A simple method of finding solutions of the nonlinear d'Alembert equation

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We consider the nonlinear d'Alembert equation

$$\Box u = F(u), \tag{1}$$

where $u = u(x)$ and $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\Box = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$$

and $F(u)$ is an arbitrary differentiable function. In equation (1) we make the local change of variable

$$u = \Phi(w), \tag{2}$$

where $w(x)$ is a new unknown function and Φ is a function to be determined later. On making this change, (1) becomes

$$\dot{\Phi}\Box w + \ddot{\Phi} w_\mu w^\mu = F(\Phi), \tag{3}$$

where

$$\dot{\Phi} = \frac{d\Phi}{dw}, \quad w_\mu w^\mu = \left(\frac{dw}{dx_0}\right)^2 - \left(\frac{dw}{dx_1}\right)^2 - \cdots - \left(\frac{dw}{dx_n}\right)^2.$$

Equation (3) is equivalent to the following equation

$$\dot{\Phi}\left(\Box w - \lambda \frac{\dot{P}_n}{P_n}\right) + \ddot{\Phi}(w_\mu w^\mu - \lambda) + \lambda\left(\ddot{\Phi} + \dot{\Phi}\frac{\dot{P}_n}{P_n}\right) - F(\Phi) = 0, \tag{4}$$

where $P_n(w)$ is an arbitrary polynomial of degree n in w, and $\lambda = -1, 0, 1$. Choosing Φ such that

$$\lambda\left(\ddot{\Phi} + \dot{\Phi}\frac{\dot{P}_n}{P_n}\right) = F(\Phi) \tag{5}$$

equation (4) becomes

$$\dot{\Phi}\left(\Box w - \lambda \frac{\dot{P}_n}{P_n}\right) + \ddot{\Phi}(w_\mu w^\mu - \lambda) = 0. \tag{6}$$

From this it is clear that a solution of the system

$$\Box w = \lambda \frac{\dot{P}_n}{P_n}, \quad w_\mu w^\mu = \lambda \tag{7}$$

J. Phys. A: Math. Gen., 1992, **25**, L871–L877.

is also a solution of (6), and in this way we obtain a solution of (1) provided Φ satisfies (5). There remains, of course, the problem of the existence of solutions of (7). We have the following result.

Theorem 1. *For $n = 3$ the system*

$$\Box w = H(w), \quad w_\mu w^\mu = \lambda$$

with $\lambda = -1, 0, 1$ is compatible if and only if

$$H(w) = \frac{\lambda N}{w + C},$$

where $N = 0, 1, 2, 3$ and C is an arbitrary constant.

This result follows from theorem 2 of [2]. In theorem 1 of [3], it is further shown that if the system in theorem 1 above is compatible, then it is necessarily of the type given in equation (7). Moreover, as is mentioned in [3], the system (7) is always compatible (for any n) if $H(w)$ is as in theorem 1 above. Having discussed the question of compatibility, we now turn to equation (5), which gives us the appropriate choice of Φ. We do this for several cases of the function $F(u)$. Note that for the remainder of this paper, we take $n = 3$.

Case 1. $F(u) = u^k$ with $k \neq 1$. If $P_n = w^m$ with $m = 0, 1, 2, 3$ then (5) becomes

$$\lambda\left(\ddot{\Phi} + \frac{m}{w}\dot{\Phi}\right) = \Phi^k. \tag{8}$$

Assuming Φ to be of the form

$$\Phi(w) = \alpha w^\beta$$

with α, β constants, we obtain

$$\Phi(w) = \left(\frac{(1-k)w}{(2\lambda(1+k+m-km))^{1/2}}\right)^{2/(1-k)} \tag{9}$$

as a solution of (8).

Case 2. $F(u) = \exp u$. Again using $P_n(w) = w^m$, $m = 0, 1, 2, 3$, equation (5) becomes

$$\lambda\left(\ddot{\Phi} + \frac{m}{w}\dot{\Phi}\right) = \exp\Phi. \tag{10}$$

and we seek solutions Φ with the help of me ansatz

$$\exp(\Phi(w)) = \alpha w^\beta$$

with α, β constants. We obtain

$$\Phi(w) = \log\left(\frac{2\lambda(m-1)}{w^2}\right), \quad m = 2, 3. \tag{11}$$

Case 3. $F(u) = -\ddot{\Psi}(u)/\dot{\Psi}^3(u)$. Here we take Ψ to be an arbitrary differentiable function such that $\ddot{\Psi} \neq 0$. If we take $P_n(w) = \lambda_0 = \text{const}$, then (5) becomes

$$\lambda\ddot{\Psi} = -\frac{\ddot{\Psi}(\Phi)}{\dot{\Psi}^3(\Phi)} \tag{12}$$

and this gives us

$$\sqrt{\lambda}\int \frac{d\Phi}{(c_1 + \dot{\Psi}^{-2}(\Phi))^{1/2}} = w + c_2, \tag{13}$$

where c_1, c_2 are two constants of integration. On choosing these two constants of integration to be zero, we obtain

$$w = \sqrt{\lambda}\Psi(\Phi)$$

as a solution of (12), and the change of variable (2) is then given by

$$w = \sqrt{\lambda}\Psi(u). \tag{14}$$

In this case of F, we have replaced Φ by another function Ψ; we now took at some cases of Ψ.

Case 3(a). $F(u) = \lambda_1 \sin u$, where $\lambda_1 = \text{const}$. On setting

$$\lambda_1 \sin u = -\frac{\ddot{\Psi}(u)}{\dot{\Psi}^3(u)}$$

we obtain

$$\Psi(u) = \int \frac{du}{(c_1 - 2\lambda_1 \cos u)^{1/2}} + c_2. \tag{15}$$

For $c_1 = 2\lambda_2$, $c_2 = 0$, $\lambda_1 > 0$ we find

$$\Psi(u) = \frac{1}{\sqrt{\lambda_1}} \log\tan(u/4)$$

and for $c_1 = -2\lambda_1$, $c_2 = 0$, $\lambda_1 < 0$ one obtains

$$\Psi(u) = \frac{1}{\sqrt{-\lambda_1}} \tanh^{-1}\tan(u/4).$$

On putting $\lambda = |\lambda_1|$ in (14), the change of variable then takes on the form

$$u = \begin{cases} 4\arctan\exp(w/\sqrt{\lambda_1}) & \text{for } \lambda_1 > 0, \\ 4\arctan\tanh(w/\sqrt{-\lambda_1}) & \text{for } \lambda_1 < 0. \end{cases}$$

Case 3(b). $F(u) = \sinh u$. Integrating the equation

$$\sinh u = -\frac{\ddot{\Psi}(u)}{\dot{\Psi}^3(u)}$$

we obtain

$$\Psi(u) = \int \frac{du}{(c_1 + 2\cosh u)^{1/2}} + c_2. \tag{16}$$

For $c_1 = 2$, $c_2 = 0$ one finds

$$\Psi(u) = 2\arctan\tanh(u/4)$$

and for $c_1 = -2$, $c_2 = 0$ one gets

$$\Psi(u) = \log\tanh(u/4).$$

Then (14) gives, with $\lambda > 0$

$$u = 4\tanh^{-1}\tan(w/\sqrt{\lambda}),$$
$$u = 4\tanh^{-1}\exp(w/\sqrt{\lambda}).$$

Case 3(c). $F(u) = \sin u/\cos^3 u$. In this case, the equation

$$\frac{\sin u}{\cos^3 u} = -\frac{\ddot{\Psi}(u)}{\dot{\Psi}^3(u)}$$

yields

$$\Psi(u) = \int \frac{du}{(c_1 + 2\tan^2 u)^{1/2}} + c_2. \tag{17}$$

Again, we choose values for the integration constants. For $c_1 = 1$, $c_2 = 0$ we find

$$\Psi(u) = \sin u$$

and for $c_1 = c_2 = 0$ we obtain

$$\Psi(u) = \log\sin u.$$

Using (14), with $\lambda > 0$ the change of variable (2) is given by the equations

$$u = \arcsin(w/\sqrt{\lambda})$$

and

$$u = \arcsin\exp(w/\sqrt{\lambda}).$$

We present our results in table 1.

Table 1. Summary of results obtained in cases 1–3(c).

$F(u)$	Solution of (1)	System (7)
u^k, $k \neq 1$	$u = \left(\frac{(1-k)w}{\sqrt{2\lambda(1+k+m-km)}}\right)^{2/(1-k)}$	
	$1+k+m-km \neq 0$, $m = 0,1,2,3$	$\Box w = m\lambda/w$
$\exp u$	$u = \log\left(\frac{2\lambda(m-1)}{w^2}\right)$, $m = 2,3$	$w_\mu w^\mu = \lambda$
$-\ddot{\Psi}(u)/\dot{\Psi}^3(u)$	$\Psi(u) = \frac{w}{\sqrt{\lambda}}$, $\lambda > 0$	
$\lambda_1 \sin u$	$u = 4\arctan\exp(w/\sqrt{\lambda_1})$, $\lambda_1 > 0$	
	$u = 4\arctan\tanh(w/\sqrt{-\lambda_1})$, $\lambda_1 < 0$	$\Box w = 0$
$\sinh u$	$u = 4\tanh^{-1}\tan(w/\sqrt{2\lambda})$, $\lambda > 0$	$w_\mu w^\mu = \lambda$
	$u = 4\tanh^{-1}\exp(w/\sqrt{2\lambda})$, $\lambda > 0$	
$\sin u/\cos^3 u$	$u = 4\arcsin(w/\sqrt{\lambda})$, $\lambda > 0$	
	$u = 4\arcsin\exp(w/\sqrt{\lambda})$, $\lambda > 0$	

We now present some results from [1] concerning the general solutions of the system

$$\Box w = \frac{m\lambda}{w}, \quad w_\mu w^\mu = \lambda, \quad n = 3. \tag{18}$$

Theorem 2. *The general solution of the system (18) for $m = 3$, $\lambda = 1$ is given by*

$$w^2 = [x_\mu + A_\mu(\tau)][x^\mu + A^\mu(\tau)],$$

where the function $\tau(x)$ is defined implicitly by the equation

$$[x_\mu + A_\mu(\tau)]B^\mu(\tau) = 0$$

and A_μ, B_μ are arbitrary differentiable functions of one variable satisfying the conditions

$$\dot{A}_\mu B^\mu = 0, \quad B_\mu B^\mu = 0.$$

Theorem 3. *The general solution of the system (18) for $m = 0$, $\lambda = -1$ is given by*

$$w = A_\mu(\tau)x^\mu + f_1(\tau), \tag{19}$$

where the function $\tau = \tau(x)$ is implicitly defined by the equation

$$B_\mu(\tau)x^\mu + f_2(\tau) = 0 \tag{20}$$

and A_μ, B_μ, f_1, f_2 are arbitrary functions of one variable satisfying

$$A_\mu A^\mu = -1, \quad \dot{A}_\mu B^\mu = 0, \quad A_\mu B^\mu = 0, \quad B_\mu B^\mu = 0. \tag{21}$$

The abtive theorems give us some general information about the solutions of (18) in particular cases. Of course, these results express the solution in terms of other functions, but now we know how to generate these functions: we have to choose A_μ, B_μ, f_1, f_2 appropriately so as to define both τ and then w (as we do below in a particular case). In this way, we have a systematic way of obtaining solutions of (18). Our approach to the solution of the nonlinear d'Alembert equation is based on a change of variable as in (2), and a decomposition of the equation (4) for the new variable w into 'component' equations (5), (6) and (7). Equation (5) involves the change of variable and the nonlinearity $F(u)$, whereas (7) provides us with a system which can be dealt with using theorems 1–3. The d'Alembert equation with nonlinearity $\sin u$ was discussed in [4], where a change of variable together with a decomposition of the ensuing equation was also used. The result obtained in [4] can be obtained with our results, as follows. In (20), (21) put

$$A_\mu = \beta_\mu, \quad B_\mu = \theta_\mu,$$

where $\theta_\mu = \alpha_\mu - \gamma_\mu$, α_μ, β_μ, γ_μ are constant vectors satisfying

$$\alpha_\mu\alpha^\mu = -\beta_\mu\beta^\mu = -\gamma_\mu\gamma^\mu = 1, \quad \alpha_\mu\beta^\mu = \alpha_\mu\gamma^\mu = \beta_\mu\gamma^\mu = 0.$$

Equation (20) then defines τ through

$$f_2(\tau) = -\theta_\mu x^\mu$$

and on choosing f_2 invertible we obtain the solution of (19)

$$w = \beta_\mu x^\mu + f(\theta_\mu x^\mu).$$

When $F(u) = -\sin u$ we obtain

$$u = 4\arctan\exp[\beta_\mu x^\mu + f(\theta_\mu x^\mu)], \tag{22}$$

where f is an arbitrary differentiable function. Equation (22) is the solution obtained in [4]. As can be seen, our method gives a useful way of obtaining exact solutions of nonlinear equations.

WF acknowledges support by the Swedish Natural Sciences Research Council, grant number R-RA 9423-307.

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