = \left(c^{-1}[u]\right)^{-\frac{3}{2}}\varphi\left(\ln c^{-1}[u]\right)u_{111}, \tag{11}$$

where $c^{-1}[u]$ is the function inverce to $c(u)$ and it is determined implicitly from the formula

$$u = \int z^{-\frac{3}{2}}\varphi(\ln z)dz. \tag{12}$$

Example 1. From the theorem 1 and the corollary 1 under $\varphi(\alpha) = 1$ we get the following invariant equations

$$z_0 = \partial_1^2\left(-\frac{1}{2}z^{-\frac{3}{2}}z_1\right) = \partial_1^3\left(z^{-\frac{1}{2}}\right), \tag{13}$$

$$u_0 = u^3 u_{111}. \tag{14}$$

So, Eq. (13) is known as Harry–Dym equation. Letting $\varphi(\alpha) = \cos\alpha$, we obtain the equation

$$z_0 = \partial_1^2\left(z^{-\frac{3}{2}}\cos\ln z z_1\right) \tag{15}$$

and corresponding to it equation of the class (1)

$$u_0 = \left(c^{-1}[u]\right)^{-\frac{3}{2}}\cos\ln\left(c^{-1}[u]\right)u_{111}. \tag{16}$$

Here $c^{-1}[u]$ is determined implicitly by formula

$$u = \frac{4}{5}\left[\sin\ln z - \frac{1}{2}\cos\ln z\right]z^{-\frac{1}{2}}. \tag{17}$$

So, we establish that the equations

$$u_0 = u^{\frac{3}{2}}u_{111}, \quad z_0 = \partial_1^3\left(z^{-2}\right), \quad w_0 = w_{11}^{-3}w_{111} \tag{18a,b,c}$$

are reduced to the linear equation

$$v_0 = v_{111} \quad (\lambda = 1) \tag{19}$$

and that, in particular, the Harry–Dym equation and connected with it equations

$$u_0 = u^3 u_{111}, \quad z_0 = \partial_1^3\left(z^{-\frac{1}{2}}\right), \quad w_0 = w_{11}^{-\frac{3}{2}}w_{111} \tag{20a,b,c}$$

are invariant with respect to corresponding nonlocal transformations.

3. The nonlocal superposition and the generating solutions.

Theorem 2. *The solutions superposition formula for Eq. (18a)*

$$u_0 = u^{\frac{3}{2}}u_{111} \tag{18a}$$

has the form

$$\overset{(3)}{u}(x_0, x_1) = \overset{(1)}{u}(x_0, \tau^1) + \overset{(2)}{u}(x_0, \tau^2) + 2\sqrt{\overset{(1)}{u}(x_0, \tau^1)\,\overset{(2)}{u}(x_0, \tau^2)}, \tag{21a}$$

$$\frac{d\tau^1}{\sqrt{\overset{(1)}{u}(x_0, \tau^1)}} = \frac{d\tau^2}{\sqrt{\overset{(2)}{u}(x_0, \tau^2)}}, \tag{21b}$$

$$\tau^1 + \tau^2 = x_1, \tag{21c}$$

$$\tau_0^1 = \frac{1}{2}\frac{\sqrt{\overset{(1)}{u}(x_0,\tau^1)\overset{(2)}{u}(x_0,\tau^2)}}{\sqrt{\overset{(1)}{u}(x_0,\tau^1)}+\sqrt{\overset{(2)}{u}(x_0,\tau^2)}}\left[\overset{(1)}{u}_{11}(x_0,\tau^1)+\overset{(2)}{u}_{11}(x_0,\tau^2)\right]. \tag{21d}$$

Let us illustrate the efficiency of the formula (21).

Example 2. Let us take, as initial, the simplest stationary solutions of Eq. (18a)

$$\overset{(1)}{u}(\overset{1}{x}_1) = (\overset{1}{x}_1)^2, \quad \overset{(2)}{u}(\overset{2}{x}_1) = 4(\overset{2}{x}_1)^2.$$

Replace $\overset{1}{x}_1$ and $\overset{2}{x}_1$ in this solutions for parameters τ^1, τ^2

$$\overset{(1)}{u} = (\tau^1)^2, \quad \overset{(2)}{u} = 4(\tau^2)^2.$$

The differential Eq. (21b) takes the form

$$\frac{d\tau^2}{d\tau^1} = 2\frac{\tau^2}{\tau^1} \tag{22}$$

and has the general solution

$$\tau^2 = -\frac{(\tau^1)^2}{2\lambda(x_0)}. \tag{23}$$

Here $\lambda(x_0)$ is an arbitrary smooth function. The equation for τ^1

$$(\tau^1)^2 - 2\lambda\tau^1 + 2\lambda x_1 = 0 \tag{24}$$

we obtain making use of (21c) and replacing in (23) τ^2 for the expression $x_1 - \tau^1$. From (24) we find

$$\overset{(3)}{u}(x_0,x_1) = [\tau^1 + 2\tau^2]^2 = [2x_1 - \tau^1]^2 = \left[2x_1 - \lambda \pm \sqrt{\lambda^2 - 2\lambda x_1}\right]^2. \tag{25}$$

The function $\lambda(x_0)$ can be defined more precisely from the condition that τ^1 is the solution of Eq. (21d). As a result, we get the equation for $\lambda(x_0)$

$$\dot{\lambda} = -6\lambda.$$

Therefore

$$\lambda = c\exp(-6x_0),$$

where c is an arbitrary constant. So, the new solution $\overset{(3)}{u}$, which is constructed from $\overset{(1)}{u}$ and $\overset{(2)}{u}$, is of the form

$$\overset{(3)}{u}(x_0,x_1) = \left[2x_1 - ce^{-6x_0} \pm \sqrt{c^2e^{-12x_0} - 2cx_1e^{-6x_0}}\right]^2. \tag{26}$$

Example 3. Let us choose, as initial, the following two solutions of Eq. (18a):

$$\overset{(1)}{u} = x_1^2, \quad \overset{(2)}{u} = 9x_1^2$$

and rewrite them in variables τ^1 and τ^2

$$\overset{(1)}{u} = (\tau^1)^2, \quad \overset{(2)}{u} = 9(\tau^2)^2.$$

Unlike the previous example when solving ODE (21b), one obtains the cubic equation for τ^1

$$(\tau^1)^3 - \lambda\tau^1 + \lambda x_1 = 0, \quad \lambda = \lambda(x_0). \tag{27}$$

The real solution of the Eq. (27) can be written in the form

$$\tau^1 = -3\lambda^{-1}\cos\frac{1}{3}\arccos\lambda x_1, \quad \lambda = \frac{1}{2}3\sqrt{3}\lambda^{-\frac{1}{2}}(x_0). \tag{27a}$$

The solution $\overset{(3)}{u}$

$$\overset{(3)}{u}(x_0, x_1) = [3x_1 - 2\tau^1]^2 = 9\left[x_1 - \frac{2}{3}\tau^1\right]^2 = 9\left[x_1 + 2\lambda^{-1}\cos\frac{1}{3}\arccos\lambda x_1\right]^2 \tag{28}$$

we find from the formula (21a). The condition on $\lambda(x_0)$ is of the form

$$\dot{\lambda} = 12\lambda.$$

Hence

$$\lambda = c\exp(12x_0).$$

c is an arbitrary constant. Finally, one can write solution $\overset{(3)}{u}$ in the form

$$\overset{(3)}{u}(x_0, x_1) = 9\left[x_1 + 2ce^{-12x_0}\cos\frac{1}{3}\arccos\left(cx_1e^{12x_0}\right)\right]^2. \tag{29}$$

4. The non-group generating of solutions. For equations of the class (11) we can write formula of generating solutions. Let $\overset{(1)}{u}(x_0, x_1)$ be a known partial solution of nonlinear Eq. (11) and $\overset{(2)}{u}(x_0, x_1)$ is its new solution, then the following assertion holds true.

Theorem 3. *The formula of generating solutions for Eq. (11), giving by nonlocal symmetry (4a), (5a), (6), (7a), (4a) has the form*

$$\overset{(2)}{u}(x_0, x_1) = \left[x_1\tau - \int [\overset{(1)}{u}(x_0, \tau)]^{-2}d\tau\right]^{\frac{1}{2}}, \tag{30a}$$

$$= [\overset{(1)}{u}(x_0, \tau)]^{-1}, \tag{30b}$$

$$x_1 = \int [\overset{(1)}{u}(x_0, \tau)]^{-2}d\tau, \tag{30c}$$

$$\tau_0 = \partial_1\left(\tau_1^{-\frac{3}{2}}\tau_{11}\right). \tag{30d}$$

Let us demonstrate the efficiency of the formula (30) for Eq. (20a) on several simple examples.

Example 4. Let $\overset{(1)}{u}$. Then

$$\overset{(2)}{u}(x_0,\tau) = \left[x_1\tau - \int\left(\int d\tau\right)d\tau\right]^{\frac{1}{2}}, \quad x_1 = \int d\tau = \tau + \lambda_1(x_0).$$

$\lambda_1(x_0)$ is an arbitrary function. Calculating the integral in the first equality and resolving the second one with respect to τ, we get

$$\overset{(2)}{u}(x_0,\tau) = \left[x_1\tau - \frac{1}{2}\tau^2 - \lambda_1\tau + \lambda_2(x_0)\right]^{\frac{1}{2}}, \tag{31}$$

$$\tau = x_1 - \lambda_1(x_0). \tag{32}$$

Having excluded parameter τ from equalities of the system (31), (32) we get the solution $\overset{(2)}{u}(x_0,x_1)$ in explicit form

$$\overset{(2)}{u}(x_0,x_1) = \left(\lambda_2 - \frac{1}{2}\lambda_1^2\right)^{\frac{1}{2}} \equiv \lambda_3 = \text{const.} \tag{33}$$

Example 5. The function

$$\overset{(1)}{u}(x_0,x_1) = \frac{1}{4}(\lambda_1 - x_1)^2 \tag{34}$$

is the solution of the Eq. (20a). λ_1 is an arbitrary constant. It follows from relations (30b,c), that

$$\overset{(2)}{u}(x_0,\tau) = 4(\lambda_1 - \tau)^{-2}, \tag{35}$$

$$x_1 = \frac{16}{3}(\lambda_1 - \tau)^{-3} + \lambda_2(x_0). \tag{36}$$

Resolving (36) with respect to τ, one obtain

$$\tau = h(x_1 - \lambda_2(x_0))^{-\frac{1}{3}} + \lambda_1, \quad h = -\left(\frac{3}{16}\right)^{-\frac{1}{3}}. \tag{37}$$

Substituting τ from the formula (37) into condition (30d), one gets

$$\dot{\lambda}_2 = -1.$$

Let us substitute specified value of τ into the formula (35) and find the solution $\overset{(2)}{u}$

$$\overset{(2)}{u}(x_0,x_1) = k(x_0 + x_1)^{\frac{2}{3}}, \quad k = \left(\frac{3}{2}\right)^{\frac{2}{3}}. \tag{38}$$

5. The non-Lie ansätze. Let us consider the ansatz of the form

$$w = h(x)\dot{\varphi}(\omega) + f(x)\varphi(\omega) + g(x), \quad x = (x_0,x_1), \quad \dot{\varphi}(\omega) = \frac{d\varphi}{d\omega} \tag{39}$$

for constructing of solutions for Eq. (20c):

$$w_0 - w_{11}^{-\frac{3}{2}} w_{111} = 0. \tag{20c}$$

Let us summarize the results obtained for Eq. (20c) in table.

	ω	$h(x)$	$f(x)$	$g(x)$
1	x_0	0	$\frac{x_1^{-2}}{6}\varphi^{-3}(\omega)$	$\lambda_1(x_0)x_1+\lambda_2(x_0)$,
2	$x_0+x_1^{-1}$	1	$-2x_1$	$\lambda_1(x_0)x_1+\lambda_2(x_0)$,
3	$\ln x_0+x_1^{-1}$	$x_0^{-\frac13}$	$-2x_1x_0^{-\frac13}$	$\lambda_1(x_0)x_1+\lambda_2(x_0)$,
4	$\ln x_0+\operatorname{arth}x_1$	$-x_0^{-\frac13}$	$2x_1x_0^{-\frac13}$	$x_0^{-\frac13}\Big\{-2\int\varphi(\omega)dx_1+2\int\varphi(\omega)\times$ $\times\frac{x_1^2+1}{x_1^2-1}dx_1+8\int\left(\int\varphi(\omega)\frac{x_1}{(x_1^2-1)^2}dx_1\right)\times$ $\times dx_1+\lambda_1(x_0)x_1+\lambda_2(x_0)\Big\}$,
5	$\ln x_0-\operatorname{arctg}x_1$	$x_0^{-\frac13}$	$-2x_1x_0^{-\frac13}$	$x_0^{-\frac13}\Big\{2\int\varphi(\omega)dx_1-2\int\varphi(\omega)\times$ $\times\frac{1-x_1^2}{1+x_1^2}dx_1-8\int\left(\int\varphi(\omega)\frac{x_1}{(1+x_1^2)^2}dx_1\right)\times$ $\times dx_1+\lambda_1(x_0)x_1+\lambda_2(x_0)\Big\}$,
6	$x_0+\operatorname{arth}x_1$	1	$-2x_1$	$-2\int\varphi(\omega)dx_1+2\int\varphi(\omega)\frac{1+x_1^2}{1-x_1^2}dx_1-$ $-8\int\left(\int\varphi(\omega)\frac{x_1}{(1-x_1^2)^2}dx_1\right)dx_1+$ $+\lambda_1(x_0)x_1+\lambda_2(x_0)$,
7	$x_0-\operatorname{arctg}x_1$	1	$-2x_1$	$-2\int\varphi(\omega)dx_1-2\int\varphi(\omega)\frac{1-x_1^2}{1+x_1^2}dx_1+$ $+8\int\left(\int\varphi(\omega)\frac{x_1}{(1+x_1^2)^2}dx_1\right)dx_1+$ $+\lambda_1(x_0)x_1+\lambda_2(x_0)$,

The ansätze 1–3 reduce the PDE (20c) to such ODEs:

1. $\dot\varphi=0,\quad \dot\lambda_1=-4\varphi,\quad \dot\lambda_2=0,$
2. $[2x_1\partial_\omega-1]\left[2(\dddot\varphi)^{-\frac12}-\ddot\varphi\right]=0,\quad \dot\lambda_1=0,\quad \dot\lambda_2=0,$
3. $[\partial_w-2x_1]\left[\frac23\varphi+\dot\varphi-2(\dddot\varphi)^{-\frac12}\right]=0,\quad \dot\lambda_1=0,\quad \dot\lambda_2=0,$

Before to reduce the Eq. (20c) using ansätze 4–7, let us make substitution putting

$$(\dddot\varphi(\omega))^{-\frac12}=\psi(\omega).$$

As a result we get other reduced ODEs:

4. $\dfrac13\psi-\dot\psi+\psi^3(4\dot\psi-\dddot\psi)=0,$
5. $\dfrac13\psi-\dot\psi+\psi^3(4\dot\psi+\dddot\psi)=0,$
6. $\dot\psi-\psi^3(4\dot\psi-\dddot\psi)=0,$
7. $\dot\psi+\psi^3(4\dot\psi+\dddot\psi)=0,$

It is known [1], that infinitesimal operator

$$X=\xi^i(x,w)\partial_i+\eta(x,w)\partial_w\quad (i=\overline{1,n}),$$

which generates a Lie ansatz, corresponds to the equation

$$Q[w] = \xi^i(x,w)w_i - \eta(x,w) = 0. \tag{40}$$

Equations of the form (40) correspond to non-Lie ansätze 1–7:

1. $x_1 w_{111} + 4w_{11} = 0,$
2. $w_{110} + x_1^2 w_{111} + 4x_1 w_{11} = 0,$
3. $x_0 w_{110} + x_1^2 w_{111} + 4\left(x_1 - \dfrac{1}{6}\right) w_{11} = 0,$
4. $x_0 w_{110} + (x_1^2 - 1) w_{111} + 4\left(x_1 - \dfrac{1}{6}\right) w_{11} = 0,$
5. $x_0 w_{110} + (x_1^2 + 1) w_{111} + 4\left(x_1 - \dfrac{1}{6}\right) w_{11} = 0,$
6. $w_{110} + (x_1^2 - 1) w_{111} + 4x_1 w_{11} = 0,$
7. $w_{110} + (x_1^2 + 1) w_{111} - 4x_1 w_{11} = 0.$

1. Фущич В.И., Симметрия в задачах математической физики, в сб. Теоретико-алгебраические исследования в математической физике, Киев, Ин-т математики, 1981, 6–27.
2. Magri F., A simple model of the integrable Hamiltonian equation, *J. Math. Phys.*, 1978, **19**, № 5, 1156–1162.