On the non-Lie reduction of the nonlinear Dirac equation

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The method of construction of exact solutions of nonlinear spinor equations based on their conditional (non-Lie) symmetry is suggested. With the help of this method new ansätze that reduce the nonlinear Poincaré-invariant Dirac equation to ordinary differential equations are constructed. The new family of exact solutions of the nonlinear Dirac equation with scalar selfinteraction is found.

1. Introduction

It is common knowledge that the classical Lie approach to the construction of exact solutions of partial differential equations (PDEs) essentially uses invariance properties of the set of solutions of the considered equation [1, 2]. In Refs. [3–5] a natural generalization of the Lie approach was suggested that takes into account not only the symmetry of the set of solutions of PDEs as a whole, but the symmetry of their subsets as well. This is achieved by imposing on the solutions of the initial equation such additional conditions (equations) that the obtained system of PDEs is compatible and possesses wide symmetry.

Using the above idea, in the present paper we construct a family of the new exact solutions of the following nonlinear spinor equation:

$$\{i\gamma_\mu\partial_{mu} - \lambda(\bar\psi\psi)^{1/2k}\}\psi = 0, \quad \lambda, k = \mathrm{const}, \tag{1}$$

where $\psi = \psi(x_0, x_1, x_2, x_3)$ is the four-component complex function, $\bar\psi = \psi^\dagger\gamma_0$ and γ_μ are 4×4 Dirac matrices, $\partial_\mu = \partial/\partial x_\mu$, and $\mu = \overline{0,3}$. Hereafter, the summation over the repeated indices is supposed.

2. Construction of the non-Lie anzätze for the spinor field

The solution of Eq. (1) is found in the form

$$\psi(x) = \exp\{f_{\mu\nu}(x)\gamma_\mu\gamma_\nu\}\varphi(\omega), \tag{2}$$

where $\varphi(\omega)$ is a four-component function and $f_{\mu\nu}(x)$ and $\omega = \omega(x)$ are scala real functions. The functions $f_{\mu\nu}$, ω are chosen such that substitution of expression (2) into Eq. (1) yields an ordinary differential equation (ODE) for $\varphi = \varphi(\omega)$.

We shall describe ansätze (2) as reducing the PDE (1) to systems of ODEs if the functions $f_{\mu\nu}$, ω are of the following structure:

$$\begin{aligned}
&f_{00} = -f_{11} = -f_{22} = -f_{33} = \frac{1}{4}\theta_0(x_0 + x_3, x_1, x_2),\\
&f_{01} = -f_{10} = f_{13} = -f_{31} = \frac{1}{2}\theta_1(x_0 + x_3, x_1, x_2),\\
&f_{02} = -f_{20} = f_{23} = -f_{32} = \frac{1}{2}\theta_2(x_0 + x_3, x_1, x_2),\\
&f_{03} = f_{30} = f_{12} = f_{21} = 0, \quad \omega = \omega(x_0 + x_3, x_1, x_2).
\end{aligned}$$

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Substituting the ansatz

$$\psi(x) = \exp\{\theta_0 + (\theta_1\gamma_1 + \theta_2\gamma_2)(\gamma_0 + \gamma_3)\}\varphi(\omega) \tag{3}$$

into Eq. (1) and multiplying the obtained equality by

$$\exp\{-\theta_0 - (\theta_1\gamma_1 + \theta_2\gamma_2)(\gamma_0 + \gamma_3)\}$$

one has

$$\begin{aligned} &i[(\gamma_0+\gamma_3)\partial_\xi\theta_0 + \gamma_a\partial_a\theta_0 + \gamma_a\gamma_B(\partial_a\theta_B)(\gamma_0+\gamma_3) - 2\theta_a(\partial_a\theta_0)(\gamma_0+\gamma_3)]\varphi + \\ &\quad + i[(\gamma_0+\gamma_3)(\partial_\xi\omega - 2\theta_a\partial_a\omega) + \gamma_a\partial_a\omega]\varphi - \lambda e^{\theta_0/k}(\bar\varphi\varphi)^{1/2k}\varphi = 0, \end{aligned}$$

where $\xi = x_0 + x_3$, $\partial_\xi = \partial/\partial\xi$, $a = \overline{1,2}$, and $B = \overline{1,2}$.

Hence it follows that ansatz (3) reduces the initial PDEs to ODEs if the nonlinear equations hold:

$$\begin{aligned} &\partial_\xi\theta_0 - 2\theta_a\partial_a\theta_0 - \partial_a\theta_a = e^{\theta_0/k}f_1(\omega), \quad \partial_1\theta_0 = e^{\theta_0/k}f_2(\omega), \\ &\partial_a\theta_0 = e^{\theta_0/k}f_3(\omega), \quad \partial_2\theta_1 - \partial_1\theta_2 = e^{\theta_0/k}f_4(\omega), \\ &\partial_\xi\omega - 2\theta_a\partial_a\omega = e^{\theta_0/k}f_5(\omega), \quad \partial_1\omega = e^{\theta_0/k}f_6(\omega), \quad \partial_2\omega = e^{\theta_0/k}f_7(\omega). \end{aligned} \tag{4}$$

In Eqs. (4) $f_1, \ldots, f_7$ are arbitrary smooth real functions.

It is worth noting that as a result of the arbitrariness of the function $\varphi(\omega)$ substitution of the expressions

$$\omega(x), \quad \theta_\alpha(x) \tag{5}$$

and

$$\tilde f(\omega(x)), \quad \theta_\alpha(x) + \tilde f_\alpha(\omega(x)), \tag{6}$$

where $\bar f, \bar f_a \subset C^1(\mathbb{R}^1, \mathbb{R}^1)$, $\alpha = \overline{0,2}$, into formula (3) gives the same ansatz for the field $\psi(x)$. In this sebse solutions of system (4) of the forms (5) and (6) are equivalent.

System (4) contains seven equations for four functions, i.e., it is an overdetermined system. This fact makes it possible to construct its general solution.

Theorem. *The general solution of the nonlinear system of PDEs (4) determined up to the above equivalence relation is given by one of the following formulas:*

$$\begin{aligned} &\theta_0 = k\ln\omega_1, \quad \theta_1 = (2\omega_1)^{-1}(\dot\omega_1 x_1 + \dot\omega_2), \\ &\theta_2 = (2\omega_1)^{-1}[(2k-1)\dot\omega_1 x_2 + \omega_3], \quad \omega = \omega_1 x_1 + \omega_2; \\ &\theta_0 = -k\ln(x_1 + \omega_1), \\ &\theta_a = \omega_3[(x_1+\omega_1)^2 + (x_2+\omega_2)^2]^{k-1}(x_a + \omega_a) + \tfrac{1}{2}\dot\omega_a, \quad a = \overline{1,2}, \\ &\omega = (x_1+\omega_1)(x_2+\omega_2)^{-1}; \\ &\theta_0 = 0, \quad \theta_1 = R(x_1 + ix_2, x_0 + x_3) + R(x_1 - ix_2, x_0 + x_3) + \omega_1 x_1, \\ &\theta_2 = iR(x_1 + ix_2, x_0 + x_3) - iR(x_1 - ix_2, x_0 + x_3) + \omega_2 x_1, \quad \omega = x_0 + x_3. \end{aligned} \tag{7}$$

Here ω_1, ω_2 and ω_3 designate arbitrary real smoothfunctions on $x_0 + x_3$ and R designates an arbitrary analytical function on the first variable.

Let us adduce the main steps of the proof. First, an overdetermined system made up of the second, third, sixth, and seventh equations in (7) is integrated. Upon making the change of the variable $\theta = e^{-\theta_0/k}$ we rewrite this system in the form

$$\partial_a\theta = F_a(\omega), \quad \partial_a\omega = \theta^{-1}G_a(\omega), \quad F_a, G_a \subset C^1(\mathbb{R}^1,\mathbb{R}^1), \quad a = 1,2. \tag{8}$$

From the necessary and sufficient compatibility conditions of system (8), $\partial_1\partial_2\theta = \partial_2\partial_1\theta$, $\partial_1\partial_2\omega = \partial_2\partial_1\omega$, one has the following relations for $F_a(\omega)$, $G_a(\omega)$:

$$\dot F_1 G_2 = G_1\dot F_2, \quad G_2\dot G_1 - G_1F_2 = G_1\dot G_2 - G_2F_1, \tag{9}$$

where the overdot means differentiation with with respect to ω.

The procedure for the integration of the system of ODEs (9) is essentially simplified by the fact that the equivalence conditions (6) induce the equivalence relation on the set of solutions of Eqs. (9):

$$\begin{aligned}&F_a(\omega) \sim F_a(f(\omega)) - \dot g(\omega)G_a(f(\omega)),\\ &G_a(\omega) \sim (\dot f(\omega))^{-1}G_a(f(\omega))(g(\omega))^{-1},\end{aligned} \tag{10}$$

where $f, g \subset C^1(\mathbb{R}^1,\mathbb{R}^1)$, $\dot f g \neq 0$.

By integrating the system of PDEs (8) and (9) one establishes that is general solution up to the equivalence relations (6) and (10) is determined by one of the following formulas:

$$\begin{aligned}&F_1 = G_1 = 1, \quad F_2 = G_2 = 0, \quad \theta = \omega_1^{-1}, \quad \omega = \omega_1x_1 + \omega_2;\\ &F_1 = 1, \quad F_2 = 0, \quad G_1 = \omega, \quad G_2 = -\omega^2,\\ &\theta = x_1 + \omega_1, \quad \omega = (x_1+\omega_1)(x_2+\omega_2)^{-1};\\ &F_1 = F_2 = G_1 = G_2 = 0, \quad \omega = \xi, \quad \theta = 1;\\ &F_1 = F_2 = 0, \quad G_a \subset C^1(\mathbb{R}^1,\mathbb{R}^1), \quad \omega = \xi,\\ &\theta = G_1(\xi)x_1 + G_2(\xi)x_2 + \omega_3.\end{aligned} \tag{11}$$

Here ω_1, ω_2, and ω_3 are arbitrary real smooth functions on ξ.

Substitution on the sxpressions for the functions $\omega(x)$, $\theta_0(x) = -k\ln\theta(x)$ into the remainder of Eqs. (4) yields four systems of PDEs on $\theta_1(x)$ and $\theta_2(x)$. By integrating the first systems of PDEs one arrives at formulas (7). The fourth system of PDEs proves to be incompatible system.

Substitution of expressions (7) into formula (3) gives three classes of ansätze for the spinor field:

$$\begin{aligned}&\psi(x) = \omega_1^k\exp\{(2\omega_1)^{-1}[(\dot\omega_1x_1 + \dot\omega^2)\gamma_1 +\\ &\qquad + ((2k-1)\dot\omega_1x_2 + \omega_3)\gamma_2](\gamma_0+\gamma_3)\}\varphi(\omega_1x_1+\omega_2),\\ &\psi(x) = (x_1+\omega_1)^{-k}\exp\{\omega_3[(x_1+\omega_1)^2 + (x_2+\omega_2)^2]^{k-1}\\ &\qquad \times\gamma_a(x_a+\omega_a)(\gamma_0+\gamma_3) + \tfrac{1}{2}\dot\omega_a\gamma_a(\gamma_0+\gamma_3)\}\varphi((x_1+\omega_1)(x_2+\omega_2)^{-1});\\ &\psi(x) = \exp\{[(R+R^*+\omega_1x_1)\gamma_1 + (iR - iR^* + \omega_2x_1)\gamma_2](\gamma_0+\gamma_3)\}\varphi(x_0+x_3),\end{aligned} \tag{12}$$

reducing the nonlinear Dirac equation (1) to the system of ODEs

$$\begin{aligned}&i\gamma_1\dot\varphi = \lambda(\bar\varphi\varphi)^{1/2k}\varphi,\\ &i(\gamma_2-\gamma_1\omega)\dot\varphi = \lambda(\bar\varphi\varphi)^{1/2k}\varphi,\\ &i(\gamma_0+\gamma_3)\dot\varphi = \lambda(\bar\varphi\varphi)^{1/2k}\varphi.\end{aligned} \tag{13}$$

The general solution of the first system in (13) is given by the formula [6]

$$\varphi(\omega) = \exp\{i\lambda\gamma_1(\bar{\chi}\chi)^{1/2k}\omega\}\chi,$$

where χ is a constant four-componet column.

By substitutiong the above expression into the first ansatz in (12) we obtain the new family of exact solutions of the nonlinear spinor equation (1) containing the three arbitrary functions $\omega_n(x_0+x_3)$, $n=\overline{1,3}$

$$\begin{aligned}\psi(x) = \omega_1^k \exp\{(2\omega_1)^{-1}[(\dot\omega_1 x_1 + \dot\omega_2)\gamma_1 + ((2k-1)\dot\omega_1 x_2 + \omega_3)\gamma_2](\gamma_0+\gamma_3)\} \times \\ \times \exp\{i\lambda\gamma_1(\bar{\chi}\chi)^{1/2k}(\omega_1 x_1 + \omega_2)\}\chi.\end{aligned} \tag{14}$$

Let us emphasize that ansätze (12) are noninvariant under the three-parameter subgroups of the symmetry group admitted by Eq. (1) (in the case involved it is the extended Poincaré group $\tilde{P}(1,3)$ (see Ref. [6])) and, consequently, they cannot be obtained in the framework of the traditional Lie approach.

3. Conditional invariance of nonlinear Dirac equation

Let us now construct the non-Lie ansätze (12) using the conditional invariance of the nonlinear Dirac equation (1).

Definition. *Equation (1) is conditionally invariant with respect to the operators*

$$Q_\tau = \varepsilon_{\tau\mu}(x)\partial_\mu + \eta_\tau(x), \quad \tau=\overline{1,N}, \tag{15}$$

where $\varepsilon_{\tau\mu}(x)$ are real scalar functions and $\eta_\tau(x)$ are variable 4×4 matrices if the system

$$\{i\gamma_\mu\partial_\mu - \lambda(\bar\psi\psi)^{1/2k}\}\psi = 0, \quad Q_\tau\psi = 0, \quad \tau=\overline{1,N} \tag{16}$$

is invariant in the Lie sense under the one-parameter transformations groups generated by the operators Q_τ.

Described another way, Eq. (1) possesses conditional symmetry if the set of its solutions contains the nonempty subset that does not coincide with the whole set having nontrivial symmetry.

We shall point out the explicit form of the operators Q_n, $n=\overline{1,3}$ such that (14) satisfies system (16). For this purpose it is necessary to solve the following system of algebraic equations on the functions $\varepsilon_{\nu\mu}$, η_ν:

$$\begin{aligned}&\varepsilon_{n\mu}\partial_\mu\omega = 0,\\ &\eta_n - [\varepsilon_{n\mu}\partial_\mu \exp\{\theta_0 + (\gamma_1\theta_1+\gamma_2\theta_2)(\gamma_0+\gamma_3)\}] \times \\ &\qquad \times \exp\{-\theta_0 - (\gamma_1\theta_1+\gamma_2\theta_2)(\gamma_0+\gamma_3)\}.\end{aligned} \tag{17}$$

Here ω, θ_0, θ_1, and θ_2 are scalar functions determined by the first set of formulas in (7) and $n=\overline{1,3}$.

Solving Eqs. (17) one has

$$\begin{aligned}&Q_1 = \frac{1}{2}(\partial_0 - \partial_3),\\ &Q_2 = \omega_1\partial_2 + \frac{1}{2}(1-2k)\dot\omega_1\gamma_2(\gamma_0+\gamma_3),\\ &Q_3 = \frac{1}{2}\omega_1(\partial_0+\partial_3) - \dot\omega_1(x_1\partial_1 + x_2\partial_2) - \dot\omega_2\partial_2 - k\dot\omega_1 + \\ &\qquad + (2\omega_1)^{-1}[(2\dot\omega_1\dot\omega_2 - \omega_1\ddot\omega_2)\gamma_1 + 2(\omega_3\dot\omega_1 - \omega_1\dot\omega_3)\gamma_2](\gamma_0+\gamma_3) + \\ &\qquad + (2\omega_1)^{-1}(2\dot\omega_1^2 - \omega_1\ddot\omega_1)(\gamma_1 x_1 + (2k-1)\gamma_2 x_2)(\gamma_0+\gamma_3).\end{aligned} \tag{18}$$

It is evident that the operators Q_2 and Q_3 are not linear combinations of the generators of the extended Poincaré group; consequently, they do not belong to the Lie algebra of the symmetry group of Eq. (1). By direct verification one can be convinced that the following relations hold:

$$\tilde{Q}_1 L = 0,$$
$$\tilde{Q}_2 L = 2(2k-1)\dot{\omega}_1\gamma_2 Q_1\psi + 2k\dot{\omega}_1\omega_1^{-1}(\gamma_0+\gamma_3)Q_2\psi + \frac{1}{2}(2k-1)\dot{\omega}_1\gamma_2(\gamma_0+\gamma_3)L,$$
$$\begin{aligned}\tilde{Q}_3 L = {} & 2\omega_1^{-1}[(\omega_1\ddot{\omega}_1 - 2\dot{\omega}_1^2)(\gamma_1 x_1 + (2k-1)\gamma_2 x_2) + (\omega_1\ddot{\omega}_2 - 2\dot{\omega}_1\dot{\omega}_2)\gamma_1 + \\ & + 2(\omega_1\dot{\omega}_3 - \omega_3\dot{\omega}_1)\gamma_2]Q_1\psi + 2\omega_1^{-2}[(1-k)(2\dot{\omega}_1^2 - \omega_1\ddot{\omega}_1)x_2 + \omega_1\dot{\omega}_3 - \\ & - \omega_3\dot{\omega}_1]Q_2\psi + 2\dot{\omega}_1\omega_1^{-1}(\gamma_0+\gamma_3)Q_3\psi - \\ & - \{\dot{\omega}_1 + (2\omega_1)^{-1}(2\dot{\omega}_1^2 - \omega_1\ddot{\omega}_1)(\gamma_1 x_1 + (2k-1)\gamma_2 x_2)(\gamma_0+\gamma_3) + \\ & + (2\omega_1)^{-1}[(2\dot{\omega}_1\dot{\omega}_2 - \omega_1\ddot{\omega}_2)(\gamma_1 + 2(\omega_3\dot{\omega}_1 - \omega_1\dot{\omega}_3)\gamma_2](\gamma_0+\gamma_3)\}L,\end{aligned}$$

where $\tilde{Q}_a$ designates the first prolongation of the operator Q_a,

$$L = i\gamma_\mu\partial_\mu\psi - \lambda(\bar{\psi}\psi)^{1/2k}\psi.$$

In addition, the commutational relations of the form

$$[Q_1, Q_2] = [Q_1, Q_3] = 0, \quad [Q_2, Q_3] = -2\dot{\omega}_1 Q_2$$

hold true.

Hence follows that the nonlinear Dirac equation (1) is conditionally invariant with respect to the operators (18).

In the same way it is established that the second and third ansätze in (12) can be obtained by using conditional invariance of Eq. (1).

In conclusion, let us note that ansatze (12) reduce to ODEs the more general spinor equations

$$\{i\gamma_\mu\partial_\mu - [f_1((\bar{\psi}\psi)(\bar{\psi}\gamma_4\psi)^{-1}) + f_2((\bar{\psi}\psi)(\bar{\psi}\gamma_4\psi)^{-1})(\bar{\psi}\psi)^{1/2k}\}\psi = 0,$$

where $f_a \subset C^1(\mathbb{R}^1, \mathbb{C}^1)$.

4. Discussion

We emphasize once more that ansatze for the spinor field ψ constructed above cannot be obtained with the help of symmetry reduction by subgroups of the invariance group of Eq. (1), These ansätze can be constructively described within the framework of the conception of “conditional invariance” introduced for the first time in Refs. [5] and [7] (see Appendix 4 of Ref. [7]). It seems impossible to obtain the complete description of conditional symmetry of the nonlinear Dirac equation (1) since (1) since the determining equations on the coefficients of the infinitesimal operators, unlike the classical case, are nonlinear equations.

Conditional symmetry of some other nonlinear mathematical physics equations has been investigated in Refs. [8–11]. Let us also mention that the wide classes of exact solutions of Eq. (1) that correspond to its Lie symmetry were constructed in Ref. [12].

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