Conditional symmetry and exact solutions of equations of nonstationary filtration

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Досліджена умовна іваріантість, одержані неліївські анзаци та побудовані точні розв'язки рівняння нестаціонарної фільтрації з нелінійною правою частиною. Результати узагальнено для n-вимірного нелінійного рівняння теплопровідності.

In describing filtration processes of gas the following nonlinear equation is widely used [1]

$$\frac{\partial v}{\partial x_0} + \frac{\partial^2 \varphi(v)}{\partial x_1^2} + \frac{N}{x_1}\frac{\partial \varphi(v)}{\partial x_1} = \Phi(v), \tag{1}$$

where $v = v(x)$, $x = (x_0, x_1) \in \mathbb{R}_2$, $N = \mathrm{const}$; $\varphi(v)$, $\Phi(v)$ are given smooth functions. Substitution $u = \varphi(v)$ reduces equation (1) to equivalent equation

$$H(u)u_0 + u_{11} + \frac{N}{x_1}u_1 = F(u), \tag{2}$$

where $u_0 = \frac{\partial u}{\partial x_0}$, $u_1 = \frac{\partial u}{\partial x_1}$, $u_{11} = \frac{\partial^2 u}{\partial x_1^2}$.

Lie symmetry of equation (2) under $N = 0$ was studied in [2, 3] and its conditional symmetry was studied in [4–7].

In present paper we study conditional symmetry of equation (2) with $N \neq 0$. Operators of conditional symmetry are used to construct ansätze which reduce (2) to ordinary differential equations (ODE). By means of this method we obtain exact solutions of equations (2) and then exact solutions of multidimensional nonlinear heat equation. Below we will use terms and definitions given in [4–7].

Theorem 1. *Equation (2) is Q-conditionally invariant under the operator*

$$Q = A(x,u)\partial_0 + B(x,u)\partial_1 + C(x,u)\partial_u, \tag{3}$$

iff function A, B, C satisfy the following system of equations:

Case I. $A \neq 0$ (without lose of generality one can put $A = 1$)

$$\begin{gathered} B_{uu} = 0, \quad C_{uu} = 2\left(B_{1u} + HBB_u - \frac{N}{x_1}B_u\right), \quad 3B_uF = 2(C_{1u} + HB_uC) - \\ - \left(HB_0 + B_{11} - \frac{N}{x_1}B_1 + \frac{N}{x_1^2}B + 2HBB_1 + \dot{H}BC\right), \\ C\dot{F} - (C_u - 2B_1)F = HC_0 + C_{11} + \frac{N}{x_1}C_1 + 2NCB_1 + \dot{H}C^2; \end{gathered} \tag{4}$$

Доповіді АН України, 1992, № 6, С. 20–24

Case II. $A = 0$, $B = 1$,

$$C\dot{F} - \left(C_u + \frac{\dot{H}}{H}C\right)F = HC_0 + C_{11} + \frac{N}{x_1}C_1 - \frac{N}{x_1^2}C + 2CC_{1u} + C^2C_{uu} - \\ - C\frac{\dot{H}}{H}\left(CC_u + C_1 + \frac{N}{x_1}C\right). \tag{5}$$

In formulas (4), (5) and everywhere below subscripts mean differentiation with respect to corresponding arguments.

To prove the theorem one should use the method described in [4–7].

To find the general solution of equations (4), (5) is impossible, but we succeeded in obtaining several partial solutions.

Theorem 2. *Equation (2) is Q-conditionally invariant under operator (3) with* $H(u) = 1$, $A = 1$, $D_u \neq 0$ *it is leeally equivalent to the equation*

$$u_0 + u_{11} + \frac{3}{2x_1}u_1 = \lambda u^3 \quad (\lambda = \text{const}), \tag{6}$$

and in this case operator (3) takes the form

$$Q = \partial_0 + \frac{3}{2}\left(\sqrt{2\lambda}u + \frac{1}{x_1}\right)\partial_1 + \frac{3}{4}u\left(2\lambda u^2 - \frac{1}{x_1^2}\right)\partial_u. \tag{7}$$

To prove the theorem one has to solve equations (4) under $H(u) = 1$, $B(u) \neq 0$. By means of operator (7) we construct an implicit ansatz

$$15\left(x_0 - \frac{x_1^2}{3}\right)\omega + 4\sqrt{2\lambda}x_1^{\frac{5}{2}} = \varphi(\omega), \quad \omega = \frac{1 + \sqrt{2\lambda}ux_1}{u\sqrt{x_1}}, \tag{8}$$

which reduces equation (6) to the ODE $\ddot{\varphi} = 0$. Having solved this latter one and taking into account (8), we obtain the following solution of equation (6), $u(x_0, x_1)$ is a new solution

$$u = -\frac{5}{\sqrt{2\lambda}}\frac{x_1^3 - 3x_0}{x_1^3 - 15x_0x_1 + c_1\sqrt{x_1}} \quad (c_1 = \text{const}). \tag{9}$$

All inequivalent ansätze of Lie type are given by one of formulae

$$u = \varphi(x_1), \quad u = x_0^{-\frac{1}{2}}\varphi(x_0^{-\frac{1}{2}}x_1). \tag{10}$$

It is obvious that (9) does not belong to (10).

The above solutions of equation (6) can be multiplied by means of formulae of generating solutions using Lie symmetry:

$$u(x_0, x_1) = \theta_1 f(\theta_1^2 x_0 + \theta_0, \theta_1 x_1), \tag{11}$$

where θ_0, θ_1 are group parameters, $f(x_0, x_1)$ is a known solution of equation (6), $u(x_0, x_1)$ is a new solution.

Theorem 3. *Equation*

$$\frac{1}{u}u_0 + u_{11} + \frac{N}{x_1}u_1 = \frac{1}{u}(\lambda_1 u + \lambda_2) \quad (\lambda_1, \lambda_2 = \text{const}), \tag{12}$$

is Q-conditionally invariant under operator

$$Q = \partial_0 + (N+1)\frac{u}{x_1}\partial_1 + (\lambda_1 u + \lambda_2)\partial_u. \tag{13}$$

Proof. To prove the theorem it is sufficient to show that the following relation holds true

$$\tilde{Q}S = \tilde{\lambda}_1 S + \tilde{\lambda}_2 Qu, \tag{14}$$

where

$$S = \frac{1}{u}u_0 + u_{11} + \frac{N}{x_1}u_1 - \frac{1}{u}(\lambda_1 u + \lambda_2),$$
$$Qu = u_0 + (N+1)\frac{u}{x_1}u_1 - (\lambda_1 u + \lambda_2),$$

$\tilde{Q}$ is corresponding prolongation of operator Q: $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ are some functions.

On acting operator $\tilde{Q}$ on S we get after rather tedious calculations,

$$\tilde{Q}S = \left[\lambda_1 + \frac{N+1}{x_1^2}(2u + 3x_1u_1)\right]S -$$
$$- \left[\frac{N+1}{x_1 u}u_1 - \frac{N+1}{x_1^2 u}(2u + 3x_1u_1) - \frac{\lambda_1 u + \lambda_2}{u^2}\right]Qu.$$

So, the theorem is proved.

Operator (13) results in the ansatz

$$\frac{x_1^2}{2(N+1)} - \int\frac{udu}{\lambda_1 u + \lambda_2} = \varphi(\omega), \quad \omega = x_0 - \int\frac{du}{\lambda_1 u + \lambda_2}, \tag{15}$$

which reduces equation (12) to the ODE

$$-\ddot{\varphi} = \lambda_1\dot{\varphi} + \lambda_2. \tag{16}$$

Having integrated equation (16) and using once more (15) one finds solution of equation (12)

$$\begin{aligned} &\lambda_1 u + \lambda_2 = \frac{\lambda_1\lambda_2 x_0 + \frac{1}{2}\lambda_1^2(N+1)^{-1}x_1^2}{1 + \lambda_3\exp(-x_0)}, \quad \lambda_1 \neq 0, \\ &u = \frac{\lambda_2 x_0^2 + \lambda_3 + (N+1)^{-1}x_1^2}{2x_0}, \quad \lambda_1 = 0, \end{aligned} \tag{17}$$

where λ_3 is a constant of integration.

It is not difficult to verify that solutions (17) cannot be obtained by means of Lie ansätze (analogously to above solutions of equation (6)).

The rest results obtained for equation (2) are collected in table, where λ_1, λ_2, λ_3 are arbitrary constants, $W = W(u)$ an arbitrary smooth function.

Let us give some solutions of equation (2) obtained as a result of integration of reduced equations listed in the table.

$$1. \quad u = \lambda\left(\frac{N-3}{\lambda_2}\frac{x_0}{x_1} + \frac{x_1}{2}\right)^{N-1},$$
$$\lambda = \left[\frac{\lambda_3}{(N-1)(N-2)}\right]^{\frac{N-1}{2}}, \quad \lambda_1 \neq 0, \quad N \neq 1, 2;$$

$$
\begin{aligned}
&2. \quad u = x_1^{-2}\varphi(\omega), \quad \text{where} \quad \omega = x_0 - \frac{x_1^2}{2},\\
& \varphi(\omega) \text{ satisfies ODE } \dot\varphi = -\lambda_1 \ln\varphi + \lambda_3\omega;\\
&3. \quad u = -2\ln x_1^{-1}\sqrt{2\lambda_3}\left(\frac{x_0}{\lambda_2} - \frac{x_1^2}{4}\right), \quad \lambda_1 = 0,\ \lambda_2 \neq 0,\ \lambda_3 > 0;\\
& u = \ln\frac{\lambda_4 x_1^2}{\exp\frac{\lambda_1\lambda_4}{\lambda_2}\left(\frac{4x_0}{\lambda_2} - x_1^2\right) - 1}, \quad \lambda_1 \neq 0,\ \lambda_2 \neq 0,\ \lambda_3 = 0,\ \lambda_4 \neq 0;\\
&4. \quad u = -\ln\frac{\lambda_1}{\lambda_2}\left(\frac{4x_0}{x_1^2} - 1\right), \quad \lambda_1 \neq 0,\ \lambda_2 \neq 0,\ \lambda_3 = 0;\\
& W(u) = x_1\left(\frac{1}{\lambda_2} + \exp\left(\frac{-\lambda_2}{\lambda_1}x_0\right)\right)^{-\frac12}, \quad \lambda_1 \neq 0,\ \lambda_2 \neq 0;\\
&5. \quad W(u) = x_1\sqrt{\frac{\lambda_1}{2x_0}}, \quad \lambda_2 = 0; \quad u = \lambda_1 x_0 + \frac{\lambda_2 x_1^2}{2(N+1)};\\
&6. \quad u = \lambda_1 x_0 + \lambda_3\frac{x_1^2}{2};\\
&7. \quad u = \exp\left(\exp\lambda_3 x_0 + \frac{\lambda_3}{4}x_1^2 + \frac{N+1}{2}\right).
\end{aligned}
\tag{18}
$$

Numeration in (18) corresponds to that of ansätze in the table.

Note, that substituting $x_1 \to r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ and putting $N = n-1$ we find that equation (2) coincides with reduced nonlinear heat equation

$$
H(u)u_0 + \Delta u = F(u), \tag{19}
$$

where $u = u(x_0, \vec x)$, $\Delta = \frac{\partial^2}{\partial_1^2} + \cdots + \frac{\partial^2}{\partial n_n}$. Equation (19) is reduced to (2) by means of the $O(n)$-invariant ansatz $u = u(x_0, r)$. Therefore, many results obtained above for equation (2) can be used straightforwardly for finding operators of conditional symmetry and corresponding solutions of multidimensional equation (19). We summarize them in the following statement.

Theorem 4. *Nonlinear heat equation (19) is Q-conditionally invariant under the set of operators $\{AO(n), Q\}$ if:*

$$
\begin{aligned}
&1) \quad H(u) = \lambda_1 u^{\frac{2}{2-n}} + \lambda_2, \quad F(u) = \lambda_3^{\frac{4-n}{2-n}},\\
& Q = \lambda_2\vec x^{\,2}\partial_0 + (4-n)x_a\partial_a + (4-n)(2-n)u\partial_u, \quad \lambda_2 \neq 0,\ n \neq 2,\ 4;\\
&2) \quad H(u) = \frac{\lambda_1}{u}, \quad F(u) = \lambda_3,\ n = 4, \quad Q = \vec x^{\,2}\partial_0 + x_a\partial_a - 2u\partial_u;\\
&3) \quad H(u) = \lambda_1\exp u + \lambda_2, \quad F(u) = \lambda_3\exp u, \quad n = 2,\\
& Q = \lambda_2\vec x^{\,2}\partial_0 + 2x_a\partial_a + 4\partial_u, \quad \lambda_2 \neq 0;\\
&4) \quad H(u) = 1, \quad F(u) = \lambda_3 u\ln u, \quad Q = x_a\partial_a + \frac{\lambda_3}{2}\vec x^{\,2}u\partial_u;\\
&5) \quad H(u) = \frac{1}{u}, \quad F(u) = \frac{1}{u}(\lambda_1 u + \lambda_2), \quad Q = \partial_0 + \frac{n}{\vec x^{\,2}}ux_a\partial_a + (\lambda_1 u + \lambda_2)\partial_u.
\end{aligned}
\tag{20}
$$

№	$H(u)$	$F(u)$	Operator Q	Ansatz	Reduced equation
1	$\lambda_1 u^{\frac{2}{1-N}} + \lambda_2,\ \lambda_2 \neq 0$	$\lambda_3 u^{\frac{3-N}{1-N}},\ N \neq 1,\ 3$	$\lambda_2 x_1^2\partial_0 + (3-N)x_1\partial_1 +$ $+(3-N)(1-N)u\partial u$	$u = x_1^{1-N}\varphi(\omega)$, $\omega = \frac{x_0}{\lambda_2} + \frac{x_1^2}{2(N-3)}$	$\frac{\lambda_1}{\lambda_2}{}^{\frac{2}{1-N}}\dot\varphi + \frac{\ddot\varphi}{(N-3)^2} = \lambda_3\varphi^{\frac{3-N}{1-N}}$
2	$\frac{\lambda_1}{u},\ N = 3$	λ_3	$x_1^2\partial_0 + x_1\partial_1 - 2u\partial u$	$u = x_1^{-2}\varphi\left(x_0 - \frac{x_1^2}{2}\right)$	$\lambda_1\frac{\dot\varphi}{\varphi} + \ddot\varphi = 3$
3	$\lambda_1 e^u + \lambda_2,\ \lambda_2 \neq 0$	$\lambda_3 e^u,\ N = 1$	$\lambda_2 x_1^2\partial_0 + 2x_1\partial_1 + 4\partial u$	$u = \varphi(\omega) + 2\ln x_1$, $\omega = \frac{x_0}{\lambda_2} - \frac{x_1^2}{4}$	$\frac{\lambda_1}{\lambda_2}e^{\varphi}\dot\varphi + \frac{1}{4}\ddot\varphi = \lambda_3 e^{\varphi}$
4	$\frac{\lambda_1}{\dot W^2}\left(N - \frac{W\ddot W}{\dot W^2}\right)$	$\frac{\lambda_2}{W\dot W}\left(N - \frac{W\ddot W}{\dot W^2}\right)$	$x_1\partial_1 + \frac{W}{\dot W}\partial_u$	$W(u) = x_1\varphi(x_0)$	$\lambda_1\dot\varphi = \lambda_3\varphi - \varphi^3$
5	$W(u),\ N \neq -1$	$\lambda_1\omega + \lambda_2,\ \lambda_2 \neq 0$	$\partial_1 + \frac{\lambda_2}{N+1}x_1\partial_u$	$u = \varphi(x_0) - \frac{\lambda_2 x_1^2}{2(N+1)}$	$\dot\varphi = \lambda_1$
6	$W(u),\ N = -1$	$\lambda_1 W(u)$	$\partial_1 + \lambda_3 x_1\partial_u$	$u = \varphi(x_0) + \lambda_3\frac{x_1^2}{2}$	$\dot\varphi = \lambda_1$
7	1	$\lambda_3 u\ln u$	$\partial_1 + \frac{\lambda_3}{2}x_1 u\partial_u$	$u = \varphi(x_0)e^{\frac{\lambda_3 x_1^2}{4}}$	$\dot\varphi + \lambda_3\frac{N+1}{2}\varphi = \lambda_3\varphi\ln\varphi$

Remark. Basis elements of $AO(n)$ have the form

$$I_{ab} = x_a\partial_b - x_b\partial_a, \quad a,b = \overline{1,n}. \tag{21}$$

Repeated indices are to be summed over $1, 2, \ldots, n$.

Let us give some exact solutions of equation (19) obtained by means of operators (20):

$$\lambda_1 u + \lambda_2 = \frac{\lambda_1\lambda_2 x_0 + \frac{\lambda_1^2 \vec{x}^{\,2}}{2n}}{1 + \lambda_3 \exp(-x_0)} \quad (\lambda_1 \neq 0)$$

is solution of equation

$$\frac{1}{u}u_0 + \Delta u = \frac{1}{u}(\lambda_1 u + \lambda_2)$$

and

$$u = \ln \frac{\vec{x}^{\,2}}{2\lambda_3}\left(\frac{x_0}{\lambda^2} - \frac{\vec{x}^{\,2}}{4}\right)^{-2}$$

is solution of equation under $n = 2$.

$$\lambda_2 u_0 + \Delta u = \lambda_3 \exp u$$

under $n = 2$.

1. Кочина П. Я., Гидродинамика и теория фильтрации. Избр. тр., М., Наука, 1991, 209 с.
2. Овсянников Л.В., Групповые свойства уравнений нелинейной теплопроводности, *Докл. АН СССР*, 1959, **125**, № 3, 492–495.
3. Дородницын В.А., Князева Н.В., Свирщевский С.Р., Групповые свойства уравнения теплопроводности с источником в двумерном и трехмерном случаях, *Дифференц. уравнения*, 1983, **19**, № 17, 1215–1224.
4. Фущич В.И., Штелень В.М., Серов Н.И., Симметрийный анализ и точные решения нелинейных уравнений математической физики, Киев, Наук. думка, 1989, 336 с.
5. Фущич В.И., Серов Н.И., Амеров Т.К., Условная инвариантность нелинейного уравнения ттеплопроводности, *Докл. АН УССР, Сер. А*, 1990, № 11, 15–18.
6. Серов Н.И., Условная инвариантность и точные решения нелинейного уравнения теплопроводности, *Укр. мат. журн.*, 1990, **42**, № 10, 1370–1376.
7. Фущич В.И., Условная симметрия уравнений нелинейной математической физики, *Укр. мат. журн.*, 1991, **43**, № 11, 1456–1471.