On the reduction of the nonlinear multi-dimensional wave equations and compatibility of the d'Alembert–Hamilton system

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The necessary conditions of the compatibility of the d'Alembert–Hamilton system in Minkowsky space $\mathbb{R}(1,n)$ are established. The problem of reduction of $P(1,n)$-invariant wave equations to ordinary differential equations is discussed.

1. Since Euler the method of reduction of partial differential equations (PDE) to ordinary differential equations (ODE) is one of the most effective ways to construct the exact solutions of PDE.

The papers [1–5] contain the symmetry reduction to ODE of the d'Alembert equation

$$\Box u = G(u), \quad \Box \equiv \partial^2_{x_0} - \partial^2_{x_1} - \cdots - \partial^2_{x_n} \tag{1}$$

(where $G(u)$ is an arbitrary smooth function). So the many-dimensional PDE [1] with the ansatz

$$u = \varphi(\omega), \tag{2}$$

where $\varphi \in C^2(\mathbb{R}^1,\mathbb{R}^1)$; $\omega = \omega(x) \in C^2(\mathbb{R}^{n+1},\mathbb{R}^1)$, the new variable, is reduced to the ODE of the form

$$\omega_\mu\omega_\mu\ddot{\varphi}(\omega) + (\Box\omega)\dot{\varphi}(\omega) = G(\varphi), \tag{3}$$

where $\omega_\mu \equiv \partial\omega/\partial x_\mu$, $\mu = 0,\ldots,n$. Hereafter summation over repeated indices is understood in the Minkowsky space $\mathbb{R}(1,n)$ with the metric $g_{\mu\nu} = \mathrm{diag}(1,-1,\ldots,-1)$.

In [3–5] using the symmetry properties of Eq. (1) and the subgroup structure of the $P(1,n)$ group the new variables $\omega = \omega(x)$ for Eq. (3) had been constructed.

Equation (3) depends on ω and does not depend on "old" variables x. $\omega(x)$ are invariants of the corresponding subgroups of the Poincaré group $P(1,n)$.

In the present paper we suggest the approach to the problem of reduction of PDE to ODE more general than one based on the employment of the symmetry properties of PDE [1–5].

Definition. *We say that the ansatz (2) reduces PDE (1) to ODE (3) when the new variable $\omega = \omega(x)$ satisfies both*

$$\Box\omega = F_1(\omega), \quad \omega_\mu\omega_\mu = F_2(\omega), \tag{4}$$

where $F_1(\omega)$, $F_2(\omega)$ are arbitrary smooth functions. Further we call Eqs. (4) the d'Alembert–Hamilton system.

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Evidently for every $\omega(x)$ satisfying the system (4) ODE (1) depends on ω only. Thus the problem of finding of the ansatze (2) reducing PDE (1) to ODE leads to the construction of solutions of the d'Alembert–Hamilton system (4).

Before solving the system (4) it is necessary to clear the matter of its compatibility, i.e., to describe all functions F_1, F_2 for the system (4) to possess nontrivial solutions.

In the three-dimensional case ($n = 2$) the compatibility of the system (4) was investigated by Collins [6] with the geometry methods. The compatibility of the d'Alembert–Hamilton system in the four-dimensional space $\mathbb{R}(1,3)$ was investigated in detail in [7]. We had generalized the results of [7] for the case of $(1+n)$-dimensional system of PDE (4) using the classical Hamilton–Cayley theorem.

2. The system (4) with the change of dependent variable $z = z(\omega)$ transforms to the following system of PDE

$$\Box\omega = F(\omega), \quad \omega_\mu\omega_\mu = \lambda, \quad \lambda = \mathrm{const}, \tag{5}$$

Eq. (3) having the form

$$\lambda\ddot{\varphi} + F(\omega)\dot{\varphi} = G(\varphi). \tag{6}$$

Before formulating the main result we adduce some preliminary statements.

Lemma 1. *The solutions of the system (5) satisfy the equalities*

$$\begin{gathered}\omega_{\mu\nu_1}\omega_{\nu_1\mu} = -\lambda\dot{F}(\omega), \quad \omega_{\mu\nu_1}\omega_{\nu_1\nu_2}\omega_{\nu_2\mu} = \frac{1}{2!}\lambda^2\ddot{F}(\omega), \quad \ldots, \\ \omega_{\mu\nu_1}\omega_{\nu_1\nu_2}\ldots\omega_{\nu_N\mu} = \frac{(-\lambda)^N}{N!}F^{(N)}(\omega), \quad N \geq 1,\end{gathered} \tag{7}$$

where $\omega_{\mu\nu} \equiv \partial^2\omega/\partial x_\mu\partial x_\nu$, $\mu, \nu = 0, \ldots, n$.

Proof. We prove the lemma with the method of mathematical induction by N.

Having differentiated twice the second equation of the system (5) with respect to x_α, x_β we obtain the relation

$$\omega_{\mu\alpha\beta}\omega_\mu + \omega_{\mu\alpha}\omega_{\mu\beta} = 0. \tag{8}$$

Convoluting (8) with the metric tensor $g^{\alpha\beta}$ we come to the equality

$$\omega_{\mu\alpha}\omega_{\mu\alpha} + \omega_\mu\Box\omega_\mu = 0.$$

Since $\Box\omega_\mu = (\partial/\partial x_\mu)F(\omega) = \omega_\mu\dot{F}(\omega)$, on the solutions of the system (5) the last expression can be rewritten in the form

$$\omega_{\mu\alpha}\omega_{\mu\alpha} + \lambda\dot{F}(\omega) = 0.$$

Thus the basic statement of induction is proved.

Let us suppose that the lemma holds for $N = k$. We prove that whence its statement follows for $N = k + 1$.

Convoluting (8) with the tensor

$$\omega_{\alpha\nu_2}\omega_{\nu_2\nu_3}\cdots\omega_{\nu_k\beta}$$

we get the equality

$$\omega_{\mu\alpha}\omega_{\alpha\nu_2}\cdots\omega_{\nu_k\beta}\omega_{\beta\mu} + \omega_\mu\omega_{\alpha\beta\mu}\omega_{\alpha\nu_2}\cdots\omega_{\nu_k\beta} = 0. \tag{9}$$

Since

$$\omega_\mu \omega_{\alpha\beta\mu}\omega_{\alpha\nu_2}\dots\omega_{\nu_k\beta} = \frac{1}{k+1}\omega_\mu(\omega_{\beta\alpha}\omega_{\alpha\nu_2}\cdots\omega_{\nu_k\beta})_\mu =$$
$$= \frac{1}{k+1}\omega_\mu\left(\frac{(-\lambda)^k}{k!}F^{(k)}(\omega)\right)_\mu = -\frac{(-\lambda)^{k+1}}{(k+1)!}F^{(k+1)}(\omega)$$

(we used the assumption of induction) then it follows from (9) that

$$\omega_{\mu\nu_1}\omega_{\nu_1\nu_2}\cdots\omega_{\nu_{k+1}\mu} = \frac{(-\lambda)^{k+1}}{(k+1)!}F^{(k+1)}(\omega)$$

The Lemma is proved.

Lemma 2. *On the solutions of the system (5) the equality*

$$\det\|\omega_{\mu\nu}\| = 0 \tag{10}$$

holds.

The proof follows from the fact that (10) is the criterium of functional dependence of $\omega_0,\dots,\omega_n$.

Theorem 1. *For the system (5) to be compatible it is necessary that*

$$F(\omega) = \lambda\dot f(\omega)f^{-1}(\omega), \tag{11}$$

f satisfying the condition

$$f^{(n+1)}(\omega) \equiv \frac{d^{n+1}f(\omega)}{d\omega^{n+1}} = 0. \tag{12}$$

Proof. Let us first consider the case $\lambda \neq 0$. For an arbitrary $(n+1)\times(n+1)$-matrix $W = \|w_{\mu\nu}\|$ by virtue of the Hamilton–Cayley theorem the equality

$$\sum_{k=0}^{n-1}(-1)^k\sum M_k \operatorname{tr}(W^{n-k}) + (-1)^n n\det W = 0 \tag{13}$$

is true. $\sum M_k$ in (13) is the sum of basic minors of the order k of the matrix W, which is calculated with the recurrent formula

$$\begin{aligned}&\sum M_k = \left[\sum_{l=0}^{k-1}(-1)^l\sum M_l\operatorname{tr}(W^{k-l})\right](-1)^{k-l}k^{-1}, \quad k\geq 1;\\ &\sum M_0 = 1.\end{aligned} \tag{14}$$

We take the matrix elements of W as

$$w_{\mu\nu} = \sum_{\alpha=0}^{n} g_{\alpha\nu}\omega_{\mu\alpha},$$

then from Lemmas 1, 2

$$\operatorname{tr}(W^k) = \frac{(-\lambda)^{k-1}}{(k-1)!}F^{(k-1)}(\omega), \quad \det W = 0. \tag{15}$$

The substitution of formula (15) into (14) gives the ODE for determination of the function $F = F(\omega)$. Let us show that this ODE reduces using the nonlocal change of variable (11) to the form (12).

Let

$$Y_N = \sum_{k=0}^{N} (-1)^k \sum M_k \operatorname{tr}(W^{N-k+1})$$

then $\sum M_k = ((-1)^{k-1}/k) Y_{k-1}$; whence

$$Y_N = \sum_{k=0}^{N} \frac{(-1)^{N-k+1}}{k(N-k)!} \lambda^{N+1-k} \left(\frac{\dot f}{f}\right)^{(N-k)} Y_{k-1}.$$

Using the method of mathematical induction we prove that

$$Y_N = \frac{(-1)^{N+1}}{N!} \lambda^{N+1} \frac{f^{(N+1)}}{f}. \tag{16}$$

For $N = 1, 2, 3$ this equality follows from the results of [7]. Let us assume that (16) holds for every $m \in \mathbb{N}$, $m \le N-1$. We show that whence it follows that (16) is true for $m = N$.

Indeed

$$Y_N = \sum_{k=0}^{N} \frac{(-1)^{k+1}}{k(k-1)!} \lambda^k \frac{f^{(k)}}{f} \frac{(-1)^{N-k}}{(N-k)!} \lambda^{N+1-k} \left(\frac{\dot f}{f}\right)^{(N-k)} =$$

$$= \frac{(-1)^{N+1} \lambda^{N+1}}{N!} \sum_{k=0}^{N} C_N^k \frac{f^{(k)}}{f} \left(\frac{\dot f}{f}\right)^{(N-k)} = \frac{(-1)^{N+1} \lambda^{N+1}}{N!} \frac{f^{(N+1)}}{f},$$

the same as what was to be proved.

From the equality (10) $Y_n = (-1)^{n+1} n \det W = 0$ whence by virtue of (15), (16) we obtain

$$f^{(n+1)} = 0.$$

Let us consider now the case $\lambda = 0$. Using Lemmas 1, 2 we have

$$\operatorname{tr}(W^k) = 0, \quad k = \overline{2, n}; \quad \det W = 0.$$

Taking into account these equalities we can rewrite the Hamilton–Cayley identity in the form

$$Y_n = 0,$$

where $Y_n = (-1)^{n+1} (F/n!)$. Whence we conclude that $F = 0$. The theorem is proved.

Consequence. *The system* $\Box u = F(u)$, $u_\mu u_\mu = 0$ *is compatible iff* $F(u) = 0$.

Proof. The necessity of the above statement follows from the Theorem 1. The sufficiency is proved by the fact that the function $u(x) = x_0 + x_1$ satisfies both the d'Alembert ($\Box u = 0$) and the Hamilton ($u_\mu u_\mu = 0$) equations.

Let us note that this consequence was proved in [9] by another technique.

Theorem 2. *The system of PDE (5) is invariant with respect to the conformul group of transformations of the Minkowskv space* $\mathbb{R}(1,n)$ *iff [7, 8]*

$$F(\omega) = \lambda n(\omega + C)^{-1}, \quad c = \mathrm{const}, \quad \lambda > 0. \tag{17}$$

The proof is carried out by S. Lie's method.

Let us note that the formula (17) is obtained from (11) when $f = (\omega + c)^n$. So Theorem 2 demonstrates the deep connection between the symmetry of overdetermined system of PDE (5) and its compatibility.

Note. It is well known that PDE (1) is invariant under the conformal group $C(1,n)$ iff $G(u) = cu^{(n+3)/(n-1)}$ (see, e.g., [3, 10, 11]). Thus the additional condition $u_\mu u_\mu = \lambda$ picks out the subset of solutions of Eq. (1) which admits a wider symmetry group than the set of its solutions in a whole. In other words the nonlinear d'Alembert equation is conditionally invariant under the conformal group if $G(u) = \lambda n(u+c)^{-1}$ (the notion of conditional invariance of PDE was introduced in [12–14]; see also [15, 16]).

The sufficient conditions of the compatibility of the d'Alembert–Hamilton system (5) are

$$F(\omega) = |\lambda| N(\omega + c)^{-1}, \tag{18}$$

where $c = \mathrm{const}$, $N = 1-n, 2-n, \ldots, 0, 1, \ldots, n$.

As shown by Collins [6] the above conditions are the necessary and sufficient ones for the system (5) to be compatible if $n = 1, 2$. In the Appendix we list exact solutions of the d'Alembert Hamilton system under (18) for $n = 3$ obtained in [3, 5, 7–9, 17]. Let us emphasize that solutions numbered (5)–(7), (9) are not invariants of the Poincaré group $P(1,n)$. Nevertheless they satisfy the d'Alembert–Hamilton system and, consequently, can be used to reduce Eq. (1) to ODE via ansatz (2).

In conclusion we briefly consider the reduction of the arbitrary Poincaré-invariant wave equation to ODE. As it was established in [18] every $P(1,n)$-invariant PDE for the scalar function $u = u(x)$ can be represented in the form where $H(R_1, \ldots, R_n; S_1, \ldots, S_n, u) = 0$

$$R_j = u_{\mu_1} u_{\mu_1\mu_2} \cdots u_{\mu_{j-1}\mu_j} u_{\mu_j}, \quad S_j = u_{\mu_1\mu_2} u_{\mu_2\mu_3} \cdots u_{\mu_j\mu_1},$$

and H is some continuous function.

It turns out that ansatz (2), where $\omega = \omega(x)$ satisfies system (5), reduces every PDE of the form (19) to ODE.

Using Lemma 2 we obtain

$$S_j(\varphi(\omega)) = \lambda^j \ddot{\varphi}^j + \dot{\varphi}^j S_j(\omega) = \lambda^j \ddot{\varphi}^j + \dot{\varphi}^j \frac{(-\lambda)^{j-1}}{(j-1)!} F^{(j-1)}(\omega),$$

$$R_j(\varphi(\omega)) = \dot{\varphi}^2 \ddot{\varphi}^{j-1} \lambda^j, \quad j = \overline{1,n}.$$

Substituting these formulae to (19) we get

$$H(R_1, \ldots, R_n, S_1, \ldots, S_n u)|_{u=\varphi(\omega)} = \tilde{H}(\omega, \varphi, \dot{\varphi}, \ddot{\varphi}).$$

Thus knowing the exact solutions of the d'Alembert–Hamilton system we can construct using ansatz (2) the exact solutions of the arbitrary Poincaré-invariant equation (19).

Appendix

Exact solutions of the d'AIembert–Hamilton system (5)
in $1 + 3$-dimensional Minkowsky space

N	$*$	$F(\omega)$	$\omega = \omega(x)$
1	1	0	x_0
2	1	ω^{-1}	$\left(x_0^2 - x_1^2\right)^{1/2}$
3	1	$2\omega^{-1}$	$\left(x_0^2 - x_1^2 - x_2^2\right)^{1/2}$
4	1	$3\omega^{-1}$	$\left(x_0^2 - x_1^2 - x_2^2 - x_3^2\right)^{1/2}$
5	-1	0	$x_1 \cos(h_1) + x_2 \sin(h_2) + h_2$
6	-1	0	$x_0 - x_1 \cos(g_1) - x_2 \sin(g_1) - g_2 = 0$
7	-1	$-\omega^{-1}$	$\left[(x_1 + h_1)^2 + (x_2 + h_2)^2\right]^{1/2}$
8	-1	$-2\omega^{-1}$	$\left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}$
9	0	0	h_1

Note. Here h_1, h_2 are arbitrary smooth functions on $x_0 + x_3$ and g_1, g_2 are arbitrary smooth functions on $\omega + x_3$.

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