Generating solutions for nonlinear equations by the Legendre transformation

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Предложен новый метод размножения решений нелинейных уравнений, основанный на использовании их инвариантности относительно преобразований Лежандра.

1. The superposition principle for nonlinear equations is not fulfilled. That is why it is important to have a method of generating solutions starting from known ones, i.e. an algorithm of constructing solution when the partial solutions are known.

In the present paper we suggest a new method of generating solutions for nonlinear equations based on the use of their invariance with respect to Legendre transformation. For this aim, first of all, it is necessary to describe the equations which possess such property. Here we do not consider solution of this general problem, and list some first and second order equations invariant with respect to the Legendre transformation and use this property to construct new solutions from known ones.

In [1] there was accomplished the group-theoretical classification for nonlinear second order differential equations invariant under the Lorentz group $O(1,n)$ and the Poincaré group $P(1,n)$ and different their extensions. These equations have the important property due to which there exists simple algorithm of constructing new solution from known ones, or, in another words, there exists the procedure of generating solutions. We use the Legendre transformation for further classification of Lorentz and Poincaré invariant equations. In other words, we distinguish from the set of Lorentz and Poincaré invariant those equations which are additionally invariant with respect to Legendre transformation.

2. The Legendre-invariant equations. The Legendre transformation in the space of n independent variables we write in the form

$$u = y_\mu v_\mu - v, \quad v_\mu = \frac{\partial v}{\partial x_\mu}, \quad x_\mu = v_\mu, \quad \det(v_{\mu\nu}) \neq 0. \tag{1}$$

So, the first and second order derivatives are changing as

$$u_\mu = y_\mu, \quad u_{\mu\nu} = \det{}^{-1}(v_{\lambda\sigma}) a_{\mu\nu}(v_{\lambda\sigma}) \quad (\mu,\nu,\lambda,\sigma = \overline{0,n-1}). \tag{2}$$

If not declare another, summation is understood over repeated indexes,

$$\det(u_{\lambda\sigma}) = \det \begin{pmatrix} u_{00} & \cdots & u_{0,n-1} \\ \cdots & \cdots & \cdots \\ u_{n-1,0} & \cdots & u_{n-1,n-1} \end{pmatrix}, \tag{3}$$

$a_{\mu\nu}(v_{\lambda\sigma})$ is an algebraic supplement to element $v_{\mu\nu}$.

The differential equation $L(x,u) = 0$ is called invariant with respect to Legandre transformation (1) when the following condition

$$L(x,u) = \lambda L(y,v) \tag{4}$$

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is fulfilled. Here λ is an arbitrary parameter or function of y, v, v_μ.

We list several sets of nonlinear equations invariant under the Lorentz group $O(1,n)$ and the Poincaré group $P(1,n)$ as well as the Legendre transformation. Consider equations

$$\begin{aligned}
&\det(u_{\mu\nu})\Box u \pm \operatorname{tr}(u_{\mu\nu}) = 0, \quad \Box u \pm \det(u_{\mu\nu}) \operatorname{tr}(u_{\mu\nu}) = 0,\\
&\det{}^m(u_{\mu\nu}) \pm \det{}^{-mn}(u_{\lambda\sigma}) \det{}^m(a_{\mu\nu}(u_{\lambda\sigma})) = 0,
\end{aligned} \tag{5}$$

$$\begin{aligned}
&u_\mu u^\mu + x^2 + c = 0, \quad (x^2 = x_\mu x^\mu),\\
&(x_\mu + u_\mu)(x^\mu + u^\mu) = x^2 + 2x_\mu u^\mu + u_\mu u^\mu = 0,\\
&u^\mu u^\nu \det(u_{\lambda\sigma}) \pm x^\mu x^\nu a_{\mu\nu}(u_{\lambda\sigma}) = 0, \quad x^\mu x^\nu u_{\mu\nu} \det(u_{\lambda\sigma}) \pm u^\mu u^\nu a_{\mu\nu}(u_{\lambda\sigma}) = 0,\\
&f(x^2) \det{}^m(u_{\lambda\sigma}) \pm f(u_\mu u^\mu) \det{}^{-nm}(u_{\lambda\sigma}) \det{}^m(a_{\mu\nu}(u_{\lambda\sigma})) = 0,\\
&f(u_\mu u^\mu) \det{}^m(u_{\lambda\sigma}) \pm f(x^2) \det{}^{-nm}(u_{\lambda\sigma}) \det{}^m(a_{\mu\nu}(u_{\lambda\sigma})) = 0,\\
&\left[1 + (x^2 - u_\mu u^\mu)\right] u_{11} + 2(u_0 - x_0)(u_1 - x_1)u_{10} -\\
&\quad - \left[1 - (x^2 - u^\mu u^\mu)\right] u_{00} = 0 \quad (\mu = 0, 1),
\end{aligned} \tag{6}$$

where $\Box u = g_{\mu\nu} u^{\mu\nu}$ is d'Alambert operator,

$$u_\mu u^\mu = g_{\mu\nu} u_\mu u_\nu, \quad \operatorname{tr}(u_{\mu\nu}) \stackrel{\text{def}}{=} g_{\mu\nu} u_{\mu\nu},$$

$(\mu, \nu = \overline{0, n-1})$, $g_{\mu\nu}$ is metric tensor of the space with signature $(+, -, \ldots, -)$, f is an arbitrary smooth function.

Let us list several equations which are invariant nor under the Lorentz group $O(1,n-1)$ not under Poincaré group $P(1,n-1)$, but are invariant with respect to Legendre transformation

$$\begin{aligned}
&\alpha^\mu(x_\mu + u_\mu) = 0, \quad \alpha^\mu(u_\nu)x_\mu + \alpha^\mu(x_\nu)u_\mu = 0,\\
&g_{\mu\nu}[\alpha_\mu(x_\lambda) + \alpha_\mu(u_\lambda)][\alpha_\nu(x_\lambda) + \alpha_\nu(u_\lambda)] = 0,\\
&\alpha^\mu x_\mu \det(u_{\lambda\sigma})\Box u \pm \alpha^\mu u_\mu \operatorname{tr}(u_{\lambda\sigma}) = 0,\\
&\alpha^\mu(u_\nu)x_\mu \det(u_{\lambda\sigma})\Box u \pm \alpha^\mu(x_\nu)u_\mu \operatorname{tr}(u_{\lambda\sigma}) = 0,\\
&\alpha^\mu x_\mu \Box u \pm \alpha^\mu u_\mu \det(u_{\lambda\sigma}) \operatorname{tr}(u_{\lambda\sigma}) = 0,\\
&\alpha^\mu(u_\nu)x_\mu \det{}^m(u_{\lambda\sigma}) \pm \alpha^\mu(x_\nu)u_\mu \det{}^{-nm}(a_{\mu\nu}(u_{\lambda\sigma})) = 0,\\
&u_{00}u_{11} - u_{10}^2 = u_{00}^{-1} u_{11}.
\end{aligned} \tag{7}$$

α^μ are vector components. The method of generating solutions for equations (7) is unknown.

To prove invariance of equations (5), (6), (7) under Legendre transformation (1) it is necessary to verify fulfilment of relations (4). For example, equation

$$u_\mu u^\mu + x^2 + c = 0$$

is transformed under Legendre transformation into $y^2 + v_\mu v^\mu + c = 0$.

3. The generating solutions. Suppose we have a solution of nonlinear equation

$$u = \overset{(1)}{u}(x), \quad x = (x_0, x_1, \ldots, x_{n-1}), \quad \det(\overset{(1)}{u}_{\mu\nu}) \neq 0. \tag{8}$$

Let us rewrite this solution, replacing x_μ for parameters θ_μ $(\mu = \overline{0, n-1})$. The formula of generating solutions resulting from Legendre transformation takes the form

$$\overset{(2)}{u} = \theta_\mu \overset{(1)}{u}_\mu(\theta) - \overset{(1)}{u}(\theta), \quad \overset{(1)}{u}_\mu(\theta) = x_\mu \quad (\theta = (\theta_0, \theta_1, \ldots, \theta_{n-1})). \tag{9}$$

To find exact solution $\overset{(2)}{u}(x)$ it is necessary to eliminate parameters θ in (9). So, the formula (9) gives the method for generating solutions.

Example 1. Consider an equation from the list given above

$$u_{00}u_{11} - u_{10}^2 = u_{00}^{-1}u_{11}. \tag{10}$$

This equation admits Lie algebra $\langle P_0, P_1, P_2, J_0, J_1, D_1, D_2\rangle$:

$$\begin{aligned} &P_0 = \partial_0, \quad P_1 = \partial_1, \quad P_2 = \partial_u, \quad J_0 = x_0\partial_u, \quad J_1 = x_1\partial_u, \\ &D_1 = x_1\partial_1, \quad D_2 = x_0\partial_0 + 2u\partial_u \end{aligned} \tag{11}$$

and consequently Lie symmetry of equation (10) gives the following formula of generating solutions:

$$\overset{(2)}{u} = e^{-2b}(\overset{(1)}{u}(e^b x_0 + \theta_0, e^a x_1 + \theta_1) + \varkappa_0 x_0 + \varkappa_1 x_1 + \theta_2), \tag{12}$$

where a, b, $\varkappa_0$, $\varkappa_1$, θ_0, θ_1, θ_2 are group parameters.

One of partial solutions of equation (10) have the form

$$\overset{(1)}{u} = \frac{x_0^2}{2}\operatorname{cth} x_1. \tag{13}$$

It is easily convinced that $\det(\overset{(1)}{u}_{\mu\nu}) \neq 0$. Let us construct a new solution of equation (10) by using the Legandre transformation. Rewrite (13), replacing x_μ, $(\mu = 0, 1)$ for parameters θ_μ, i.e.

$$\overset{(1)}{u} = \frac{\theta_0^2}{2}\operatorname{cth}\theta_1.$$

and make use of formulas (9) to obtain

$$\begin{aligned} &\overset{(2)}{u} = \frac{1}{2}\theta_0^2\operatorname{cth}\theta_1 - \frac{1}{2}\theta_1\theta_0^2\operatorname{sh}^{-2}\theta_1, \\ &\theta_0\operatorname{cth}\theta_1 = x_0, \\ &-\frac{1}{2}\theta_0^2\operatorname{sh}^{-2}\theta_1 = x_1. \end{aligned} \tag{14}$$

Exclude parameters by expressing them through x_μ from two lest equations of system (14). We obtain

$$-\frac{x_0^2}{2x_1} = \operatorname{ch}^2\theta_1,$$

or

$$\operatorname{th}\theta_1 = \frac{1}{x_0}\sqrt{x_0^2 + 2x_1}, \quad \theta_0 = \sqrt{x_0^2 + 2x_1}.$$

Substituting θ_0 and θ_1 into first equation of system (14), we get

$$\overset{(2)}{u} = \frac{1}{2}x_0^2\frac{\sqrt{x_0^2+2x_1}}{x_0} + x_1 \operatorname{arth}\frac{\sqrt{x_0^2+2x_1}}{x_0}. \tag{15}$$

Designating

$$\omega = \sqrt{1+2\frac{x_1}{x_0}},$$

we write the solution in another form

$$\overset{(2)}{u} = \frac{1}{2}x_0^2\omega + x_1 \operatorname{arth}\omega. \tag{15$'$}$$

It is not difficult to verify that function (15) satisfies the equation (10). Note that this solution cannot be constructed from (13) by generating it according to formula (12).

Example 2. Consider the equation

$$u_\mu u^\mu + x^2 + c = 0 \quad (\mu = \overline{0, n-1}). \tag{16}$$

c is an arbitrary constant. The maximal invariance algebra of equation (16) is $\langle P_0, J_{0a}, J_{ab}\rangle$:

$$P_0 = \partial_0, \quad J_{0a} = x_a\partial_0 + x_0\partial_a, \quad J_{ab} = x_a\partial_b - x_b\partial_a \quad (a, b = \overline{1, n-1}). \tag{17}$$

An ansatz of the form [1]

$$\begin{aligned} &u = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_\mu x^\mu = x^2, \quad \omega_2 = (\beta_\mu x^\mu)^2 + (\gamma_\mu x^\mu)^2, \\ &\beta^2 = \gamma^2 = -1, \quad \beta\gamma = 0, \end{aligned} \tag{18}$$

reduces n-dimensional PDE (16) to 2-dimensional PDE

$$\omega_1\varphi_1^2 - \omega_2\varphi_2^2 + 2\omega_2\varphi_1\varphi_2 + \frac{1}{4}(\omega_1 + c) = 0. \tag{19}$$

Construct the solution of equation (19) by using Legendre transformation

$$z = \omega_1\varphi_1 + \omega_2\varphi_2 - \varphi, \quad y_1 = \varphi_1, \quad y_2 = \varphi_2. \tag{20}$$

Then the equation (19) is transformed to the linear

$$\left(y_1^2 + \frac{1}{4}\right) z_1 - (y_2 - 2y_1y_2)z_2 + \frac{1}{4}c = 0. \tag{21}$$

The general solution of (21) is:

$$z = \Phi(2\operatorname{arctg} 2y_1 - 2\operatorname{arctg} 2(y_1 - y_2)) - \frac{1}{2}c \operatorname{arctg} 2y_1. \tag{22}$$

One gets the general solution of equation (19) by inverting (20)

$$\varphi = y_1z_1 + y_2z_2 - z, \quad \omega_1 = z_1, \quad \omega_2 = z_2,$$

or, substituting z from (22),

$$\omega_1 = \dot\Phi\left[\frac{1}{y_1^2+\frac14} - \frac{1}{(y_1-y_2)^2+\frac14}\right] - \frac14\frac{c}{y_1^2+\frac14},$$
$$\omega_2 = \dot\Phi\frac{1}{(y_1-y_2)^2+\frac14},$$
$$\varphi = y_1\omega_1 + y_2\omega_2 - \Phi + \frac12 c\,\mathrm{arctg}\,2y_1,$$

where y_1, y_2 play the role of parameters and must be eliminate for obtaining the solution $\varphi(\omega_1,\omega_2)$ in explicit form. To demonstrate the solutions of equation (16) generating process we choose Φ in simplest form

$$\Phi(\alpha) = k\alpha,$$

where k is an arbitrary constant. Then

$$z = 2k(\mathrm{arctg}\,2y_1 - \mathrm{arctg}\,2(y_1-y_2)) - \frac12 c\,\mathrm{arctg}\,2y_1.$$

So, corresponding partial solution of equation (21) have the form

$$\begin{aligned}\overset{(1)}{u} &= (\omega_1+\omega_2)\sqrt{\frac{k-\frac14 c}{\omega_1+\omega_2} - \frac14} - \omega_2\sqrt{\frac{k}{\omega_2}-\frac14} - \\ &\quad - 2\left(k-\frac14 c\right)\mathrm{arctg}\,2\sqrt{\frac{k-\frac14 c}{\omega_1+\omega_2}-\frac14} + 2k\,\mathrm{arctg}\,2\sqrt{\frac{k}{\omega_2}-\frac14}.\end{aligned} \tag{23}$$

Writing $\varphi(\omega_1,\omega_2)$ with ω_1 and ω_2 determined by (18) in variables θ_μ and make use of Legendre transformation (1), we obtain second solution

$$\begin{aligned}\overset{(2)}{u} &= \left[4\left(k-\frac14 c\right) - \omega_1 - \omega_2\right]\sqrt{\frac{k-\frac14 c}{4\left(k-\frac14 c\right)-\omega_1-\omega_2} - \frac14} - \\ &\quad - [4k-\omega_2]\sqrt{\frac{k}{4k-\omega_2}-\frac14} - \\ &\quad - 2\left(k-\frac14 c\right)\mathrm{arctg}\,2\sqrt{\frac{k-\frac14 c}{4\left(k-\frac14 c\right)-\omega_1-\omega_2}-\frac14} + \\ &\quad + 2k\,\mathrm{arctg}\,2\sqrt{\frac{k}{4k-\omega_2}-\frac14}.\end{aligned} \tag{24}$$

If solution (22) of the equation (21) is $\Phi = 0$, we get

$$\begin{aligned}\overset{(1)}{u} &= -\frac12\omega_1\sqrt{\frac{c}{\omega_1}-\frac14} + c\,\mathrm{arctg}\,2\sqrt{\frac{c}{\omega_1}-\frac14}, \\ \overset{(2)}{u} &= (c-\omega_1)\sqrt{\frac{\omega_1}{c-\omega_1}} - c\,\mathrm{arctg}\sqrt{\frac{\omega_1}{c-\omega_1}}.\end{aligned} \tag{25}$$

Choosing invariants of equation (16) the functions

$$\omega_1 = \alpha_\mu x^\mu, \quad \omega_2 = x_\mu x^\mu = x^2,$$

we obtain the solutions

$$\overset{(1)}{u} = -\omega_1\sqrt{\omega_1^2 - \omega_2}, \quad \overset{(2)}{u} = -\omega_1\sqrt{\omega_1^2 - \omega_2}. \tag{26}$$

i.e. this solution is Legandre-invariant solution of equation (16).

Example 3. The equation

$$\det(u_{\mu\nu})\Box u + \mathrm{tr}\,(u_{\mu\nu}) = 0 \tag{27}$$

is invariant with respect to Legandre transformation. An ansatz

$$u = \varphi(\omega), \quad \omega = x_\mu x^\mu = x^2 \tag{28}$$

reduces (27) to ODE

$$8\dot\varphi\ddot\varphi^2 + \omega\ddot\varphi[4(n+1)\dot\varphi^2 - n + 1] + 2n\dot\varphi^3 - n\dot\varphi = 0.$$

Following the formula (9), from (28)

$$\overset{(1)}{u} = \varphi(\theta^2), \quad \theta^2 = \theta_\mu\theta^\mu \quad (\mu = \overline{0, n-1}),$$

we obtain

$$\overset{(2)}{u} = \theta_\mu\varphi_\mu(\theta^2) - \varphi(\theta^2), \quad \varphi_\mu(\theta^2) = x_\mu. \tag{29}$$

Hence we find

$$\varphi_\mu = 2\theta_\mu\dot\varphi, \quad x^2 = 4\theta^2(\dot\varphi)^2, \tag{30}$$

and formula (29) take the form

$$\overset{(2)}{u} = \frac{1}{2}x^2\dot\varphi^{-1}(\psi(x^2)) - \varphi(\psi(x^2)). \tag{31}$$

1. Фущич В.И., Штелень В.М., Серов Н.И., Симметрийный анализ и точные решения нелинейных уравнений математической физики, Киев, Наук. думка, 1989, 336 с.