On the connection between solutions of Dirac and Maxwell equations, dual Poincaré invariance and superalgebras of invariance and solutions of nonlinear Dirac equations

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The connection between solutions of massless Dirac and Maxwell equations is established. It is shown that the massless Dirac equation is invariant under three different representation of the Poincaré algebra corresponding to spins $\frac{1}{2}$ and 1 and 0, and under three superalgebras. All generators of these symmetry algebras and superalgebras are local (differential operators of first order). A system of two Dirac equations with masses m and $-m$ has analogous symmetry properties. Invariant nonlinear generalizations of this system are described. We construct the complete set of $P(1,3)$-inequivalent ansätze of codimension 1 for all representations of Poincaré algebra discused. These ansätze are used for reduction and finding exact solutions of some nonlinear Dirac equations.

1. Introduction

It is well known that the Dirac equation describes a particle with spin-$\frac{1}{2}$, or a fermionic field, because it is invariant with respect to the representation $D(\frac{1}{2},0)\oplus D(0,\frac{1}{2})$ of the Poincaré algebra $AP(1,3)$. In this paper we will show that the massless Dirac equation as well as the system of two coupled Dirac equations with masses m and $-m$ are invariant not only with respect to the spin-$\frac{1}{2}$ representation of $AP(1,3)$ but also under integer spin representations of $AP(1,3)$. This means that Dirac equations describe not only fermionic fields but also bosonic ones.

In section 2 we obtain formulae of connection between solutions of the massless Dirac equation and Maxwell equations for a vacuum, so that one can construct solutions of the Dirac equation knowing solutions of the Maxwell equations and vice versa. Further, we show that the massless Dirac equation is invariant under three different representations of the Poincaré algebra $AP(1,3)$ and under three superalgebras. All generators of these symmetries are differential operators of first order and belong to the maximal in the sense of Lie invariance algebra of the equation. We shall call invariance of an equation, with respect to different representations of the Poincaré algebra, dual Poincaré invariance.

In section 3 we study dual Poincaré invariance of the Dirac equation with non-zero mass and prove that the system of two coupled Dirac equations with masses m and $-m$ possesses this symmetry. It is worthwhile to note that the same Dirac system was studied by Fushchych [1, 2] and by Petroni et al [12, 13]. Fushchych [1, 2] had shown that the most symmetric (including discrete symmetries) spinor representation of the Poincaré algebra is realized only on the system of two coupled Dirac equations and such a realization is impossible on a single Dirac equation with non-zero mass. We prove that the Dirac system under study is also invariant under two superalgebras. Nonlinear dual Poincaré invariant generalizations of the equations are considered.

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In section 4 we construct the complete set of the $P(1,3)$-inequivalent ansätze of codimension 1 for all representations of $AP(1,3)$ discussed in the previous sections. These ansätze reduce corresponding Poincaré invariant equation to a system of ordinary differential equations (ODEs). Here we essentially used results on the subalgebraic classification of $AP(1,3)$ of Patera et al [11] and Grundland et al [7]. It will be noted that the $P(1,3)$-inequivalent ansätze of codimenions 1 and 3 for the spin-$\frac{1}{2}$ Dirac field are fully described in Fushchych and Zhdanov [5], Fushchych and Shtelen [4] and Fushchych et al [6]. Using ansätze constructed, we make reductions and find exact solutions of some nonlinear Dirac equations. An example solution of a linear Dirac equation is considered. This solution is obtained by making use of the vector representation of $AP(1,3)$ of the coupled Dirac equations. It has an unusual structure and can be obtained as the invariant solution of the non-Lie symmetry operator of second order. In conclusion we give operators which transform the fermionic ansätze into bosonic ones.

The massless Dirac equation and Maxwell equations

Consider the massless Dirac equation

$$i\gamma\partial\psi \equiv i\gamma^\mu\partial_\mu\psi = 0, \tag{2.1}$$

where $\psi = \psi(x)$ is a four-component complex function (column), $x = \{x^0 = t, \boldsymbol{x}\} \in R(1,3)$, $\mu = \overline{0,3}$, $\partial_\mu = \partial/\partial x^\mu$ and γ^μ are 4×4 Dirac matrices,

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{2.2}$$

There is a connection between solutions of (2.1) and the Maxwell equations for a vacuum [15]:

$$\begin{aligned} \dot{\boldsymbol{E}} &\equiv \frac{\partial \boldsymbol{E}}{\partial t} = \mathrm{rot}\,\boldsymbol{H}, \quad \mathrm{div}\,\boldsymbol{E} = 0, \\ \dot{\boldsymbol{H}} &\equiv \frac{\partial \boldsymbol{H}}{\partial t} = -\mathrm{rot}\,\boldsymbol{H}, \quad \mathrm{div}\,\boldsymbol{H} = 0, \end{aligned} \tag{2.3}$$

where $\boldsymbol{E} = (E_1, E_2, E_3)$ and $\boldsymbol{H} = (H_1, H_2, H_3)$ are vectors of electric and magnetic field. To establish this connection let us decompose an arbitrary solution of (2.1) intoreal and imaginary parts using the notation of Ljolje [9]:

$$\psi = \psi_{\mathrm{real}} + i\psi_{\mathrm{imag}} = \begin{pmatrix} -D_1 \\ D_3 \\ -B_2 \\ -G \end{pmatrix} + i \begin{pmatrix} D_2 \\ -F \\ -B_1 \\ B_3 \end{pmatrix}. \tag{2.4}$$

Theorem 1. *Let ψ defined by (2.4) be an arbitrary solution of the massless Dirac equation (2.1). Then the functions*

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{D} + \nabla \int_{t_0}^{t} G(\tau, \boldsymbol{x}) d\tau + \nabla \tilde{G}(t_0, \boldsymbol{x}), \\ \boldsymbol{H} &= \boldsymbol{B} + \nabla \int_{t_0}^{t} F(\tau, \boldsymbol{x}) d\tau + \nabla \tilde{F}(t_0, \boldsymbol{x}), \end{aligned} \tag{2.5}$$

where $\tilde{G}(t_0, \boldsymbol{x})$ and $\tilde{F}(t_0, \boldsymbol{x})$ satisfy the Poisson equations

$$\Delta \tilde{G}(t_0, \boldsymbol{x}) = \frac{\partial G(\tau, \boldsymbol{x})}{\partial \tau}\Big|_{\tau = t_0}, \quad \Delta \tilde{F}(t_0, \boldsymbol{x}) = \frac{\partial F(\tau, \boldsymbol{x})}{\partial \tau}\Big|_{\tau = t_0}, \tag{2.6}$$

t_0 is an arbitrary constant, are solutions of the Maxwell equations (2.3).

Prof. First of all we note that after substitution of (2.4) into (2.1) and separation into real and imaginary parts we get Maxwell equations with currents

$$\begin{aligned} &\dot{\boldsymbol{D}} - \operatorname{rot} \boldsymbol{B} = -\nabla G, \quad \operatorname{div} \boldsymbol{D} = -\dot{G}, \\ &\dot{\boldsymbol{B}} + \operatorname{rot} \boldsymbol{D} = -\nabla F, \quad \operatorname{div} \boldsymbol{B} = -\dot{F}, \end{aligned} \tag{2.7}$$

where $\boldsymbol{D} = (D_1, D_2, D_3)$, $\boldsymbol{B} = (B_1, B_2, B_3)$ and the dot means differentiation with respect to t. So, the Dirac equation (2.1) and the system (2.7) are fully equivalent. Therefore, taking into account (2.7) and the well known fact that every component of the ψ-function (2.4) obeying (2.1) satisfies the wave equation $\Box \psi = 0$ (in particular, $\Delta G(\tau, \boldsymbol{x}) = \partial^2 G(\tau, \boldsymbol{x}) / \partial \tau^2$) we find after substitution of (2.5) into (2.3)

$$\begin{aligned} &\dot{\boldsymbol{E}} - \operatorname{rot} \boldsymbol{H} = \dot{\boldsymbol{D}} + \nabla G - \operatorname{rot} \boldsymbol{B} = 0, \\ &\operatorname{div} \boldsymbol{E} = \operatorname{div} \boldsymbol{D} + \int_{t_0}^{t} \Delta G(\tau, \boldsymbol{x}) d\tau + \Delta \tilde{G}(t_0, \boldsymbol{x}) = \\ &\qquad = \operatorname{div} \boldsymbol{D} + \int_{t_0}^{t} \frac{\partial^2 G(\tau, \boldsymbol{x})}{\partial \tau^2} d\tau + \Delta \tilde{G}(t_0, \boldsymbol{x}) = \\ &\qquad = \operatorname{div} \boldsymbol{D} + \dot{G} - \frac{\partial G(\tau, \boldsymbol{x})}{d\tau}\Big|_{\tau = t_0} + \Delta \tilde{G}(t_0, \boldsymbol{x}) = 0. \end{aligned}$$

In the last equality we have used (2.6). In the same spirit one can prove the validity of the theorem for the second pair of Maxwell equations (2.3). Thus, the theorem is proved. ■

The inverse statement also holds true.

Theorem 2. *Let there be given a solution $\boldsymbol{E}$, $\boldsymbol{H}$ of the Maxwell equations (2.3) and two solutions F and G of the scalar wave equation*

$$\Box F = \Box G = 0. \tag{2.8}$$

Then the ψ-function (2.4) witn components F, G and

$$\begin{aligned} D_a &= E_a - \partial_a \left(\int_{t_0}^{t} G(\tau, \boldsymbol{x}) d\tau + \tilde{G}(t_0, \boldsymbol{x}) \right), \\ B_a &= H_a - \partial_a \left(\int_{t_0}^{t} F(\tau, \boldsymbol{x}) d\tau + \tilde{F}(t_0, \boldsymbol{x}) \right), \end{aligned} \tag{2.9}$$

where $a = 1, 2, 3$, $\tilde{G}(t_0, x)$ and $\tilde{F}(t_0, x)$ are determined from (2.6), is a solution of the massless Dirac equation (2.1).

Proof. Let us use the equivalence between the Dirac equation (2.1) and the system (2.7). Having substituted (2.9) into (2.7) and taking into account (2.3), (2.8) and (2.6), we get

$$\dot{\boldsymbol{D}} - \operatorname{rot}\boldsymbol{B} + \nabla G = \dot{\boldsymbol{E}} - \nabla G + \nabla G - \operatorname{rot}\boldsymbol{H} = 0,$$
$$\operatorname{div}\boldsymbol{D} + \dot{G} = \operatorname{div}\boldsymbol{E} + \int_{t_0}^{t} \Delta G(\tau, \boldsymbol{x})d\tau - \Delta\tilde{G}(t_0, \boldsymbol{x}) + \dot{G} = 0.$$

Analogously one has to act to prove the theorem for the rest of the equations of system (2.7).

Theorem 2 has an important corollary: choosing $F = G = 0$ we get from (2.9) $\boldsymbol{D} = \boldsymbol{E}$, $\boldsymbol{B} = \boldsymbol{H}$, and in this case formula (2.4) takes the particularly simple form

$$\psi = \begin{pmatrix} -E_1 + iE_2 \\ E_3 \\ -H_2 - iH_1 \\ iH_3 \end{pmatrix}. \tag{2.10}$$

So, if $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfy the Maxwell equations (2.3), then ψ given by (2.10) automatically satisfies the Dirac equation (2.1), and one can consider relation (2.10) as a representation of the spinor field ψ by an electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$. It is appropriate to note that if $\boldsymbol{E}$ and $\boldsymbol{H}$ are transformed under Lorentz boost as an electromagnetic Maxwell field, then the ψ-function (2.10) is not transformed like a Dirac spinor (this point will be discussed in detail below). It will be also noted that, according to theorem 1, the procedure of obtaining solutions of the vacuum Maxwell equations (2.3) from those of the massless Dirac equation (2.1) and the associated Poisson equations (2.6) is unique to within a gauge transformation, whereas the inverse procedure, Maxwell $\to$ Dirac, involves ambiguities due to the arbitrary choice of additional scalar fields F and G satisfying (2.8). When we construct solutions of Maxwell equations via solutions of the massless Dirac equation using formulae (2.5), then we have arbitrariness in determining $\tilde{F}$ and $\tilde{G}$. But this arbitrariness can be considered as gauge transformetions $\boldsymbol{E} \to \boldsymbol{E}' = \boldsymbol{E} + \nabla f(\boldsymbol{x})$, $\boldsymbol{H} \to \boldsymbol{H}' = \boldsymbol{H} + \nabla g(\boldsymbol{x})$ (f and g are arbitrary scalar functions satisfying the Laplace equation $\Delta f = \Delta g = 0$), which leave invariant the Maxwell equations (2.3). An analogous situation is when considering the inverse procedure (formulae (2.9), Dirac equation in the form (2.7)).

Consider an example. Let us take solutions of the Maxwell equations (2.3) and wave equations (2.8) in the form

$$\boldsymbol{E} = \boldsymbol{\alpha} \times \boldsymbol{x}, \quad \boldsymbol{H} = -2\boldsymbol{\alpha}t, \quad F = G = 3t^2 + \boldsymbol{x}^2, \quad (\boldsymbol{\alpha} = \mathrm{const}).$$

Then, by means of (2.9) and (2.4) one easily finds the following solution of the Dirac equation (2.1):

$$\psi = \begin{pmatrix} -[(\boldsymbol{\alpha} \times \boldsymbol{x})_1 - 2tx_1] + i[(\boldsymbol{\alpha} \times \boldsymbol{x})_2 - 2tx_2] \\ [(\boldsymbol{\alpha} \times \boldsymbol{x})_3 - 2tx_3] - i(3t^2 + \boldsymbol{x}^2) \\ 2t(\alpha_2 + x_2) + 2it(\alpha_1 + x_1) \\ -(3t^2 + \boldsymbol{x}^2) - 2it(\alpha_3 + x_3) \end{pmatrix}.$$

In terms of $\boldsymbol{D}$, $\boldsymbol{B}$, F, G from (2.4)

$$\bar{\psi}\psi = \boldsymbol{D}^2 - \boldsymbol{B}^2 + F^2 - G^2 \qquad (2.11)$$

and in the case of solution ψ considered above we have

$$\bar{\psi}\psi = \boldsymbol{\alpha}^2\boldsymbol{x}^2 - (\boldsymbol{\alpha}\cdot\boldsymbol{x})^2 - 4t^2(\boldsymbol{\alpha}^2 + 2\boldsymbol{\alpha}\cdot\boldsymbol{x}).$$

Let us make up a four-component ψ-function as

$$\psi = i\gamma\partial \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}, \qquad (2.12)$$

where $\varphi_0, \dots, \varphi_3$ are arbitrary solutions of the wave equation, that is $\Box\varphi_\mu = 0$. Since $(i\gamma\partial)^2 = \Box$, then the ψ-function (2.12) automatically satisfies the Dirac equation (2.1) for any set of φ_μ, $\Box\varphi_\mu = 0$. So, (2.12) and (2.4), (2.5) give the following chain of solutions: scalar wave equation $\to$ massless Dirac equation $\to$ vacuum Maxwell equations.

It will be noted that Shtelen [14] and Fushchych et al [6] described a simple prescription for obtaining solutions of linear partial differential equations with nontrivial symmetry. It consists of the following. Let there be given a solution u of the wave equation ($\Box u = 0$). Then the functions

$$u_1 = \mathcal{K}u, \qquad u_2 = \mathcal{K}u_1, \quad \dots \qquad (2.13)$$

where $\mathcal{K} = 2\boldsymbol{c}\boldsymbol{x}\boldsymbol{x}\partial - \boldsymbol{x}^2\boldsymbol{c}\partial + 2\boldsymbol{c}\boldsymbol{x}$ (generator of conformal transformations) and c_μ are arbitrary constant, will be also solutions $u = 1$ we get from (2.13)

$$u_1 = \boldsymbol{c}\boldsymbol{x}, \quad u_2 = (\boldsymbol{c}\boldsymbol{x})^2 - \frac{1}{4}\boldsymbol{c}^2\boldsymbol{x}^2, \quad u_3 = (cx)^3 - \frac{1}{2}(cx)c^2x^2, \quad \dots \qquad (2.14)$$

For further analysis it is convenient to consider the Dirac equation (2.1) together with its conjugation and write it uniformly as

$$i\Gamma^\mu\partial_\mu\Psi = 0, \qquad (2.15)$$

where $\Psi = \Psi(x) = \mathrm{column}\,(\Psi\tilde{\Psi})$, $\tilde{\Psi} = \gamma_0\Psi^*$, Γ^μ are 8×8 matrices,

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0_4 \\ 0_4 & -(\gamma^\mu)^T \end{pmatrix}, \qquad (2.16)$$

γ^μ are Dirac matrices (2.2), 0_4 is a 4×4 zero matrix.

Symmetry properties of (2.15) were studied first by Dirac who showed that the equation is conformally invariant. Later, Pauli and Touschhek found that this equation also admits an eight-parameter group, G_8, of component transformations. And, finally, Ibragimov [8] proved that a 23-parameter group, $G_{23} = C(1,3)\otimes G_8$, is the maximal in the sence of the Lie invariance group of the equation. Relativistic invariance of (2.15) is usually understood as invariance with respect to the spinor representation

$$D\left(\frac{1}{2},0\right) \oplus D\left(0,\frac{1}{2}\right) \oplus D\left(\frac{1}{2},0\right) \oplus D\left(0,\frac{1}{2}\right) \qquad (2.17)$$

of the Poincaré group $P(1,3)$ (it means that Ψ is transformed under the Lorentz boost as a spinor). However, the invariance of (2.15) under the Pauli–Touschek eight-parameter group allows two additional representations of $AP(1,3)$, which are realized on the set of solutions of (2.15), namely

$$D(1,0) \oplus D(0,1) \oplus D(0,0) \oplus D(0,0) \tag{2.18}$$

and

$$D\left(\frac{1}{2},\frac{1}{2}\right) \oplus D\left(\frac{1}{2},\frac{1}{2}\right). \tag{2.19}$$

The explicit form of basis elements of $AP(1,3)$ for representations (2.17)–(2.19) is

$$AP^{(k)}(1,3) = \left\langle P_\mu = \frac{\partial}{\partial x^\mu},\ J^{(k)}_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + S^{(k)}_{\mu\nu} \right\rangle, \tag{2.20}$$

where $k = 1,2,3$ corresponds to (2.17)–(2.19), respectively;

$$x_\mu = g_{\mu\nu} x^\nu, \quad g_{\mu\nu} = \{1,-1,-1,-1\}\delta_{\mu\nu}$$

and matrices $S^{(k)}_{\mu\nu}$ are

$$\begin{gathered} S^{(1)}_{\mu\nu} = -\frac{1}{4}[\Gamma_\mu,\Gamma_\nu], \quad S^{(2)}_{\mu\nu} = S^{(1)}_{\mu\nu} + Q_{\mu\nu}, \quad S^{(3)}_{01} = S^{(2)}_{01}, \quad S^{(3)}_{02} = S^{(2)}_{02}, \\ S^{(3)}_{03} = S^{(2)}_{03} - 2Q_{03}, \quad S^{(3)}_{12} = S^{(2)}_{12}, \quad S^{(3)}_{13} = S^{(2)}_{13} - 2Q_{13}, \quad S^{(3)}_{23} = S^{(2)}_{23} - 2Q_{23}. \end{gathered} \tag{2.21}$$

Here Γ_μ are the same as in (2.16); $Q_{\mu\nu}$ are six basis elements of the Pauli–Touschek algebra, they are 8×8 matrices of the form

$$\begin{gathered} Q_{01} = \frac{1}{2}\begin{pmatrix} 0_4 & -i\gamma^0\gamma^2 \\ -i\gamma^0\gamma^2 & 0_4 \end{pmatrix}, \quad Q_{02} = \frac{1}{2}\begin{pmatrix} 0_4 & -\gamma^0\gamma^2 \\ -\gamma^0\gamma^2 & 0_4 \end{pmatrix}, \\ Q_{03} = \frac{1}{2}\begin{pmatrix} -\gamma_5 & 0 \\ 0_4 & \gamma_5 \end{pmatrix}, \quad Q_{12} = \frac{i}{2}\begin{pmatrix} I_4 & 0_4 \\ 0_4 & -I_4 \end{pmatrix}, \\ Q_{13} = \frac{1}{2}\begin{pmatrix} 0_4 & -\gamma^1\gamma^3 \\ -\gamma^1\gamma^3 & 0_4 \end{pmatrix}, \quad Q_{23} = \frac{i}{2}\begin{pmatrix} 0_4 & \gamma^1\gamma^3 \\ -\gamma^1\gamma^3 & 0_4 \end{pmatrix}, \end{gathered} \tag{2.22}$$

where

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0_2 & I_2 \\ I_2 & 0_2 \end{pmatrix}$$

I_2, I_4 are 2×2 and 4×4 unit matrices. It will be noted that the action of operators (2.20) is defined in the space of the eight-component function introduced in (2.15).

Invariance of (2.15) under $AP^{(2)}(1,3)$ results in the possibility of representing this equation in the form (2.7), and invariance of (2.15) under $AP^{(3)}(1,3)$ allows us to rewrite it as [9]

$$\begin{gathered} \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}(\partial^\rho B^\sigma - \partial^\sigma B^\rho) = 0, \\ \partial_\nu A^\nu = \partial_\nu B^\nu = 0, \end{gathered} \tag{2.23}$$

where

$$\psi = \psi_{\rm real} + i\psi_{\rm imag} = \begin{pmatrix} -A^2 \\ -B^0 \\ -B^1 \\ B^3 \end{pmatrix} + i \begin{pmatrix} -A^1 \\ A^3 \\ B^2 \\ -A^0 \end{pmatrix}. \tag{2.24}$$

Now consider the following three sets of symmetry operators of (2.15):

$$SA^{(k)} = \langle P_\mu, J^{(k)}_{\mu\nu}, \Gamma_4, I; Q_{\mu\nu}\rangle, \tag{2.25}$$

where P_μ, $J^{(k)}_{\mu\nu}$ and $Q_{\mu\nu}$ are defined in (2.20) and (2.22), Γ_μ are given in (2.16), $\Gamma_4 = \Gamma^0\Gamma^1\Gamma^2\Gamma^3$. There sets of operators form Lie algebra as well as superalgebras. Operators P_μ, $J^{(k)}_{\mu\nu}$, Γ_μ, I are even and $Q_{\mu\nu}$ are odd in corresponding superalgebras. To prove this statement we write down commutation and anticommutation relations for these operators.

Operators P_μ and $J^{(k)}_{\mu\nu}$ satisfy standard commutation relations of the Poincaré algebra $AP(1,3)$

$$\begin{aligned} &[P_\mu, P_\nu] = 0, \quad [P_\sigma, J_{\mu\nu}] = g_{\sigma\mu}P_\nu - g_{\sigma\nu}P_\mu, \\ &[J_{\mu\nu}, J_{\rho\sigma}] = g_{\nu\rho}J_{\mu\sigma} + g_{\mu\sigma}J_{\nu\rho} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\nu}, \end{aligned} \tag{2.26}$$

Γ_4 and I commute with all elements of $SA^{(k)}$. Further, it is convenient to introduce the notation

$$R_a = Q_{0a}, \quad T_a = \frac{1}{2}\varepsilon_{abc}Q_{bc}, \quad N^{(k)}_a = J^{(k)}_{0a}, \quad M^{(k)}_a = \frac{1}{2}\varepsilon_{abc}J^{(k)}_{bc}. \tag{2.27}$$

It is easy to check that

$$\begin{aligned} &\{R_a, R_b\} \equiv R_aR_b + R_bR_a = \frac{1}{2}\sigma_{ab}, \\ &\{T_a, T_b\} = -\frac{1}{2}\delta_{ab}I, \quad \{R_aT_b\} = \delta_{ab}\Gamma_4. \end{aligned} \tag{2.28}$$

Operators R_a, T_a from $SA^{(1)}$ commute with all even operators of $SA^{(1)}$. For $SA^{(2)}$ we have

$$\begin{aligned} &[P_\mu, R_a] = [P_\mu, T_a] = 0, \quad [N^{(2)}_a, R_b] = [R_a, R_b] = \varepsilon_{abc}T_c, \\ &[N^{(2)}_a, T_b] = [R_a, T_b] = -\varepsilon_{abc}R_c, \quad [M^{(2)}_a, R_b] = [T_a, R_c] = -\varepsilon_{abc}R_c, \\ &[M^{(2)}_a, T_b] = [T_a, T_b] = -\varepsilon_{abc}T_c. \end{aligned} \tag{2.29}$$

Subalgebra $SA^{(3)}$ is isomorphis to $SA^{(2)}$. The isomorphism is achived by means of the transformations

$$R_3 \to R'_3 = -R_3, \quad T_1 \to T'_1 = -T_1, \quad T_2 \to T'_2 = -T_2. \tag{2.30}$$

So, the structure of superalgebras (2.25) is fully described. The superalgebras (2.25) do not belong to the semi-simple family, but the quotient by their radical is simply $SO(1,3)$.

3. Dirac equations with non-zero mass possessing dual Poincaré invariance

The Dirac equation for a massive particle (field)

$$(i\gamma^\mu \partial_\mu - m)\psi = 0, \tag{3.1}$$

where γ^μ are given in (2.2) and m is an arbitrary real constant (mass of the particle), is invariant under a 14-parameter group only [8], which includes the Poincaré group, and identical, phase and two charge-type transformations. As always, we are factoring out an infinite-demensional ideal, present for any linear equation, and corresponding to the linear superposition principle. It is to be emphasized that we are considering group action on the field of real numbers, and therefore identical $\psi' = e^\alpha \psi$ (α is an arbitrary real constant) and phase transformations $\psi' = e^{i\alpha}\psi$ should be distinguished.

The above-mentioned four-parameter group of component transformations is not sufficient to construct a non-spinor representation of $AP(1,3)$, as was done in the case of the massless field. The situation can be improved by considering the system of two Dirac equations

$$(i\gamma\partial - m)\Psi_- = 0, \quad (i\gamma\partial + m)\Psi_+ = 0. \tag{3.2}$$

The full information on Lie symmetry of this system gives the following statement.

Theorem 3. *The maximal in the sense of the Lie invariance algebra of system (3.2) is a 26-dimensional Lie algebra $A_{26} = AP^{(1)}(1,3) \oplus A_{16}$, with basis elements having the form*

$$\begin{aligned} &AP^{(1)}(1,3) = \left\langle P_\mu = \frac{\partial}{\partial x^\mu},\ \hat{J}^{(k)}_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + \hat{S}^{(1)}_{\mu\nu} \right\rangle, \\ &A_{16} = \left\langle \text{matrices } 16\times 16 \text{ of the form } \begin{pmatrix} \Lambda & \Sigma \\ \tilde{\Sigma} & \tilde{\Lambda} \end{pmatrix} \right\rangle, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &\hat{S}^{(1)}_{\mu\nu} = -\frac{1}{4}[\hat{\Gamma}_\mu, \hat{\Gamma}_\nu], \quad \hat{\Gamma}_\mu = \begin{pmatrix} \Gamma_\mu & 0_8 \\ 0_8 & -\Gamma_\mu \end{pmatrix}, \quad \langle \Lambda, \tilde{\Lambda} \rangle = \langle I, Q_{01}, Q_{02}, Q_{03} \rangle, \\ &\langle \Sigma, \tilde{\Sigma} \rangle = \langle Q_{12}, Q_{13}, Q_{23}, \Gamma_4 \rangle, \quad \Gamma_4 = \Gamma_0\Gamma_1\Gamma_2\Gamma_3 \end{aligned} \tag{3.4}$$

(matrices 8×8 Γ_μ and $Q_{\mu\nu}$ are defined in (2.16)), and acting in the space of 16-component functions

$$\hat{\Psi} = \text{column}\ (\Psi_-\Psi_+) \equiv \text{column}\ (\Psi_-, \tilde{\Psi}_- = \gamma_0\Psi^*_-, \Psi_+, \tilde{\Psi}_+ = \gamma_0\Psi^*_+). \tag{3.5}$$

Proof. First of all we write system (3.2) together with its conjugation as

$$(i\hat{\Gamma}^\mu \partial_\mu - m)\hat{\Psi} = 0, \tag{3.6}$$

where $\hat{\Psi} = \hat{\Psi}(x)$ is defined in (3.5). To prove the theorem is to find the general form of infinitesimal operator of invariance

$$Q = \xi^\mu(x)\partial_\mu + \eta(x), \tag{3.7}$$

where $\xi^\mu(x)$ are scalar functions and $\eta(x)$ is a 16×16 matrix. It can be done by means of the standard Lie algorithm (see [10]), but the simplest way is to use the invariance condition in the form

$$[L,Q]=\lambda(x)L, \tag{3.8}$$

where L is the operator of (3.6), $L\equiv i\hat\Gamma^\mu\partial_\mu-m$ and $\lambda(x)$ is some scalar smooth function. Starting from (3.8) one gets, after some simple but tedious calculations, the proof of the theorem. ■

Invariance of the system (3.6) with respect to the matrix algebra A_{16} (3.3) allows a vector representation of $AP(1,3)$, which can be realized on the set of solutions of this system. This representation is

$$\mathcal{L}(D(1,0)\oplus D(0,1))\oplus 4D(0,0). \tag{3.9}$$

It is defined by the basis elements

$$AP^{(2)}(1,3)=\langle P_\mu,\hat J^{(2)}_{\mu\nu}=\hat J^{(1)}_{\mu\nu}+\hat Q_{\mu\nu}\rangle, \tag{3.10}$$

where P_μ and $\hat J^{(1)}_{\mu\nu}$ are given in (3.3),

$$\hat Q_{\mu\nu}=\begin{cases}\begin{pmatrix}Q_{\mu\nu} & 0_8\\ 0_8 & Q_{\mu\nu}\end{pmatrix}, & \text{if } (\mu\nu)=\langle(0,1),(0,2),(1,2)\rangle,\\ \begin{pmatrix}0_8 & Q_{\mu\nu}\\ Q_{\mu\nu} & 0_8\end{pmatrix}, & \text{if } (\mu\nu)=\langle(0,3),(1,3),(2,3)\rangle\end{cases} \tag{3.11}$$

and matrices 8×8 $Q_{\mu\nu}$ are given in (2.22). Invariance of (3.6) with respect to $AP^{(2)}(1,3)$ (3.10) means that (3.6) describes not only spinor particles(fermionic fields) but also a coupled system of vector and scalar particles (bosonic fields).

Now consider the following two sets of symmetry operators of equation (3.6):

$$SA^{(i)}=\langle P_\mu,\hat J^{(i)}_{\mu\nu},\hat\Gamma_4,I;\hat Q_{\mu\nu}\rangle,\quad i=1,2, \tag{3.12}$$

where

$$\hat\Gamma_4=\begin{pmatrix}0_8 & \Gamma_4\\ \Gamma_4 & 0_8\end{pmatrix}, \tag{3.13}$$

Γ_4 is givel in (3.4). These sets of operators form Lie algebras as well as superalgebras. Superalgebras (3.12) are isomorphic to those from (2.25). The isomorphism is achieved by means of the transformations

$$P_\mu\to P_\mu,\quad J^{(i)}_{\mu\nu}\to\hat J_{\mu\nu},\quad \Gamma_4\to\hat\Gamma_4,\quad I\to I,\quad Q_{\mu\nu}\to\hat Q_{\mu\nu}. \tag{3.14}$$

In conclusion of this section let us consider a nonlinear generalization of (3.6) possessing dual Poincaré invariance.

Theorem 4. *The equation*

$$[i\hat\Gamma^\mu\partial_\mu-F(\bar\Psi\hat\Psi,\Psi M\hat\Psi)]\hat\Psi=0, \tag{3.15}$$

where $\hat\Psi$ is defined in (3.5),

$$\bar\Psi=\mathrm{row}\,(\bar\Psi_-\Psi^T_-\bar\Psi_+\Psi^T_+),\quad M=\begin{pmatrix}0_8 & I_8\\ I_8 & 0_8\end{pmatrix} \tag{3.16}$$

and F is an arbitrary smooth function, is invariant under the two Poincaré algebras (3.3) and (3.10).

Proof. One can make sure that the operator $i\hat{\Gamma}^\mu\partial_\mu$ commutes with all generators of the considered Poincaré algebras. Further, the quantities $\bar{\Psi}\hat{\Psi}$, $\bar{\Psi}M\hat{\Psi}$ are absolute invariants of these Poincaré algebras. Thus, the theorem is proved.

It will be noted that

$$\bar{\Psi}\hat{\Psi} = 2(\bar{\Psi}_-\Psi_- + \bar{\Psi}_+\Psi_+), \quad \bar{\Psi}M\hat{\Psi} = 2(\bar{\Psi}_-\Psi_+ + \bar{\Psi}_+\Psi_-), \tag{3.17}$$

where Ψ_-, Ψ_+ are four-component functions, $\bar{\Psi}_\pm = (\Psi_\pm)^+\gamma_0$. ■

4. $P^{(i)}(1,3)$-inequivalent ansätze, reduction and solutions of nonlinear Dirac equations

The nonlinear equation (3.15), as we have shown, is dual Poincaré invariant and therefore it unites fermionic and bosonic fields. Such unification opens new ways to solve the general problem of unification forces and fields.

It is important to find exact solutions of (3.15). Of course, we shall be looking for classical solutions, but these solutions may be very useful as basic ones in the corresponding quantum theory. It is to be emphasized that the standard procedure of quantization, when the complete set of solutions of a given equation is quantized according to bosonic or fermionic rules, may be misleading because our equation may have bosonic and fermionic subsets of solutions simultaneously (the simplest example is the massless Dirac equation considered in section 2). Therefore, it is more preferable to quantize separate families of solutions, having established beforehand what representation of the Poincaré algebra is realized on them.

To fing exact solutions of equations of the (3.15) we construct $P^{(i)}(1,3)$-inequivalent ansätze of codimension 1. These ansätze reduce a given equation to ODEs. The general form of such an anzatz is

$$\hat{\Psi}(x) = A(x)\phi(\omega), \tag{4.1}$$

where $A(x)$ is 16×16 matrix, ϕ is 16-component function (column) depending on the new variable ω. Matrix $A(x)$ and the new independent variable ω are determined from the equations [3]

$$\begin{aligned} &Q_k A(x) \equiv (\xi_k^\nu(x)\partial_\nu + \eta_k(x))A(x) = 0, \\ &\xi_k^\nu(x)\partial_\nu\omega(x) = 0, \quad k = 1,2,3, \end{aligned} \tag{4.2}$$

where $\langle Q_1, Q_2, Q_3\rangle$ is a three-dimensional subalgebra of $AP(1,3)$. The full description of subalgebras of $AP(1,3)$ is given in [11] and [7]. Fushchych and Shtelen [4]) (see also [6]) have used one-dimensional subalgebras of $AP(1,3)$ to construct ansätze of codimension 3 for the Dirac spinor field. Ansätze of codimension 1 for the Dirac spinor field are fully described in [5]. We present the complete set of $P^{(i)}(1,3)$-inequivalent ansätze of codimension 1 for a 16-component field (3.5) in table 1. Basis elements of $AP^{(i)}(1,3)$ are given in (3.3) and (3.10).

In table 1 α and β are arbitrary non-zero constants,

$$G_k^{(i)} = \hat{j}_{0k}^{(i)} + j_{3k}^{(i)} = (x_0 + x_3)P_k + x_k(P_0 - P_3) + \hat{S}_{0k}^{(i)} + \hat{S}_{3k}^{(i)}, \tag{4.3}$$

$\hat{S}_{\mu\nu}^{(1)}$ are given in (3.4) and $\hat{S}_{\mu\nu}^{(2)} = \hat{S}_{\mu\nu}^{(1)} + \hat{Q}_{\mu\nu}$ see (3.10) and (3.11).

Table 1. $P^{(i)}(1,3)$-inequivalent ansätze (4.1) of codimension 1 for field (3.5).

N	Algebra	$A(x)$	ω
1	$P_0,\ P_1,\ P_2$	1	x_3
2	$P_1,\ P_2,\ P_3$	1	x_0
3	$P_0-P_3,\ P_1,\ P_2$	1	x_0+x_3
4	$\hat{J}_{03}^{(i)},\ P_1,\ P_2$	$\exp[-\hat{S}_{03}^{(i)}\ln(x_0+x_3)]$	$x_0^2-x_3^2$
5	$\hat{J}_{03}^{(i)},\ P_1,\ P_0-P_3$	$\exp[-\hat{S}_{03}^{(i)}\ln(x_0+x_3)]$	x_2
6	$\hat{J}_{03}^{(i)}+\alpha P_2,\ P_0,\ P_3$	$\exp\left(-\frac{x_2}{\alpha}\hat{S}_{03}^{(i)}\right)$	x_1
7	$\hat{J}_{03}^{(i)}+\alpha P_2,\ P_0-P_3,\ P_1$	$\exp\left(-\frac{x_2}{\alpha}\hat{S}_{03}^{(i)}\right)$	$\alpha\ln(x_0+x_3)-x_2$
8	$\hat{J}_{12}^{(i)},\ P_0,\ P_3$	$\exp\left(\hat{S}_{12}^{(i)}\tan^{-1}\frac{x_1}{x_2}\right)$	$x_1^2+x_2^2$
9	$\hat{J}_{03}^{(i)}-\alpha P_0,\ P_1,\ P_2$	$\exp\left(\frac{x_0}{\alpha}\hat{S}_{12}^{(i)}\right)$	x_3
10	$\hat{J}_{12}^{(i)}+\alpha P_3,\ P_1,\ P_2$	$\exp\left(-\frac{x_3}{\alpha}\hat{S}_{12}^{(i)}\right)$	x_0
11	$\hat{J}_{12}^{(i)}-P_0+P_3,\ P_1,\ P_2$	$\exp\left(-\frac{1}{2}(x_3-x_0)\hat{S}_{12}^{(i)}\right)$	x_0+x_3
12	$G_1^{(i)},\ P_0-P_3,\ P_2$	$\exp\left[-\frac{x_1}{x_0+x_3}(\hat{S}_{01}^{(i)}+S_{31}^{(i)})\right]$	x_0+x_3
13	$G_1^{(i)},\ P_0-P_3,\ P_1+\alpha P_2$	$\exp\left[\frac{x_2-\alpha x_1}{\alpha(x_0+x_3)}(\hat{S}_{01}^{(i)}+S_{31}^{(i)})\right]$	x_0+x_3
14	$G_1^{(i)}+P_2,\ P_1,\ P_0-P_3$	$\exp\left[-x_2(\hat{S}_{01}^{(i)}+S_{31}^{(i)})\right]$	x_0+x_3
15	$G_1^{(i)}-P_0,\ P_2,\ P_0-P_3$	$\exp\left[(x_0+x_3)(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})\right]$	$2x_1+(x_0+x_3)^2$
16	$G_1^{(i)}-P_0,\ P_0-P_3,$ $P_1+\alpha P_2$	$\exp\left[(x_0+x_3)(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})\right]$	$2(x_2-\alpha x_1)-\alpha(x_0+x_3)^2$
17	$\hat{J}_{03}^{(i)}+\alpha\hat{J}_{12}^{(i)},\ P_0,\ P_3$	$\exp\left[\frac{1}{\alpha}(\hat{S}_{03}^{(i)}+\alpha\hat{S}_{12}^{(i)})\tan^{-1}\frac{x_1}{x_2}\right]$	$x_1^2+x_2^2$
18	$\hat{J}_{03}^{(i)}+\alpha\hat{J}_{12}^{(i)},\ P_1,\ P_2$	$\exp\left[-(\hat{S}_{03}^{(i)}+\alpha\hat{S}_{12}^{(i)})\ln(x_0+x_3)\right]$	$x_0^2-x_3^2$
19	$G_1^{(i)},\ G_2^{(i)},\ P_0-P_3$	$\exp\left\{-\frac{1}{x_0+x_3}[x_1(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})+\right.$ $\left.+x_2(\hat{S}_{02}^{(i)}+\hat{S}_{32}^{(i)})]\right\}$	x_0+x_3
20	$G_1^{(i)}+P_2,\ G_2^{(i)}+\alpha P_1+$ $+\beta P_2,\ P_0-P_3$	$\exp\left[\frac{\alpha x_2-x_1(x_0+x_3+\beta)}{(x_0+x_3)(x_0+x_3+\beta)-\alpha}(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})+\right.$ $\left.+\frac{x_1-x_2(x_0+x_3)}{(x_0+x_3)(x_0+x_3+\beta)-\alpha}(\hat{S}_{01}^{(i)}+\hat{S}_{32}^{(i)})\right]$	x_0+x_3
21	$G_1^{(i)},\ G_2^{(i)}+P_1+\beta P_2,$ P_0-P_3	$\exp\left[-\frac{x_1}{x_0+x_3}(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})-\right.$ $-\frac{x_2}{x_0+x_3+\beta}(\hat{S}_{02}^{(i)}+\hat{S}_{32}^{(i)})+$ $\left.+\frac{x_2}{(x_0+x_3)(x_0+x_3+\beta)}(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})\right]$	x_0+x_3
22	$G_1^{(i)},\ G_2^{(i)}+P_2,\ P_0-P_3,$	$\exp\left[-\frac{x_1}{x_0+x_3}(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})-\right.$ $\left.-\frac{x_2}{x_0+x_3+1}(\hat{S}_{02}^{(i)}+\hat{S}_{32}^{(i)})\right]$	x_0+x_3
23	$G_1^{(i)},\ \hat{J}_{03}^{(i)},\ P_2$	$\exp\left[-\frac{x_1}{x_0+x_3}(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})\right]\times$ $\times\exp[-\hat{S}_3^{(i)}\ln(x_0+x_3)]$	$x_0^2-x_1^2-x_3^2$

Table 1. (continued)

N	Algebra	$A(x)$	ω
24	$J_{03}^{(i)}+\alpha P_1+\beta P_2,\ \hat{G}_1^{(i)},\ P_0-P_3$	$\exp\left[\frac{\alpha\ln(x_0+x_3)-x_1}{x_0+x_3}(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})\right]\times$ $\times\exp[-\hat{S}_{03}^{(i)}\ln(x_0+x_3)]$	$x_2-\beta\ln(x_0+x_3)$
25	$\hat{J}_{12}^{(i)}-P_0+P_3,\ G_1^{(i)},\ G_2^{(i)}$	$\exp\Big\{-\frac{1}{x_0+x_3}[x_1(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})+$ $+x_2(\hat{S}_{02}^{(i)}+\hat{S}_{32}^{(i)})\Big\}\times$ $\times\exp\left(\frac{x\cdot x}{2(x_0+x_3)}(\hat{S}_{12}^{(i)}\right)$	x_0+x_3
26	$\hat{J}_{03}^{(i)}+\alpha\hat{J}_{12}^{(i)},\ G_1^{(i)},\ G_2^{(i)}$	$\exp\Big\{-\frac{1}{x_0+x_3}[x_1(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})+$ $+x_2(\hat{S}_{02}^{(i)}+\hat{S}_{32}^{(i)})]\Big\}\times$ $\times\exp[-(\hat{S}_{03}^{(i)}+\alpha\hat{S}_{12}^{(i)})\ln(x_0+x_3)]$	$x\cdot x$

Let us substitute ansätze (4.1) from table 1 info (3.15). As a result we obtain the following reduced ODEs:

(1) $\hat{\Gamma}^2\dot{\phi}+iR\phi=0,$

(2) $\hat{\Gamma}^0\dot{\phi}+iR\phi=0,$

(3) $(\hat{\Gamma}^0+\hat{\Gamma}^3)\dot{\phi}+iR\phi=0,$

(4) $-(\hat{\Gamma}^0+\hat{\Gamma}^3)\hat{S}_{03}^{(i)}\phi+[\omega(\hat{\Gamma}^0+\hat{\Gamma}^3)+(\hat{\Gamma}^0+\hat{\Gamma}^3)]\dot{\phi}+iR\phi=0,$

(5) $-(\hat{\Gamma}^0+\hat{\Gamma}^3)\hat{S}_{03}^{(i)}\phi+\hat{\Gamma}^2\dot{\phi}+iR\phi=0,$

(6) $-\dfrac{1}{\alpha}\hat{\Gamma}_2\hat{S}_{03}^{(i)}\hat{\Gamma}^1\dot{\phi}+iR\phi=0,$

(7) $-\dfrac{1}{\alpha}\hat{\Gamma}_2\hat{S}_{03}^{(i)}\phi+[\alpha(\hat{\Gamma}^0+\hat{\Gamma}^3)e^{\omega/\alpha}-\hat{\Gamma}^2]\dot{\phi}+iR\phi=0,$

(8) $\dfrac{1}{\sqrt{\omega}}\hat{\Gamma}_1\hat{S}_{12}^{(i)}\phi+2\sqrt{\omega}\hat{\Gamma}^2\phi+iR\phi=0,$

(9) $\dfrac{1}{\alpha}\hat{\Gamma}_0\hat{S}_{12}^{(i)}\phi+\hat{\Gamma}^3\dot{\phi}+iR\phi=0,$

(10) $-\dfrac{1}{\alpha}\hat{\Gamma}^3\hat{S}_{12}^{(i)}\phi+\hat{\Gamma}^0\dot{\phi}+iR\phi=0,$

(11) $-\dfrac{1}{2}(\hat{\Gamma}_0-\hat{\Gamma}^3)\hat{S}_{12}^{(i)}\phi+(\hat{\Gamma}^0+\hat{\Gamma}^3)\dot{\phi}+iR\phi=0,$

(12) $-\dfrac{1}{\omega}\hat{\Gamma}_1(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})\phi+(\hat{\Gamma}^0+\hat{\Gamma}^3)\dot{\phi}+iR\phi=0,$

(13) $\dfrac{1}{\alpha\omega}(\hat{\Gamma}_2-\alpha\hat{\Gamma}^1)(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})\phi+(\hat{\Gamma}^0+\hat{\Gamma}^3)\dot{\phi}+iR\phi=0,$

(14) $-\hat{\Gamma}^2(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})\phi+(\hat{\Gamma}^0+\hat{\Gamma}^3)\dot{\phi}+iR\phi=0,$

(15) $(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})(\hat{\Gamma}^0+\hat{\Gamma}^3)\phi+2\hat{\Gamma}^1\dot{\phi}+iR\phi=0,$

(16) $(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})(\hat{\Gamma}^0+\hat{\Gamma}^3)\phi+2(\hat{\Gamma}^2-\alpha\hat{\Gamma}^1)\dot{\phi}+iR\phi=0,$

$$
\begin{aligned}
&(17)\quad -\frac{1}{\alpha\sqrt{\omega}}\hat{\Gamma}_1(\hat{S}_{03}^{(i)}+\alpha\hat{S}_{12}^{(i)})\phi+2\sqrt{\omega}\hat{\Gamma}^2\dot{\phi}+iR\phi=0,\\
&(18)\quad -(\hat{\Gamma}^0+\hat{\Gamma}^3)(\hat{S}_{03}^{(i)}+\alpha\hat{S}_{12}^{(i)})\phi+[\omega(\hat{\Gamma}^0+\hat{\Gamma}^3)+(\hat{\Gamma}^0-\hat{\Gamma}^3)]\dot{\phi}+iR\phi=0,\\
&(19)\quad -\frac{1}{\omega}[\hat{\Gamma}_1(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})+\hat{\Gamma}^2(\hat{S}_{02}^{(i)}+\hat{S}_{32}^{(i)})]\phi+(\hat{\Gamma}^0+\hat{\Gamma}^3)\dot{\phi}+iR\phi=0,\\
&(20)\quad [\omega(\omega+\beta)-\alpha]^{-1}\{[\alpha\hat{\Gamma}^2-(\omega+\beta)\hat{\Gamma}^1](S_{01}^{(i)}+S_{31}^{(i)})+\\
&\qquad\quad +(\hat{\Gamma}^1-\omega\hat{\Gamma}^2)(\hat{S}_{02}^{(i)}+\hat{S}_{32}^{(i)})\}\phi+(\hat{\Gamma}^0+\hat{\Gamma}^3)\dot{\phi}+iR\phi=0,\\
&(21)\quad [(-\hat{\Gamma}^1+(\omega+\beta)^{-1}\hat{\Gamma}^2)\omega^{-1}(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})-\\
&\qquad\quad -(\hat{\Gamma}^2(\omega+\beta)^{-1}(\hat{S}_{02}^{(i)}+\hat{S}_{32}^{(i)})]\phi+(\hat{\Gamma}^0+\hat{\Gamma}^3)\dot{\phi}+iR\phi=0,\\
&(22)\quad \left[-\frac{1}{\omega}\hat{\Gamma}^1(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})-\frac{\hat{\Gamma}^2}{\omega+1}(\hat{S}_{02}^{(i)}+\hat{S}_{32}^{(i)})\right]\phi+(\hat{\Gamma}^0+\hat{\Gamma}^3)\dot{\phi}+iR\phi=0,\\
&(23)\quad -[(\hat{\Gamma}^0+\hat{\Gamma}^3)\hat{S}_{03}^{(i)}+\hat{\Gamma}^1(\hat{S}_{31}^{(i)}+\hat{S}_{31}^{(i)})]\phi+[\omega(\hat{\Gamma}^0+\hat{\Gamma}^3)+\hat{\Gamma}^0-\hat{\Gamma}^3]\dot{\phi}+iR\phi=0,\\
&(24)\quad -[(\hat{\Gamma}^0+\hat{\Gamma}^3)\hat{S}_{03}^{(i)}+\hat{\Gamma}^1(\hat{S}_{31}^{(i)}+\hat{S}_{31}^{(i)})]\phi+[\hat{\Gamma}^2-\beta(\hat{\Gamma}^0+\hat{\Gamma}^3)]\dot{\phi}+iR\phi=0,\\
&(25)\quad \left[2\hat{S}_{03}^{(i)}(\hat{\Gamma}^0+\hat{\Gamma}^3)\omega^{-1}+\frac{1}{2}\hat{S}_{12}^{(i)}(\hat{\Gamma}^0-\hat{\Gamma}^3)\right]\phi+(\hat{\Gamma}^0+\hat{\Gamma}^3)\dot{\phi}+iR\phi=0,\\
&(26)\quad -[(\hat{\Gamma}^0+\hat{\Gamma}^3)\hat{S}_{03}^{(i)}+(\hat{S}_{01}^{(i)}+\hat{S}_{31}^{(i)})\hat{\Gamma}^1+(\hat{S}_{02}^{(i)}+\hat{S}_{32}^{(i)})\hat{\Gamma}^2]\phi+\\
&\qquad\quad +[(\hat{\Gamma}^0+\hat{\Gamma}^3)\omega+(\hat{\Gamma}^0-\hat{\Gamma}^3)]\dot{\phi}+iR\phi=0.
\end{aligned}
\tag{4.4}
$$

Enumerations (1)–(26) in (4.4) correspond to those of the ansätze in table 1; the dot denotes differentiation with respect to the corresponding ω and $R=F(\bar{\phi}\phi,\bar{\phi}M\phi)$.

Below we obtain some solutions of reduces ODEs (4.4) in the case of a non-standard representation of $AP(1,3)$ realized by matrices $\hat{S}_{\mu\nu}^{(2)}=\hat{S}_{\mu\nu}^{(1)}+\hat{Q}_{\mu\nu}$ (see (3.4), (3.10) and (3.11)). The cases with $\hat{S}_{\mu\nu}^{(1)}$ are analogous to those considered in [3, 4, 5, 6].

First of all we note that the condition of compatability for equations (3), (12)–(14), (19) and (22) in (4.4) results in $R\equiv F(\bar{\phi}\phi,\bar{\phi}M\phi)=0$ and therefore such cases are rather trivial.

Consider equation (5) in (4.4), choosing

$$
R=\lambda\rho^{1/2k},\quad \rho\equiv\bar{\phi}\phi, \tag{4.5}
$$

where $\lambda,k\neq 0$ are arbitrary real constants. From equation (5) we find as a corollary (or condition of compatibility)

$$
\frac{d^2\rho}{d\omega^2}=4\lambda\rho^{1/2k}\left(\frac{2\lambda k}{1+2k}\rho^{1+1/2k}+c_0\right),\quad k\neq-\frac{1}{2}, \tag{4.6}
$$

c_0 is an arbitrary real constant. A particular solution of (4.6) is

$$
\rho(\omega)=\left(c-\frac{2\lambda}{1+2k}\omega\right)^{-2k}, \tag{4.7}
$$

c is an arbitrary real constant. Let us go back to equation (5) in (4.4). Using (4.7) we obtain a linear ODE and its general solution has the form

$$\begin{aligned}\phi = \exp\left[-\frac{i(1-2k)}{2}\hat{\Gamma}^2\ln\left(c-\frac{2\lambda\omega}{1+2k}\right)\right]\exp\left\{-\frac{1+2k}{4\lambda}(\hat{\Gamma}^0+\hat{\Gamma}^3)\hat{S}_{03}\times\right.\\ \times\left[\hat{\Gamma}^2\left(\frac{[c-2\lambda\omega/(1+2k)]^{2k+2}}{2k+2}-\frac{[c-2\lambda\omega/(1+2k)]^{-2k}}{2k}\right)\times\right.\\ \left.\left.+i\left(\frac{[c-2\lambda\omega/(1+2k)]^{2k+2}}{2k+2}+\frac{1}{2k[c-2\lambda\omega/(1+2k)]^{2k}}\right)\right]\right\}\chi,\end{aligned}\tag{4.8}$$

where χ is an arbitrary 16-component constant column satisfying the conditions

$$\bar{\chi}\chi=0,\quad \bar{\chi}(\hat{\Gamma}^0+\hat{\Gamma}^3)\hat{Q}_{03}\hat{\Gamma}^2\chi=-i\bar{\chi}(\hat{\Gamma}^0+\hat{\Gamma}^3),\quad \bar{\chi}\hat{Q}_{03}\chi=\frac{2\lambda k}{1+2k}.\tag{4.9}$$

Let us write down the general solution of equation (5) in (4.4) in the case of the spinor representation ($i=1$). It can be found without difficulty and has the form

$$\phi=\exp\left\{\omega\hat{\Gamma}^2\left[\frac{1}{2}(\hat{\Gamma}^0+\hat{\Gamma}^3)+i\lambda(\bar{\chi}\chi)^{1/2k}\right]\right\}\chi,\tag{4.10}$$

where χ is an arbitrary 16-component column.

Consider equation (15) in (4.4). In this case, by analogy with (5) in (4.4) considered above, we find

$$\frac{d^2\rho}{d\omega^2}=\lambda\rho^{1/2k}\left(\frac{2\lambda k}{1+2k}\rho^{1+1/2k}+c_0\right),\quad k\neq-\frac{1}{2}$$

and then

$$\begin{aligned}\phi(\omega)=\exp[i\hat{\Gamma}^1\beta(\omega)]\exp\left[\frac{1}{2}\hat{\Gamma}^1(\hat{Q}_{01}+\hat{Q}_{31})(\Gamma^0+\Gamma^3)\times\right.\\ \left.\times\left(\int^\omega\cosh\beta(y)dy+i\hat{\Gamma}^1\int^\omega\sinh\beta(y)dy\right)\right]\chi,\end{aligned}\tag{4.11}$$

where

$$\begin{aligned}&\bar{\chi}\chi=0,\\ &\bar{\chi}\hat{\Gamma}^1(\hat{Q}_{01}+\hat{Q}_{31})(\hat{\Gamma}^0+\hat{\Gamma}^3)\chi=i\bar{\chi}(\hat{Q}_{01}+\hat{Q}_{31})(\hat{\Gamma}^0+\hat{\Gamma}^3)\chi=\frac{2\lambda k}{1+2k},\\ &\beta(\omega)=-(1+2k)\ln\left(c-\frac{\lambda\omega}{1+2k}\right).\end{aligned}$$

Analogously, in the case of equation (16) in (4.4) we have

$$\frac{d^2\rho}{d\omega^2}=\frac{\lambda}{1+\alpha^2}\rho^{1/2k}\left(\frac{2\lambda k}{1+2k}\rho^{1+1/2k}+c\right)$$

and

$$\phi(\omega) = \exp\left(\frac{i\beta(\omega)}{2(1+\alpha^2)}(\hat\Gamma^2 - \alpha\hat\Gamma^1)\right) \times$$
$$\times \exp\left[\frac{1}{2(1+\alpha^2)}(\hat Q_{01} + \hat Q_{31})(\hat\Gamma^0 + \hat\Gamma^3) \times \right. \tag{4.12}$$
$$\left.\left((\hat\Gamma^1 - \alpha\hat\Gamma^2)\int^\omega \cosh\frac{\beta(y)}{\sqrt{1+\alpha^2}}dy + i\sqrt{1+\alpha^2}\int^\omega \sinh\frac{\beta(y)}{\sqrt{1+\alpha^2}}dy\right)\right]\chi,$$

where

$$\beta(\omega) = -(1+2k)\ln\left(c - \frac{\lambda\omega}{(1+2k)\sqrt{1+\alpha^2}}\right),$$
$$\bar\chi\chi = 0,$$
$$\bar\chi(\hat Q_{03} + \hat Q_{31})(\hat\Gamma^0 + \hat\Gamma^3)(\hat\Gamma^1 - \alpha\hat\Gamma^2)\chi =$$
$$= i\sqrt{1+\alpha^2}\bar\chi(\hat Q_{01} + \hat Q_{31})(\hat\Gamma^0 + \hat\Gamma^3)\chi = \frac{2k\lambda(1+\alpha^2)}{1+2k}.$$

Now consider an example of obtaining an exact solution of the standard Dirac equation with non-zero mass

$$(i\gamma\partial - m)\Psi_- = 0 \tag{4.13}$$

using symmetry $AP^{(2)}(1,3)$ (3.10) of system (3.2) (or, to be more exact, of the equivalent system (3.6). Let us take a two-dimensional subalgebra $\langle \hat J_{23}^{(2)}, P_0 - P_1\rangle$ of $AP^{(2)}(1,3)$. The corresponding ansatz for (3.5) has the form

$$\hat\Psi(x) = \exp\left(\hat S_{23}^{(2)}\tan^{-1}\frac{x_2}{x_3}\right)\phi(\omega),$$
$$\omega = \{\omega_1, \omega_2\}, \quad \omega_1 = x_0 + x_1, \quad \omega_2 = (x_2^2 + x_2^2)^{1/2}. \tag{4.14}$$

Taking into account the identities

$$\hat S_{23}^{(2)} = \hat S_{23}^{(1)} + \hat Q_{23}, \quad [\hat S_{23}^{(1)}, Q_{23}] = 0$$

we find from (4.14) the ansatz for Ψ_-:

$$\Psi_-(x) = \frac{1}{2}\left(1 + \frac{1}{\omega^2}(x_3 - \gamma_2\gamma_3 x_2)\right)\varphi_-(\omega) -$$
$$- \frac{i}{2}\left(\gamma_2 + \frac{1}{\omega_2}(\gamma_3 x_2 - \gamma_2 x_3)\right)\gamma_1\varphi_+(\omega). \tag{4.15}$$

Further, it is convenient to introduce the notation

$$Z(\omega) = \frac{1}{2\omega_2}\varphi_- + \frac{i}{2\omega_2}\gamma_2\gamma_1\varphi_+, \quad H(\omega) = \frac{1}{2}(\varphi_- - i\gamma_2\gamma_1\varphi_+). \tag{4.16}$$

By means of (4.16) we rewrite (4.15) as

$$\psi_- = (x_3 - \gamma_2\gamma_3 x_2)Z + H. \tag{4.17}$$

After substitution of (4.17) into (4.13) we get the following system of reduced equations

$$2\gamma_3 Z + (\gamma_0+\gamma_1)\frac{\partial H}{\partial\omega_1} + \gamma_3\omega_2\frac{\partial Z}{\partial\omega_2} = -imH,$$
$$(\gamma_0+\gamma_1)\omega_2^2\frac{\partial Z}{\partial\omega_1} + \gamma_3\omega_2\frac{\partial H}{\partial\omega_2} = -im\omega_2^2 Z. \tag{4.18}$$

We shall look for solutions of this system in the form

$$Z = \omega_2^{-2}A(\omega_2)\exp[i(\gamma_0+\gamma_1)f(\omega_1)],$$
$$H = B(\omega_2)\exp[i(\gamma_0+\gamma_1)f(\omega_1)],$$

where A and B are some 4×4 matrices and f is an arbitrary differentiable function. Now one can easy solve (4.18) and write down the solution of (4.13),

$$\psi_-(x) = \left[\frac{1}{\omega_2}(\gamma_2x_2+\gamma_3x_3)J_1(im\omega_2)\right]\exp[i(\gamma_0+\gamma_1)f(\omega_1)]_x, \tag{4.19}$$

where J_1 and J_0 are Bessel functions and χ is a four-component constant.

It is noteworthy that ansatz (4.15) has, due to its construction, a vector rather than spinor nature and therefore solution (4.19) of the Dirac equation (4.13) cannot be obtained within the framework of local symmetry of (4.13). Indeed, ansatz (4.15) (and therefore solution (4.19)) is invariant with respect to operators P_0-P_3 and $J_{23}^2+\frac{1}{4}$, $(J_{23}=x_2P_3-x_3P_2-\frac{1}{2}\gamma_2\gamma_3)$, the latter being a non-Lie one (differentional oprator of second order).

In conclusion, let use note that there is a simple connection between $P^{(2)}(1,3)$-invariant ansätze and $P^{(1)}(1,3)$ invariant ones. Since

$$\hat S^{(2)}_{\mu\nu} = \hat S^{(1)}_{\mu\nu} + \hat Q_{\mu\nu}$$

(see (3.4), (3.10) and (3.11)), we can write

$$\hat\Psi^{(2)}(x) = \exp(f(x)Q)\Psi^{(1)}(x), \tag{4.20}$$

where $f(x)$ is some smooth function, Q is an element of six-dimensional Pauli–Touschek algebra (3.11). It is natural to consider relation (4.20) as a conection between bosonic and fermionic fields.

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